COLLOQUIUM MATHEMATICUM

VOL. 131 2013 NO. 2

ADMISSIBILITY FOR QUASIREGULAR REPRESENTATIONS OF EXPONENTIAL SOLVABLE LIE GROUPS

BY

VIGNON OUSSA (Bridgewater, MA)

Abstract. Let N be a simply connected, connected non-commutative nilpotent Lie group with Lie algebra $\mathfrak{n}$ of dimension n. Let H be a subgroup of the automorphism group of N. Assume that H is a commutative, simply connected, connected Lie group with Lie algebra $\mathfrak{h}$. Furthermore, assume that the linear adjoint action of $\mathfrak{h}$ on $\mathfrak{n}$ is diagonalizable with non-purely imaginary eigenvalues. Let $\tau = \mathrm{Ind}_H^{N\rtimes H} 1$. We obtain an explicit direct integral decomposition for τ, including a description of the spectrum as a submanifold of $(\mathfrak{n}+\mathfrak{h})^*$, and a formula for the multiplicity function of the unitary irreducible representations occurring in the direct integral. Finally, we completely settle the admissibility question for τ. In fact, we show that if $G = N \rtimes H$ is unimodular, then τ is never admissible, and if G is non-unimodular, then τ is admissible if and only if the intersection of H and the center of G is equal to the identity of the group. The motivation of this work is to contribute to the general theory of admissibility, and also to shed some light on the existence of continuous wavelets on non-commutative connected nilpotent Lie groups.

1. Introduction. Let π be a unitary representation of a locally compact group X, acting in some Hilbert space $\mathcal{H}$. We say that π is *admissible* if there exists some function $\phi \in \mathcal{H}$ such that the operator W_ϕ defines an isometry on $\mathcal{H}$, and $W_\phi : \mathcal{H} \to L^2(X)$, $W_\phi\psi(x) = \langle \psi, \pi(x)\phi\rangle$. For continuous wavelets on the real line, the admissibility of the quasiregular representation $\mathrm{Ind}_{(0,\infty)}^{\mathbb{R}\rtimes(0,\infty)} 1$ of the '$ax+b$' group which is a unitary representation acting in $L^2(\mathbb{R})$ leads to the well-known Calderón condition.

Given any locally compact group, a great deal is already known about the admissibility of its left regular representation [10]. For example, it is known that the left regular representation of the '$ax+b$' group is admissible. The left regular representation of $\mathbb{R} \rtimes (0,\infty)$ admits a decomposition into a direct sum of two unitary irreducible representations acting in $L^2((0,\infty))$, each with infinite multiplicities. Thus, the Plancherel measure of this affine group is supported on two points. It is also known that the quasiregular representation $\mathrm{Ind}_{(0,\infty)}^{\mathbb{R}\rtimes(0,\infty)} 1$ is unitarily equivalent to a subrepresentation of the left regular representation, and thus is admissible.

2010 *Mathematics Subject Classification*: Primary 22E27; Secondary 22E30.
Key words and phrases: admissibility, representation, solvable, Lie group.

DOI: 10.4064/cm131-2-7

Several authors have studied the admissibility of various representations; see [1], and also [13], where Guido Weiss and his collaborators obtained an almost complete characterization of groups of the type $H \leq \mathrm{GL}(n, \mathbb{R})$ for which the quasiregular representation $\tau = \mathrm{Ind}_H^{\mathbb{R}^n \rtimes H} 1$ is admissible. It is known that if τ is admissible then the stabilizer subgroup of the action of H on characters belonging to the unitary dual of $\mathbb{R}^n$ must be compact almost everywhere. However, this condition is not sufficient to guarantee the admissibility of τ. In [11], a complete characterization of dilation groups $H \leq \mathrm{GL}(n, \mathbb{R})$ is given. On non-commutative nilpotent domains, Liu and Peng answered the question for $\tau = \mathrm{Ind}_H^{N \rtimes H} 1$, where N is the Heisenberg group, and H is a 1-parameter dilation group. They have also constructed some explicit continuous wavelets on the Heisenberg group (see [16]).

In 2007, Currey considered $\tau = \mathrm{Ind}_H^{N \rtimes H} 1$, where N is a connected, simply connected non-commutative nilpotent Lie group, and H is a commutative, connected, simply connected Lie group such that $G = N \rtimes H$ is completely solvable and $\mathbb{R}$-split. He settled the admissibility question for τ under the restriction that the stabilizer subgroup inside H is trivial, and he also gave an explicit construction of some continuous wavelets (see [7]). However, he did not address the case where the stabilizer of the action of H on the unitary dual of N is non-trivial, leaving this problem open. In 2011, we provided some answers for the admissibility of monomial representations for completely solvable exponential Lie groups [8]. We now know that when N is not commutative, the stabilizer of the action of H on the dual of N does not have to be compact in order for τ to be admissible. Also, we were recently informed that new results on the subject of admissibility were obtained by Cordero and Tabacco [3], and Filippo De Mari and Ernesto De Vito [9] for a different class of groups.

The purpose of this paper is to extend the results of Currey [5]. Firstly, we make no assumption that the little group inside H is trivial. Secondly, the class of groups considered in this paper is larger than the class considered by Currey. This class of groups also contains exponential solvable Lie groups which are not completely solvable. We consider the situation where the action of $\mathfrak{h}$ on $\mathfrak{n}$ has roots of the type $\alpha + i\beta$, with $\alpha \neq 0$. Let us be more precise. Let N be a simply connected, connected non-commutative nilpotent Lie group with real Lie algebra $\mathfrak{n}$. Let H be a subgroup of the automorphism group of N, which we denote by $\mathrm{Aut}(N)$. Assume that H is isomorphic to $\mathbb{R}^r$ with Lie algebra $\mathfrak{h}$. Furthermore, assume that the linear adjoint action of $\mathfrak{h}$ on $\mathfrak{n}$ is diagonalizable with non-purely imaginary complex eigenvalues. We form the semidirect product Lie group $G = N \rtimes H$ such that G is an exponential solvable Lie group with Lie algebra $\mathfrak{g}$. More precisely, there exist basis elements such that $\mathfrak{h} = \mathbb{R}A_1 \oplus \cdots \oplus \mathbb{R}A_r$, and basis elements Z_i for the complexification of $\mathfrak{n}$ such that Z_i are eigenvectors for the linear operator $\operatorname{ad} A_k, k = 1, \ldots, r$. Furthermore, we have $\operatorname{ad} A_k Z_j = [A_k, Z_j] = \gamma_j(A_k) Z_j$ with weight $\gamma_j(A_k) = \lambda(A_k)(1 + i\alpha_j)$, $\lambda \in \mathfrak{h}^*$, a real-valued linear functional, and $\alpha_j \in \mathbb{R}$. We observe that G is an exponential solvable Lie group, and is therefore type I. We define the action of H on N multiplicatively, and the multiplication law for G is obtained as follows: $(n, h)(n', h') = (nh \cdot n', hh')$. The Haar measure of G is $|\det \operatorname{Ad}(h)|^{-1} dndh$, where dn, dh are the canonical Haar measures on N, H respectively. We will denote by L the left regular representation of G acting in $L^2(G)$. We consider the quasiregular representation $\tau = \operatorname{Ind}_H^G 1$ acting in $L^2(N)$ as follows:

$$\tau(n,1)f(m) = f(n^{-1}m), \qquad \tau(1,h)f(m) = |\det(\operatorname{Ad} h)|^{-1/2} f(h^{-1}m).$$

