COLLOQUIUM MATHEMATICUM

VOL. 131 2013 NO. 2

GROUPS WITH FINITELY MANY CONJUGACY CLASSES OF NON-NORMAL SUBGROUPS OF INFINITE RANK

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Abstract. It is proved that if a locally soluble group of infinite rank has only finitely many non-trivial conjugacy classes of subgroups of infinite rank, then all its subgroups are normal.

1. Introduction. A famous theorem of B. H. Neumann [13] proves that all conjugacy classes of subgroups of a group G are finite if and only if the centre $Z(G)$ has finite index in G. This result suggests that the size of conjugacy classes of subgroups has a strong influence on the structure of a group, and this phenomenon was confirmed in a paper by A. V. Izosov and N. F. Sesekin [11], dealing with groups having only finitely many infinite classes of conjugate subgroups. On the other hand, it is known that there exist infinite simple groups in which all proper non-trivial subgroups are conjugate (see [10]).

Recall also that a group G is said to have *finite* (*Prüfer*) *rank* r if every finitely generated subgroup of G can be generated by at most r elements, and r is the least positive integer with such property. A classical theorem of A. I. Mal'tsev [12] states that a locally nilpotent group of infinite rank must contain an abelian subgroup of infinite rank. The behaviour of subgroups of infinite rank in a (generalized) soluble group has been investigated in a series of recent papers (see for instance [4]–[7]). In particular, M. J. Evans and Y. Kim [8] have proved that if G is a (generalized) soluble group in which all subgroups of infinite rank are normal, then either G is a Dedekind group or it has finite rank.

The aim of the present paper is to provide a further contribution to this topic, characterizing groups having few conjugacy classes of subgroups of infinite rank. It can be proved that a locally soluble group containing finitely many normal subgroups of infinite rank must have finite rank, so that in particular this holds for locally soluble groups with finitely many

2010 *Mathematics Subject Classification*: Primary 20F24; Secondary 20F16.
Key words and phrases: conjugacy class, group of infinite rank.

DOI: 10.4064/cm131-2-6

conjugacy classes of subgroups of infinite rank. Thus the main result of this paper deals with the structure of groups in which non-normal subgroups of infinite rank fall into finitely many conjugacy classes.

THEOREM. *Let G be a locally soluble group with finitely many non-trivial conjugacy classes of subgroups of infinite rank. Then either G has finite rank or it is a Dedekind group.*

Most of our notation is standard and can be found in [14].

2. Proofs. It is known that if G is any locally soluble group of finite rank, then there exists a positive integer k such that the subgroup $G^{(k)}$ is hypercentral (see [14, Part 2, Lemma 10.39]). Since the commutator subgroup of any (non-trivial) hypercentral group is a proper subgroup, we have the following consequence.

LEMMA 2.1. *Let G be a perfect (non-trivial) locally soluble group. Then G has infinite rank.*

The above lemma can be improved in the following way.

LEMMA 2.2. *Let G be a locally soluble group, and let N be a perfect non-trivial normal subgroup of G. Then N contains a proper G-invariant subgroup of infinite rank.*

Proof. The subgroup N has infinite rank by Lemma 2.1. Assume for a contradiction that all proper G-invariant subgroups of N have finite rank. Let W be the largest G-invariant subgroup of N having an ascending G-invariant series with abelian factors, and suppose that $W \neq N$. Since N is perfect, it cannot have maximal G-invariant subgroups, and hence there exists a normal subgroup K of G such that $W < K < N$. Then K has finite rank, and so the subgroup $K^{(n)}$ is hypercentral for some positive integer n. In particular, K has an ascending characteristic series with abelian factors, and so K/W contains an abelian non-trivial G-invariant subgroup. This contradiction shows that $N = W$ has an ascending G-invariant series

$$\{1\} = N_0 < N_1 < \cdots < N_\alpha < N_{\alpha+1} < \cdots < N_\lambda = N$$

with abelian factors. Moreover, λ must be a limit ordinal as $N = N'$, and so all factors of such series have finite rank. Thus the Hirsch–Plotkin radical H of N is hypercentral by a result of Charin (see [14, Part 2, p. 39]), and hence $H' \neq H$. Thus H is a proper subgroup of N, and so it has finite rank. It follows that N contains a normal subgroup M such that the index $|N : M|$ is finite and the commutator subgroup M' of M is hypercentral (see [14, Part 2, Theorem 8.16]). But N has no proper subgroups of finite index, so that $N = M$ and hence $N = N'$ is hypercentral. This contradiction proves the lemma. ∎

It is well-known that locally soluble groups with finitely many normal subgroups are finite. A corresponding result holds for groups having only a finite number of normal subgroups of infinite rank.