In this paper, mainly motivated by the admissibility question for τ, we aim to obtain an explicit decomposition of τ, including a precise description of its spectrum, an explicit formula for the multiplicity function, the measure occurring in the decomposition of τ, and finally, we completely settle the admissibility question for τ. Here is the main result of our paper.

THEOREM 1. *Let N be a simply connected, connected non-commutative nilpotent Lie group with Lie algebra $\mathfrak{n}$ of dimension n. Let H be a subgroup of the automorphism group of N. Assume that H is a commutative simply connected, connected Lie group with Lie algebra $\mathfrak{h}$. Furthermore, assume that the linear adjoint action of $\mathfrak{h}$ on $\mathfrak{n}$ is diagonalizable with non-purely imaginary eigenvalues such that $N \rtimes H$ is an exponential solvable Lie group. Let $\tau = \operatorname{Ind}_H^{N \rtimes H} 1$.*

(1) *Assume that* $\dim(H \cap Z(G)) = 0$. *Then τ is admissible if and only if $N \rtimes H$ is non-unimodular.*
(2) *Assume that* $\dim(H \cap Z(G)) \neq 0$. *Then τ is never admissible.*

2. Preliminaries. We recall that the *coadjoint action* of G on $\mathfrak{g}^*$ is simply the dual of the adjoint action, and is also defined multiplicatively as $g \cdot l(X) = l(\operatorname{Ad}_{g^{-1}} X)$, $g \in G$, $X \in \mathfrak{g}^*$. In this paper, the group G always stands for $N \rtimes H$ as described earlier.

DEFINITION 2. Given two representations π, θ of G acting on the Hilbert spaces $\mathcal{H}_\pi, \mathcal{H}_\theta$ respectively, if there exists a bounded linear operator $T : \mathcal{H}_\pi \to \mathcal{H}_\theta$ such that $\theta(x)T = T\pi(x)$ for all $x \in G$, we say T *intertwines* π with θ. If T is a unitary operator, then we say the representations are *unitarily equivalent*, we write $\pi \simeq \theta$, and $[\pi] = [\theta]$.

LEMMA 3. *Let L be the left regular representation of G acting in $L^2(G)$. Then L is admissible if and only if G is non-unimodular.*

Lemma 3 was proved in more general terms by Hartmut Führ in Theorem 4.23 of [10]. In fact, the general statement of his proof only assumes that G is type I and connected.

LEMMA 4. *If G is non-unimodular, then τ is admissible if and only if τ is equivalent to a subrepresentation of L.*

LEMMA 5. *Let π, ρ be two type I unitary representations of G with direct integral decompositions $\pi \simeq \int_{\widehat{G}}^{\oplus} \sigma \otimes 1_{\mathbb{C}^{m_\pi}}\, d\mu(\sigma)$ and $\rho \simeq \int_{\widehat{G}}^{\oplus} \sigma \otimes 1_{\mathbb{C}^{m'_\rho}}\, d\mu'\sigma$. Then π is equivalent to a subrepresentation of ρ if and only if μ is absolutely continuous with μ' and $m_\pi \leq m'_\rho$ μ-a.e.*

A clear explanation of Lemmas 4 and 5 is given on page 126 of the monograph [10]. The following theorem is due to Lipsman, and the proof is in Theorem 7.1 of [14].

LEMMA 6. *Let $G = N \rtimes H$ be a semidirect product of locally compact groups, N normal and type I. Let $\gamma \in \widehat{N}$, H_γ the stability group. Let $\widetilde{\gamma}$ be any extension of γ to H_γ. Suppose that N is unimodular, $\widehat{N}/H$ is countably separated and $\widetilde{\gamma}$ is a type I representation for μ_N-almost every $\gamma \in \widehat{N}$. Let*

$$\widetilde{\gamma} \simeq \int_{\widehat{H_\gamma}}^{\oplus} n_\gamma(\sigma)\sigma \, d\mu_\gamma(\sigma)$$

be the unique direct integral decomposition of $\widetilde{\gamma}$. Then

$$\mathrm{Ind}_H^G 1 \simeq \int_{\widehat{N}/H}^{\oplus} \int_{\widehat{H_\gamma}}^{\oplus} \pi_{\gamma,\sigma} \otimes 1_{\mathbb{C}^{n_\gamma(\sigma)}} \, d\mu_\gamma(\sigma)\, d\dot{\mu}_N(\gamma),$$

where $\dot{\mu}_N$ is the push-forward of the Plancherel measure on μ_N on $\widehat{N}$.

It is now clear that in order to settle the admissibility question, it is natural to compare both representations. As G is a type I group, there exist unique direct integral decompositions for both L and τ. Since both representations use the same family of unitary irreducible representations in their direct integral decomposition, in order to compare the representations, it is important to obtain the direct integral decompositions for both L and τ, and to check for the containment of τ inside L. In order to have a complete picture of the results in Lemma 6, we will need the following:

1. A precise description of the spectrum of the quasiregular representation.
2. The multiplicity function of the irreducible representations occurring in the decomposition of the quasiregular representation.
3. A description of the push-forward of the Plancherel measure of N.

Our approach here will rely on the orbit method, and we will construct a smooth orbital cross-section to parametrize the dual of the group G.

3. Orbital parameters. In this section, we will introduce the reader to the theory developed by Currey, Arnal, and Dali [2] for the construction of cross-sections for coadjoint orbits in $\mathfrak{g}^*$, where $\mathfrak{g}$ is any n-dimensional real exponential solvable Lie algebra with Lie group G. First, we consider a complexification of the Lie algebra $\mathfrak{g}$, which we denote here by $\mathfrak{c} = \mathfrak{g}_{\mathbb{C}}$. Let us be more precise. We begin by fixing an ordered basis $\{Z_1, \dots, Z_n\}$ for the Lie algebra $\mathfrak{c}$, where $Z_i = \operatorname{Re} Z_i + i \operatorname{Im} Z_i$ with $\operatorname{Re} Z_i$ and $\operatorname{Im} Z_i$ belonging to $\mathfrak{g}$, such that the following conditions are satisfied:

1. For each $k \in \{1, \dots, n\}$, $\mathfrak{c}_k = \mathbb{C}\text{-}\operatorname{span}\{Z_1, \dots, Z_k\}$ is an ideal.
2. If $\mathfrak{c}_j \neq \overline{\mathfrak{c}_j}$ then $\mathfrak{c}_{j+1} = \overline{\mathfrak{c}_{j+1}}$ and $Z_{j+1} = \overline{Z_j}$.
3. If $\mathfrak{c}_j = \overline{\mathfrak{c}_j}$ and $\mathfrak{c}_{j-1} = \overline{\mathfrak{c}_{j-1}}$ then $Z_j \in \mathfrak{g}$.
4. For any $A \in \log H$, $[A, Z_j] = \gamma_j(A) Z_j \bmod \mathfrak{c}_{j-1}$ with weight
$$\gamma_j(A_k) = \lambda(A_k)(1 + i\alpha_j),$$
where $\lambda \in \mathfrak{h}^*$ is a real-valued linear functional and $\alpha_j \in \mathbb{R}$.

Such a basis is called an *adaptable basis*. We recall the procedure described in [2]. For any $l \in \mathfrak{g}^*$, we define, for any subset $\mathfrak{s}$ of $\mathfrak{c}$, $\mathfrak{s}^l = \{Z \in \mathfrak{c} : l([\mathfrak{s}, Z]) = 0\}$ and $\mathfrak{s}(l) = \mathfrak{s}^l \cap \mathfrak{s}$. Also, we set

$$\begin{aligned} i_1(l) &= \min\{j : \mathfrak{c}_j \not\subset \mathfrak{c}(l)\}, \\ \mathfrak{h}_1(l) &= \mathfrak{c}_{i_1}^l = (Z_{i_1})^l, \\ j_1(l) &= \min\{j : \mathfrak{c}_j \not\subset \mathfrak{h}_1(l)\}. \end{aligned}$$

By induction, for any $k \in \{1, \dots, n\}$, we define

$$i_k(l) = \min\{j : \mathfrak{c}_j \cap \mathfrak{h}_{k-1}(l) \not\subset \mathfrak{h}_{k-1}(l)^l\}, \tag{3.1}$$

$$\mathfrak{h}_k(l) = (\mathfrak{h}_{k-1}(l) \cap \mathfrak{c}_{i_k})^l \cap \mathfrak{h}_{k-1}(l), \tag{3.2}$$

$$j_k(l) = \min\{j : \mathfrak{c}_j \cap \mathfrak{h}_{k-1}(l) \not\subset \mathfrak{h}_k(l)\}. \tag{3.3}$$

Finally, we put $\mathbf{e}(l) = \mathbf{i}(l) \cup \mathbf{j}(l)$, where $\mathbf{i}(l) = \{i_k(l) : 1 \le k \le d\}$ and $\mathbf{j}(l) = \{j_k(l) : 1 \le k \le d\}$. An interesting well-known fact is that $\operatorname{card}(\mathbf{e}(l))$ is always even. Also, observe the sequence $\{i_k : 1 \le k \le d\}$ is increasing, and $i_k < j_k$ for $1 \le k \le d$.

Following Definition 2 of [2], let $\mathcal{P}$ be a partition of the linear dual of the Lie algebra $\mathfrak{g}$.

Definition 7. We say $\mathcal{P}$ is an *orbital stratification* of $\mathfrak{g}^*$ if the following conditions are satisfied:

(1) Each element Ω in $\mathcal{P}$ is G-invariant.
(2) For each Ω in $\mathcal{P}$, the coadjoint orbits in Ω have the same dimension.
(3) There is a linear ordering on $\mathcal{P}$ such that for each $\Omega \in \mathcal{P}$,
$$\bigcup \{\Omega' : \Omega' \le \Omega\}$$
is a Zariski open subset of $\mathfrak{g}^*$.

The elements Ω belonging to a stratification are called *layers* of the dual space $\mathfrak{g}^*$.

DEFINITION 8. Given any subset of $\mathbf{e}$ of $\{1, \dots, n\}$, we define the set

$$\Omega_{\mathbf{e}} = \{l \in \mathfrak{g}^* : \mathbf{e}(l) = \mathbf{e}\},$$

which is G-invariant. The collection of non-empty $\Omega_{\mathbf{e}}$ forms a partition of $\mathfrak{g}^*$. Such a partition is called a *coarse stratification* of $\mathfrak{g}^*$. Given $\mathbf{e}(l) = \{i_1, \dots, i_d\} \cup \{j_1, \dots, j_d\}$, we define

$$\Omega_{\mathbf{e},\mathbf{j}} = \{l \in \mathfrak{g}^* : \mathbf{e}(l) = \mathbf{e} \text{ and } \mathbf{j}(l) = \mathbf{j}\}.$$

The collection of non-empty $\Omega_{\mathbf{e},\mathbf{j}}$ forms a partition of $\mathfrak{g}^*$ called the *fine stratification* of $\mathfrak{g}^*$, and the elements $\Omega_{\mathbf{e},\mathbf{j}}$ are called *fine layers*.

We keep the notations used in [2].

1. We fix an adaptable basis, an open dense layer $\Omega_{\mathbf{e},\mathbf{j}}$. We let $\mathfrak{c}_0 = \{0\}$, and we define

$$\begin{aligned}
I &= \{0 \le j \le n + r : \mathfrak{c}_j = \overline{\mathfrak{c}_j}\},\\
j' &= \max(\{0, 1, \dots, j - 1\} \cap I)\\
j'' &= \min(\{j, j + 1, \dots, n + r\} \cap I),\\
K_0 &= \{1 \le k \le d : i''_k - i'_k = 1\},\\
K_1 &= \{1 \le k \le d : i_k \notin I \text{ and } i_k + 1 \notin \mathbf{e}\},\\
K_2 &= \{1 \le k \le d : i_k - 1 \in \mathbf{j} \setminus I\},\\
K_3 &= \{1 \le k \le d : i_k \notin I \text{ and } i_k + 1 \in \mathbf{j}\},\\
K_4 &= \{1 \le k \le d : i_k \notin I \text{ and } i_k + 1 \in \mathbf{i}\},\\
K_5 &= \{1 \le k \le d : i_k - 1 \in \mathbf{i} \setminus I\}.
\end{aligned} \tag{3.4}$$