PROPOSITION 2.3. *Let G be a locally soluble group having only finitely many normal subgroups of infinite rank. Then G has finite rank.*

Proof. Assume for a contradiction that the group G has infinite rank, and let N be a minimal element of the set of all normal subgroups of G of infinite rank. The factor group G/N has only finitely many normal subgroups, and hence it is finite, because chief factors of locally soluble groups are abelian. Since every proper G-invariant subgroup of N has finite rank, it follows from Lemma 2.2 that N' is properly contained in N. Then N' has finite rank, so that G/N' has infinite rank, and hence replacing G by G/N' we may suppose without loss of generality that N is abelian. Clearly, N has no proper subgroups of finite index, and so it is divisible.

Let T be the subgroup consisting of all elements of finite order of N, and let S be the socle of T. Then S is a proper G-invariant subgroup of N, and hence it has finite rank. It follows that T has finite rank, and replacing G by the factor group G/T it can also be assumed that N is torsion-free. Let A be a maximal free abelian subgroup of N. Then N/A is periodic and A has only finitely many conjugates in G, so that also G/A_G is periodic and hence the core A_G of A has infinite rank. This contradiction completes the proof of the statement. ■

COROLLARY 2.4. *Let G be a locally soluble group having only finitely many conjugacy classes of subgroups of infinite rank. Then G has finite rank.*

Proof. Clearly, the group G has only finitely many normal subgroups of infinite rank, and so the statement follows from Proposition 2.3. ■

In the proof of our main theorem we will also need the following result of D. I. Zaĭtsev, for a proof of which we refer to [1, Lemma 4.6.3].

LEMMA 2.5. *Let G be a group locally satisfying the maximal condition on subgroups. If X is a subgroup of G such that $X^g \leq X$ for some element g of G, then $X^g = X$.*

Zaĭtsev's lemma shows in particular that, at least within the universe of locally polycyclic groups, the condition of having finitely many non-trivial conjugacy classes of subgroups is closely related to the minimal condition on non-normal subgroups.

LEMMA 2.6. *Let G be an abelian group of infinite rank. Then G is covered by its proper subgroups of infinite rank.*

Proof. If x is any element of G, the factor group $G/\langle x\rangle$ has infinite rank, and so it contains a proper subgroup of infinite rank. It follows that x belongs to a proper subgroup of G of infinite rank, and hence G is covered by its proper subgroups of infinite rank. ∎

LEMMA 2.7. *Let G be a locally soluble group satisfying the minimal condition on non-normal subgroups of infinite rank. Then either G is a Dedekind group or it has finite rank.*

Proof. Assume for a contradiction that the statement is false, and let G be a counterexample. Since G has infinite rank but is not a Dedekind group, it is known that G must contain some non-normal subgroup of infinite rank (see [8, Theorem C]). Let M be a minimal element of the set of non-normal subgroups of G of infinite rank. Then each proper subgroup of infinite rank of M is normal in G, and hence in particular M is a Dedekind group. Clearly, M/M' has infinite rank, and so it follows from Lemma 2.6 that M is covered by its proper subgroups of infinite rank. Thus M is normal in G, and this contradiction proves the statement. ∎

COROLLARY 2.8. *Let G be a locally polycyclic group with finitely many non-trivial conjugacy classes of subgroups of infinite rank. Then either G has finite rank or it is a Dedekind group.*

Proof. Assume that

$$X_1 > X_2 > \cdots > X_n > X_{n+1} > \cdots$$

is an infinite descending sequence of non-normal subgroups of G of infinite rank. Then there exist distinct positive integers h and k such that X_h and X_k are conjugate in G, and hence $X_h = X_k$ by Lemma 2.5. This contradiction shows that the group G satisfies the minimal condition on non-normal subgroups of infinite rank, and so the statement follows from Lemma 2.7. ∎

Groups with finitely many non-normal subgroups have been completely described by N. S. Hekster and H. W. Lenstra [9]. Moreover, it has been proved in [3] that locally soluble groups with finitely many non-trivial conjugacy classes actually have only a finite number of non-normal subgroups. The description given by Hekster and Lenstra has the following consequence.

LEMMA 2.9. *Let G be an infinite group having only finitely many non-normal subgroups. Then either G is a Dedekind group or it is a periodic metabelian group of finite rank.*

Our last lemma deals with the behaviour of the commutator subgroup of a soluble group whose non-normal subgroups of infinite rank fall into finitely many conjugacy classes.