We remark here that

$$\mathbf{i} = \bigcup_{j=0}^{5} \{i_k : k \in K_j\}.$$

2. We gather some data corresponding to the fixed fine layer $\Omega_{\mathbf{e},\mathbf{j}}$. For each $j \in \mathbf{e}$, we define recursively the rational function $Z_j : \Omega \to \mathfrak{c}_{j''}$ such that, for $k \in \{1, \dots, d\}$,

$$\begin{aligned}
&V_1(l) = Z_{i_1}(l), \quad U_1(l) = Z_{j_1}(l),\\
&V_k(l) = \rho_{k-1}(Z_{i_k}(l), l), \quad U_k(l) = \rho_{k-1}(Z_{j_k}(l), l),\\
&Z_{i_k}(l) = \beta_{1,k}(l) \operatorname{Re} Z_{i_k} + \beta_{2,k}(l) \operatorname{Im} Z_{i_k},\\
&Z_{j_k}(l) = \alpha_{1,k}(l) \operatorname{Re} Z_{j_k} + \alpha_{2,k}(l) \operatorname{Im} Z_{j_k},\\
&\alpha_{1,k} = l[\operatorname{Re} Z_{j_k}, V_k(l)], \alpha_{2,k} = l[\operatorname{Im} Z_{j_k}, V_k(l)].
\end{aligned} \tag{3.5}$$

Furthermore,

$$\rho_k(Z,l) = \rho_{k-1}(Z,l) - \frac{l[\rho_{k-1}(Z,l), U_k(l)]}{l[V_k(l), U_k(l)]} V_k(l) - \frac{l[\rho_{k-1}(Z,l), V_k(l)]}{l[U_k(l), V_k(l)]} U_k(l)$$

and $\rho_0(\cdot, l)$ is the identity map.

(a) If $k \in K_0$ then $\beta_{1,k}(l) = 1$ and $\beta_{2,k}(l) = 0$.

(b) If $k \in K_1$ then

$$\beta_{1,k}(l) = l([\rho_{k-1}(Z_{j_k}, l), \operatorname{Re} Z_{i_k}]), \quad \beta_{2,k}(l) = l([\rho_{k-1}(Z_{j_k}, l), \operatorname{Im} Z_{i_k}]).$$

(c) If $k \in K_2$ then $i_k - 1 = j_k$, $\beta_{1,k}(l) = -\alpha_{2,k}(l)$ and $\beta_{2,k}(l) = -\alpha_{1,k}(l)$.

(d) If $k \in K_3$ then $\beta_{1,k}(l) = 0$ and $\beta_{2,k}(l) = 1$.

(e) If $k \in K_4$ (K_5 is covered here too) and if $Z_{j_{k+1}} = \overline{Z_{j_k}}$, then $\beta_{1,k}(l) = 1$, $\beta_{2,k}(l) = 0$, and

$$Z_{i_k+1}(l) = -\frac{l[U_k(l), \operatorname{Im} Z_{i_k}]}{l[U_k(l), \operatorname{Re} Z_{i_k}]} \operatorname{Re} Z_{i_k+1} - \operatorname{Im} Z_{i_k+1}.$$

3. Let $C_j = \ker \gamma_j \cap \mathfrak{g}$ and $\mathfrak{a}_j(l) = (\mathfrak{g}^l_{j'} \cap C_j)/(\mathfrak{g}^l_{j''} \cap C_j)$. We define $\varphi(l) \subset \mathbf{i}$ by setting $\varphi(l) = \{j \in \mathbf{e} : \mathfrak{a}_j(l) = \{0\}\}$, and we put

$$\mathbf{b}_j(l) = \frac{\gamma_j(U_k(l))}{l[Z_j, U_k(l)]}.$$

The collection of sets $\Omega_{\mathbf{e},\mathbf{j},\varphi} = \{l \in \Omega_{\mathbf{e},\mathbf{j}} : \varphi(l) = \varphi\}$ forms a partition of $\mathfrak{g}^*$, refining the fine stratification, which we call the *ultrafine stratification* of $\mathfrak{g}^*$.

4. Letting $\Omega_{\mathbf{e},\mathbf{j},\varphi}$ be a layer obtained by refining the fixed fine layer $\Omega_{\mathbf{e},\mathbf{j}}$, and gathering the data

$$Z_j(l), \mathbf{e}, \varphi(l), \mathbf{b}_j(l),$$

the cross-section for the coadjoint orbits of Ω is given by the set

$$\Sigma = \{l \in \Omega : l(Z_j(l)) = 0,\ j \in \mathbf{e} \setminus \varphi \text{ and } |\mathbf{b}_j(l)| = 1,\ j \in \varphi\}. \tag{3.6}$$

Let us now offer some concrete examples.

Example 9. Let $\mathfrak{g}$ be a Lie algebra spanned by $\{Z, Y, X, A\}$ with the following non-trivial Lie brackets:

$$[X,Y] = Z, \quad [A, X+iY] = (1+i)(X+iY), \quad [A,Z] = 2Z.$$

An adaptable basis is $\{Z, X+iY, X-iY, A\}$ and an arbitrary linear functional is written as $l = (z, x+iy, x-iy, a)$. Here $I = \{0,1,3,4\}$, $1' = 0$, $2' = 1$, $3' = 1$, $4' = 3$, $4'' = 1$, $2'' = 3$, $3'' = 3$, and $4'' = 4$. Put $\mathbf{e} = \{1,2,3,4\}$, and $\mathbf{j} = \{3,4\}$. Next, it is easy to see that $1 \in K_0$ and $2 \in K_3$. Moreover,

$$Z_{i_1}(l) = V_1(l) = Z, \quad Z_{j_1}(l) = U_1(l) = A, \quad Z_{i_2}(l) = Y,$$

$$V_2(l) = \rho_1(Y,l) = Y - \frac{x+y}{2z} Z,$$

$$Z_{j_2}(l) = X, \quad U_2(l) = \rho_1(X,l) = X - \frac{x-y}{2z} Z.$$

Then $\varphi = \{1\}$ and $\Omega_{\mathbf{e},\mathbf{j}} = \{(z, x+iy, x-iy, a) : z \neq 0\}$ and

$$\Sigma = \{(z, x+iy, x-iy, a) \in \Omega : |z| = 1,\ a = x = y = 0\}.$$

EXAMPLE 10. Let $\mathfrak{g}$ be a Lie algebra spanned by

$$\{Z_1, Z_2, Y, X_1, X_2, A\}$$

with the following non-trivial Lie brackets:

$$[X_j, Y] = Z_j, \qquad [A, X_1 + iX_2] = (1+i)(X_1 + iX_2),$$
$$[A, Z_1 + iZ_2] = (1+i)(Z_1 + iZ_2).$$

We choose an adaptable basis

$$\{Z_1 + iZ_2, Z_1 - iZ_2, Y, X_1 + iX_2, X_1 - iX_2, A\}$$

for $\mathfrak{c}$. We compute here that $I = \{0, 2, 3, 5, 6\}$, and $1' = 0$, $2' = 0$, $3' = 2$, $4' = 3$, $5' = 3$, $6' = 5$, $1'' = 2$, $2'' = 2$, $3'' = 3$, $4'' = 5$, $5'' = 5$, $6'' = 6$. Pick $\mathbf{e} = \{1, 3, 4, 6\}$ and $\mathbf{j} = \{6, 4\}$. In this example, the set K_1 contains 1, and K_0 contains 2. Next, by simple computations,

$$Z_{i_1}(l) = (z_1 - z_2)Z_1 + (z_1 + z_2)Z_2, \qquad Z_{j_1} = A, \qquad Z_{i_2}(l) = Y,$$
$$Z_{j_2} = z_1 X_1 + z_2 X_2.$$

Clearly $\varphi = \{1\}$, the corresponding layer is $\Omega_{\mathbf{e},\mathbf{j}} = \{(z, \overline{z}, y, x, \overline{x}) : z \neq 0\}$ and the corresponding cross-section is

$$\Sigma = \{(z, \overline{z}, y, x, \overline{x}) : |z| = 1,\ a = y = 0,\ \mathrm{Re}(\overline{z}x) = 0\}.$$

Now that we introduced the general construction, we will focus on N which is the Lie group of the nilradical of $\mathfrak{g}$. Since N is also an exponential solvable Lie group, formula (3.6) is valid. Let us recall the following well-known facts. The first one is due to Kirillov, and the second one is an application of the 'Mackey Machine' (see [17]).

LEMMA 11. *Let $f \in \mathfrak{n}^*$, and let $\widehat{N}$ be the set of unitary irreducible representations of N up to equivalence. Let $\mathfrak{n}^*/N = \{N \cdot f : f \in \mathfrak{n}^*\}$ be the set of coadjoint orbits. There exists a unique bijection between $\mathfrak{n}^*/N$ and $\widehat{N}$ via the Kirillov map. Thus, the construction of a measurable cross-section for the coadjoint orbits is a natural way to parametrize $\widehat{N}$.*

LEMMA 12. *The set $\widehat{G}$ of unitary irreducible representations of G is a fiber set with base $\widehat{N}/H$ and fibers $\widehat{H_\lambda}$, where H_λ is a closed subgroup of H stabilizing the coadjoint action of H on the linear functional λ.*

We aim here to construct an H-invariant cross-section for the coadjoint orbits of N in $\mathfrak{n}^*$. We consider the nilradical $\mathfrak{n}$ of $\mathfrak{g}$ instead of $\mathfrak{g}$, and we go through the procedure described earlier. We first obtain an adaptable basis $\{Z_1, \ldots, Z_n\}$ for the complexification of the Lie algebra $\mathfrak{n}$, which we denote by $\mathfrak{m}$. Notice that $\{Z_1, \ldots, Z_n, A_1, \ldots, A_{\dim(\mathfrak{h})}\}$ is then an adaptable basis for $\mathfrak{g}$.

First, fixing a dense open layer $\Omega \subset \mathfrak{g}^*$ and $f \in \Omega$, we obtain the jump indices corresponding to the generic layer of $\mathfrak{g}^*$:

$$\mathbf{i}^\circ(f) = \{i_1, \ldots, i_{d^\circ}\}, \quad \mathbf{j}^\circ(f) = \{j_1, \ldots, j_{d^\circ}\},$$
$$\mathbf{e}^\circ(f) = \{i_1, \ldots, i_{d^\circ}\} \cup \{j_1, \ldots, j_{d^\circ}\}.$$

Second, let $\Omega_{\mathbf{e}^\circ, \mathbf{j}^\circ}$ be a fixed fine layer obtained by refining Ω. Given any subset $\mathbf{e}^\circ \subseteq \{1, \ldots, n\}$, the non-empty sets $\Omega_{\mathbf{e}^\circ, \mathbf{j}^\circ}$ are characterized by the Pfaffian of the skew-symmetric matrix $M_{\mathbf{e}^\circ}(f) = [f[Z_i, Z_j]]_{i,j \in \mathbf{e}^\circ}$. Referring to the procedure described in (3.4) and (3.5), we obtain

$$Z_{i_1^\circ}(f), Z_{j_1^\circ}(f), \ldots, Z_{i_{d^\circ}}(f), Z_{j_{d^\circ}}(f),$$

and we have the polarizing sequence $\mathfrak{m} = \mathfrak{h}_0(l) \supseteq \mathfrak{h}_1(l) \supseteq \cdots \supseteq \mathfrak{h}_{d^\circ}(l)$. Thirdly, we compute the following data:

$$I, j', j'', K_0, K_1, \ldots, K_5, V_1(f), \ldots, V_{d^\circ}(f), U_1(f), \ldots, U_{d^\circ}(f), \varphi(f), b_j(f)$$

corresponding to our fine layer $\Omega_{\mathbf{e}^\circ, \mathbf{j}^\circ}$ as described in (3.4) and (3.5). Finally, gathering all the data, we notice that $\varphi(f) = \emptyset$, since according to Proposition 4.1 in [2], $\mathfrak{a}_j(l) = 0$ if and only if $\gamma_j(U_k(l)) \neq 0$ for $j = i_k$. As shown in [2], an H-invariant cross-section for the coadjoint N-orbits for $\Omega_{\mathbf{e}^\circ}$ is given by

$$\Lambda = \{f \in \Omega_{\mathbf{e}^\circ, \mathbf{j}^\circ} : f(Z_j(f)) = 0,\ j \in \mathbf{e}^\circ\}. \tag{3.7}$$

Following the proof of Theorem 4.2 in [2], we have three separate cases:

Case 1. If $j \in I$ or if $j \notin I$ and $j+1 \in \mathbf{e}^\circ$ then $f(Z_j(f)) = 0$ is equivalent to $f(Z_j) = 0$.