Lemma 2.10. *Let G be a soluble group with finitely many non-trivial conjugacy classes of subgroups of infinite rank. Then the commutator subgroup G' of G has finite rank.*

Proof. Assume that the statement is false, and choose a counterexample G with smallest derived length. If A is the last non-trivial term of the derived series of G, it follows that the statement holds for the factor group G/A, so that G'/A has finite rank and hence A has infinite rank. Let N be any G-invariant subgroup of A of infinite rank. Clearly, the group G/N has finitely many conjugacy classes of non-normal subgroups, so that it has only finitely many non-normal subgroups (see [3]). On the other hand, it follows from Corollary 2.8 that G is not locally polycyclic, and hence G/N cannot be periodic. Application of Lemma 2.9 shows that G/N is abelian, and so $G' \leq N$. Therefore $G' = A$ is abelian and all proper G-invariant subgroups of G' have finite rank.

Let X and Y be proper subgroups of G' of finite index. Then X and Y cannot be normal in G, and they fall into different conjugacy classes of G, provided that $|G' : X| \neq |G' : Y|$. It follows that G' contains only finitely many subgroups of finite index, so that the finite residual J of G' has finite index in G', and hence $J = G'$. This means that G' has no proper subgroups of finite index, i.e. G' is a divisible group. Let T be the subgroup consisting of all elements of G' of finite order. Then T is divisible and has the same rank as its socle, so that T must have finite rank. Then the factor group G/T is likewise a counterexample, and so without loss of generality it can be assumed that G' is torsion-free. For each prime number p, there exists a subgroup H_p of G' such that G'/H_p is a group of type p^∞. Clearly, the subgroups H_p have infinite rank and are pairwise non-conjugate. This contradiction completes the proof of the lemma. ∎

We are now in a position to prove the main result of the paper.

Proof of the Theorem. Assume that the statement is false, and suppose first that G is soluble. As G has infinite rank, it contains an abelian subgroup A of infinite rank (see [2]), and of course A can be chosen to be either free abelian or of prime exponent p. Moreover, the commutator subgroup G' of G has finite rank by Lemma 2.10, and so we may also choose A in such a way that $A \cap G' = \{1\}$. Observe that A cannot be normal in G, since otherwise the factor group G/A would be periodic by Lemma 2.9, and G would be locally polycyclic, contrary to Corollary 2.8.

Suppose that A is free abelian, and let q be any prime number. For each positive integer n, the above argument shows that the subgroup A^{q^n} is not normal in G, and hence there exist positive integers h and k such that $h < k$

and $A^{q^h} = (A^{q^k})^x$ for some element x of G. Then $A^{q^h}G' = A^{q^k}G'$, and so

$$A^{q^h} = (A^{q^k}G') \cap A^{q^h} = A^{q^k},$$

which is of course a contradiction.

Therefore A has prime exponent p. In this case there exist subgroups of finite index X and Y of A such that

$$|A : X| \neq |A : Y|$$

and X and Y are conjugate in G. It follows that $XG' = YG'$, and then

$$|A : X| = |AG' : XG'| = |AG' : YG'| = |A : Y|;$$

this further contradiction proves the statement when G is soluble.

Suppose now that G is an arbitrary locally soluble group for which the statement is false, and assume that $G^{(n)} = G^{(n+1)}$ for some non-negative integer n. As G is not soluble by the first part of the proof, the subgroup $G^{(n)}$ is not trivial, and hence by Lemma 2.2 it contains a proper G-invariant subgroup K of infinite rank. The factor group G/K has only finitely many conjugacy classes of non-normal subgroups, so that it has only finitely many non-normal subgroups (see [3]), and hence is soluble by Lemma 2.9. This contradiction shows that $G^{(n)} \neq G^{(n+1)}$ for each non-negative integer n.

Assume that $G^{(n)}$ has finite rank for some $n \geq 3$. Then the soluble group $G/G^{(n)}$ has infinite rank, and so it is a Dedekind group by the soluble case. This contradiction shows that each $G^{(n)}$ has infinite rank. For each non-negative integer n, the group $G^{(n)}/G^{(n+3)}$ contains a non-normal subgroup $X_n/G^{(n+3)}$, and hence $\{X_n \mid n \in \mathbb{N}\}$ is a set of non-normal subgroups of infinite rank of G which fall into infinitely many conjugacy classes. This last contradiction completes the proof of the Theorem. ∎

Acknowledgements. This work was partially supported by MIUR-PRIN 2009 (Teoria dei Gruppi e Applicazioni). The authors are members of GNSAGA (INdAM).

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Received 26 October 2012;
revised 21 April 2013 (5798)