Case 2. If $j \notin I$, $j + 1 \notin \mathbf{e}^\circ$, and $j = i_k$ then

$$f(Z_j(f)) = f([\rho_{k-1}(Z_{j_k}, f), \operatorname{Re} Z_j]) \operatorname{Re} f(Z_j) + f[\rho_{k-1}(Z_{j_k}, f), \operatorname{Im} Z_j] \operatorname{Im} f(Z_j).$$

Case 3. If $j \notin I$, $j + 1 \notin \mathbf{e}^\circ$, and $j = j_k$ then the equation $f(Z_j(f)) = 0$ is equivalent to

$$\operatorname{Re}(f[\rho_{k-1}(\overline{Z_j}, f), \operatorname{Re} Z_{i_k}] f(Z_j) = \operatorname{Re}(f[\rho_{k-1}(\overline{Z_j}, f), \operatorname{Im} Z_{i_k}] f(Z_j) = 0.$$

Remark 13. If the assumptions of Case 1 hold for all elements of $\mathbf{e}^\circ$ then

$$\Lambda = \{f \in \Omega_{\mathbf{e}^\circ, \mathbf{j}^\circ} : f(Z_j) = 0,\ j \in \mathbf{e}^\circ\}.$$

Example 14. Let $\mathfrak{g}$ be a nilpotent Lie algebra spanned by $\{Z_1, Z_2, Y_1, Y_2, X_1, X_2\}$ with the only non-trivial Lie brackets $[X_j, Y_j] = Z_j$. Choosing the adaptable basis

$$\{Z_1 + iZ_2, Z_1 - iZ_2, Y_1 + iY_2, Y_1 - iY_2, X_1 + iX_2, X_1 - iX_2\},$$

letting $\mathbf{e}^\circ = \{3,4,5,6\}$ and $\mathbf{j}^\circ = \{5,6\}$ we then get

$$\Omega_{\mathbf{e}^\circ,\mathbf{j}^\circ} = \{(z,\overline{z},y,\overline{y},x,\overline{x}) : z \neq 0\},$$
$$\Lambda = \{(z,\overline{z},y,\overline{y},x,\overline{x}) \in \Omega_{\mathbf{e}^\circ,\mathbf{j}^\circ} : x = y = 0\}.$$

Now, we will compute a general formula for a smooth cross-section of the G-orbits in some open dense set in $\mathfrak{g}^*$. Let $\lambda : \Omega_{\mathbf{e}^\circ,\mathbf{j}^\circ} \to \Lambda$ be the cross-section mapping. For each $f \in \mathfrak{n}^*$, we define $\nu(f) = \{1 \leq j \leq n : f(Z_j) \neq 0\}$. Put

$$\mathfrak{h}(f) = \bigcap_{j\in\nu(f)} \ker \gamma_j,$$

and let $\Lambda_\nu = \{f \in \Lambda : \nu(f) = \nu\}$. Observe that $\mathfrak{h}(f)$ is the Lie algebra of the stabilizer subgroup (a subgroup of H) of the linear functional f. For any $f \in \Lambda_\nu$, since we have a diagonal action, $\mathfrak{h}(f)$ is independent of f and is equal to some constant subalgebra $\mathfrak{k} \subset \mathfrak{h}$.

Lemma 15. *There exists $\nu \subseteq \{1,\dots,n\}$ such that Λ_ν is dense and Zariski open in Λ, and if we let π be the projection or restriction mapping from $\mathfrak{g}^*$ onto $\mathfrak{n}^*$, and $\Omega_\nu = \pi^{-1} \circ \lambda^{-1}(\Lambda_\nu)$, then Ω_ν is Zariski open in $\mathfrak{g}^*$.*

Proof. It suffices to let $\nu = \{1,\dots,n\} \setminus \mathbf{e}^\circ$. Notice that

$$\Lambda_\nu = \{f \in \Lambda : \nu(f) = \{1,\dots,n\} \setminus \mathbf{e}^\circ\}$$

is dense and Zariski open in Λ. Additionally, we observe that for $f \in \Lambda_\nu$ and $j \in \{1,\dots,n\} \setminus \mathbf{e}^\circ$, $f(Z_j) \neq 0$. Next, Ω_ν is Zariski open in $\mathfrak{g}^*$ since the projection map is continuous, and the cross-section mapping is rational and smooth (see [2]). ∎

Lemma 16. *If $l \in \Omega_\nu$ and $\mathbf{e}(l)$ is the set of jump indices for Ω_ν such that*

$$\mathbf{e}(l) = \{i_1,\dots,i_d\} \cup \{j_1,\dots,j_d\},$$
$$\mathbf{i}(l) = \{i_1,\dots,i_d\}, \qquad \mathbf{j}(l) = \{j_1,\dots,j_d\},$$

then $\max \mathbf{i}(l) \leq \dim \mathfrak{n}$.

Proof. Let us assume for contradiction that there exists some jump index $i_t \in \mathbf{i}(l)$ such that $Z_{i_t} \in \mathfrak{h}$. Because jump indices always come in pairs, and because $j_t > i_t$, we see that $Z_{j_t} \in \mathfrak{h}$. However, since $\mathfrak{h}$ is commutative, $l[Z_{i_t}, Z_{j_t}] = 0$. This is a contradiction. ∎

Lemma 17. *For any $l \in \Omega_\nu$ and for all $j \in (\mathbf{e}(l) \setminus \mathbf{e}^\circ) \setminus \mathbf{i}(l)$, $Z_j \in \mathfrak{h}$.*

Proof. We have

$$\mathbf{e}(l) = \mathbf{e}^\circ \mathbin{\dot\cup} \{i_{s_1},\dots,i_{s_r}\} \mathbin{\dot\cup} \{j_{s_1},\dots,j_{s_r}\}.$$

If $j \in (\mathbf{e}(l) \setminus \mathbf{e}^\circ) \setminus \mathbf{i}(l)$ then $j \in \mathbf{j}(l) \setminus \mathbf{e}^\circ$, and there exists some k such that $Z_j = Z_{j_{s_k}}$. Assume that $Z_{j_{s_k}} \in \mathfrak{n}$. Since $j_{s_k} \notin \mathbf{e}^\circ$, there must exist some jump index i_{s_k} such that $i_{s_k} < j_{s_k}$ and $l[Z_{i_{s_k}}, Z_{j_{s_k}}] \neq 0$. Since $Z_{i_{s_k}}$ also belongs to $\mathfrak{n}$, letting $\pi(l) = f$ gives $f[Z_{i_{s_k}}, Z_{j_{s_k}}] \neq 0$. Thus, both $i_{s_k}, j_{s_k} \in \mathbf{e}^\circ$, which contradicts our assumption. ■

We observe that the choice of an adaptable basis mainly relies on the choice of an adaptable basis for the nilpotent Lie algebra. Any permutation of the basis elements of $\mathfrak{h}$ will not affect the 'adaptability' of the basis. Without loss of generality, we will assume that we have the following adaptable basis for $\mathfrak{g}$:

$$\{Z_1, \ldots, Z_n, A_m, \ldots, A_{r+1}, A_r, \ldots, A_2, A_1\}$$

such that $A_r = Z_{j_{s_r}}, \ldots, A_1 = Z_{j_{s_1}}$. Additionally, we assume that the basis elements $A_r, \ldots, A_1$ with weights $\gamma_r, \ldots, \gamma_1$ are chosen such that $\mathrm{Re}(\gamma_t(A_t)) = 1$, $\gamma_t(A_{t'}) = 0$, $t \neq t'$.

Lemma 18. *For any $l \in \Omega_\nu$, $\varphi(l) = \{i_{s_1}, \ldots, i_{s_r}\}$.*

Proof. We already know that $\varphi(l) \subseteq \{i_{s_1}, \ldots, i_{s_r}\}$. We only need to show that $j \in \varphi(l)$ for any $j = i_{s_1}$. By definition, $\varphi(l) = \{j \in \mathbf{e} : \mathfrak{a}_j(l) = 0\}$, and according to Proposition 4.1 in [2], $\mathfrak{a}_j(l) = 0$ if and only if $\gamma_j(U_k(l)) \neq 0$ for $j = i_k$. In order to prove the lemma, it suffices to show that $\gamma_{i_{s_k}}(U_k(l)) = 0$. We have

$$\begin{aligned} U_k(l) &= \rho_{k-1}(Z_{j_{s_k}}(l), l) = \rho_{k-1}(A_{s_k}) = \rho_{k-1}(A_k) \\ &= \rho_{k-2}(A_k, l) - \frac{l[\rho_{k-2}(A_k, l), U_{k-1}(l)]}{l[V_{k-1}(l), U_{k-1}(l)]} V_{k-1}(l) \\ &\quad - \frac{l[\rho_{k-2}(A_k, l), V_{k-1}(l)]}{l[U_{k-1}(l), V_{k-1}(l)]} U_{k-1}(l). \end{aligned}$$

A straightforward computation shows that, for some coefficients c_t,

$$\gamma_{i_{s_k}}(U_k(l)) = \gamma_k(A_k) - c_{k-1}\gamma_k(A_{k-1}) - \cdots - c_1\gamma_1(A_1) = \gamma_k(A_k) \neq 0. \ ■$$

Proposition 19. *Let $\mathfrak{g} = \mathfrak{n} \times \mathfrak{k} \times \mathfrak{a}$ where $\mathfrak{h} = \mathfrak{k} \times \mathfrak{a}$. A cross-section for the G-orbits in Ω_ν is*

$$\Sigma = \{l \in \Omega_\nu : l = (f, k, 0),\ f \in \Sigma^\circ, k \in \mathfrak{k}^*\}.$$

Letting $\pi : \mathfrak{g}^* \to \mathfrak{n}^*$ be the projection map,

$$\pi(\Sigma) = \Sigma^\circ = \{l \in \Lambda_\nu : |l(Z_j)| = 1\ \forall j \in \{i_{s_1}, \ldots, i_{s_r}\}\}.$$

Proof. Let $\pi(l) = f$. So far, we have shown that $\mathbf{e}(l) = \mathbf{e}^\circ \mathbin{\dot\cup} \varphi(l) \mathbin{\dot\cup} \{j_{s_1}, \ldots, j_{s_r}\}$. Using the description of the cross-section in [2],

$$\Sigma = \{l \in \Omega_\nu : l(Z_j(l)) = 0 \text{ for } j \in \mathbf{e} \setminus \varphi, \text{ and } |\mathbf{b}_j(l)| = 1 \text{ for } j \in \varphi\}.$$

For $l \in \mathfrak{g}^*$, if $j \in \mathbf{e} \setminus \varphi$ then $j \in \mathbf{e}^\circ \mathbin{\dot\cup} \{j_{s_1}, \ldots, j_{s_r}\}$. For $j \in \mathbf{e}^\circ$, $l(Z_j(l)) = f(Z_j(f)) = 0$, and for $j \in \{j_{s_1}, \ldots, j_{s_r}\}$, $l(Z_j(l)) = 0$. Thus, $A_j = 0$ for

$j \in \{j_{s_1}, \ldots, j_{s_r}\}$. Next, for $j \in \varphi(l) = \{i_{s_1}, \ldots, i_{s_r}\}$,

$$|\mathbf{b}_j(l)| = \left|\frac{\gamma_j(U_k(l))}{l[Z_j, U_k(l)]}\right| = \left|\frac{\gamma_j(A_k)}{l[Z_j, A_K]}\right| = \left|\frac{1}{l(Z_j)}\right| = 1 \ \Rightarrow \ |l(Z_j)| = 1.$$

Thus, we conclude that $\Sigma = \{l \in \Omega_\nu : l = (f, k, 0),\ f \in \Sigma^\circ, k \in \mathfrak{k}^*\}$ where

$$\Sigma^\circ = \{l \in \Lambda_\nu : |l(Z_j)| = 1,\ j \in \{i_{s_1}, \ldots, i_{s_r}\}\}. \ \blacksquare$$

Throughout the remainder of this paper, we will also use the symbol $\simeq$ to denote homeomorphism between two topological spaces.

PROPOSITION 20. *Σ° is a cross-section for the H-orbits in Λ_ν. In other words,*

$$\Sigma^\circ = \pi(\Sigma) \simeq \Lambda_\nu / H.$$

Proof. The set Λ_ν is an H-invariant cross-section for the N coadjoint orbits of a fixed layer $\Omega_{\mathbf{e}^\circ, \mathbf{j}^\circ}$, while Σ is a cross-section for the G coadjoint orbits of for Ω_ν. We must show that each H-orbit of an arbitrary element inside Λ_ν meets the set Σ° in a unique point, and also that any point in Σ° belongs to the H-orbit of some linear functional belonging to Λ_ν.

We start by showing that $H \cdot f \cap \Sigma^\circ \neq \emptyset$ for $f \in \Lambda_\nu$. Given $f \in \Lambda_\nu$, we consider the element $(f, k, 0) \in \Omega_\nu$ such that $f = \pi((f, k, 0))$. We know there exists $x \in \Sigma$ such that $g \cdot (f, k, 0) = x$ for some $g \in G$. In fact, let $g = (n, 1)(1, h)$. If $(n, 1)(1, h) \cdot (f, k, 0) = x$, then $\pi((n, 1)(1, h) \cdot (f, k, 0)) = \pi(x)$, and $(n, 1)\pi((1, h) \cdot (f, k, 0)) = \pi(x) \in \Lambda_\nu$. Consequently, $(n, 1)$ stabilizes $\pi((1, h) \cdot (f, k, 0))$, implying that $\pi((1, h) \cdot (f, k, 0)) = \pi(x) \in \Lambda_\nu$. Since

$$\pi((1, h) \cdot (f, k, 0)) = \pi((h \cdot f, k, 0)) = h \cdot f,$$

we see that $h \cdot f \in \pi(\Sigma) = \Sigma^\circ$.

Next, let us assume that there exist $h, h' \in H$ such that $f \in \Lambda_\nu$ and $h \cdot f, h' \cdot f \in \Sigma^\circ$ with $h \cdot f \neq h' \cdot f$. Consider $(h' \cdot f, k, 0), (h \cdot f, k, 0) \in \Sigma$. We have

$$(h \cdot f, k, 0) = (1, h) \cdot (f, k, 0), \qquad (h' \cdot f, k, 0) = (1, h') \cdot (f, k, 0).$$

Both $(h \cdot f, k, 0), (h' \cdot f, k, 0)$ are in the G-orbit of $(f, k, 0)$, and since $(h \cdot f, k, 0)$ and $(h' \cdot f, k, 0)$ also belong to the cross-section Σ, we get $(h \cdot f, k, 0) = (h' \cdot f, k, 0)$. The latter implies that $h \cdot f = h' \cdot f$, a contradiction. We conclude that $\pi(\Sigma^\circ) = \pi(\Sigma) \simeq \Lambda_\nu / H$. $\blacksquare$

EXAMPLE 21. Let N be the Heisenberg Lie group with Lie algebra $\mathfrak{n}$ spanned by the adaptable basis $\{Z, Y, X\}$ with non-trivial Lie bracket $[X, Y] = Z$. Let H be a 2-dimensional commutative Lie group with Lie algebra $\mathfrak{h} = \mathbb{R}A \oplus \mathbb{R}B$ acting on $\mathfrak{n}$ as follows: $\mathbb{R}B = \mathfrak{z}(\mathfrak{g})$ and $[A, X] = \frac{1}{2}X$,

$[A, Y] = \frac{1}{2}Y$, $[A, Z] = Z$. Applying the procedure above, we obtain

$$\begin{aligned} \nu &= \{1\}, \\ \Lambda_\nu &= \{(z, 0, 0) \in \mathfrak{n}^* : z \neq 0\}, \\ \Omega_\nu &= \{(z, y, x, a, b) \in \mathfrak{g}^* : z \neq 0,\ y, x, a, b \in \mathbb{R}\}, \\ \Sigma &= \{(\pm 1, 0, 0, 0, b) : b \in \mathbb{R}\}, \\ \Sigma^\circ &= \{(\pm 1, 0, 0) \in \mathfrak{n}^*\}. \end{aligned}$$

EXAMPLE 22. Let $\mathfrak{g} = (\mathbb{R}Z_1 \oplus \mathbb{R}Z_2 \oplus \mathbb{R}Y_1 \oplus \mathbb{R}Y_2 \oplus \mathbb{R}X_1 \oplus \mathbb{R}X_2) \oplus \mathbb{R}A$ with

$$\mathfrak{n} = \mathbb{R}Z_1 \oplus \mathbb{R}Z_2 \oplus \mathbb{R}Y_1 \oplus \mathbb{R}Y_2 \oplus \mathbb{R}X_1 \oplus \mathbb{R}X_2$$

and non-trivial Lie brackets

$$\begin{aligned} [X_1 + iX_2, Y_1 + iY_2] &= Z_1 + iZ_2, \\ [X_1 - iX_2, Y_1 - iY_2] &= Z_1 - iZ_2, \\ [A, X_1 + iX_2] &= (1+i)/2(X_1 + iX_2), \\ [A, Y_1 + iY_2] &= (1+i)/2(Y_1 + iY_2), \\ [A, Z_1 + iZ_2] &= (1+i)(Z_1 + iZ_2). \end{aligned}$$

Then

$$\begin{aligned} \nu &= \{1, 2\}, \\ \Lambda_\nu &= \{(z, \overline{z}, 0, 0, 0, 0) : z \neq 0\}, \\ \Omega_\nu &= \{(z, \overline{z}, y, \overline{y}, x, \overline{x}, a) : z \neq 0,\ y, x \in \mathbb{C},\ a \in \mathbb{R}\}, \\ \Sigma &= \{(z, \overline{z}, 0, 0, 0, 0, 0) : z \neq 0\}, \\ \Sigma^\circ &= \{(z, \overline{z}, 0, 0, 0, 0) : z \neq 0\}. \end{aligned}$$

Now that we have a precise description of the orbital parametrization of the unitary dual of the group, we will take a closer look at the quasiregular representation τ of G.

4. Decomposition of the quasiregular representation. In this section, we will provide a precise decomposition of τ as a direct integral of irreducible representations of G. As a result, we will be able to compare the quasiregular representation with the left regular representation of G, and to completely settle the question of admissibility for τ.

There is a well-known algorithm available for the computation of the Plancherel measure of N. It is simply obtained by computing the Pfaffian of a certain skew-symmetric matrix. More precisely, the Plancherel measure on Λ_ν is

$$d\mu(\lambda) = |\det(M_{\mathbf{e}^\circ}(\lambda))|^{1/2} d\lambda = |\mathbf{Pf}(\lambda)| d\lambda,$$

where $M_{\mathbf{e}^\circ}(\lambda) = (\lambda[Z_i, Z_j])_{1 \le i,j \le \mathbf{e}^\circ}$. In this section, we will focus on the decomposition of the quasiregular representation $\tau = \operatorname{Ind}_H^G 1$, which is a unitary representation of G realized as acting in $L^2(N)$ in the following way:

$$(\tau(n,1)\phi)(m) = \phi(n^{-1}m),$$
$$(\tau(1,h)\phi)(m) = |\delta(h)|^{-1/2}\phi(h^{-1}\cdot m), \qquad \text{with } \delta(h) = \det(\operatorname{Ad} h).$$

Let $\mathbf{F}$ be the Fourier transform defined on $L^2(N) \cap L^1(N)$, which we extend to $L^2(N)$. Define

$$\widehat{\tau}(\cdot) = \mathbf{F} \circ \tau(\cdot) \circ \mathbf{F}^{-1}.$$

Definition 23. Let $\lambda \in \Lambda_\nu$ a linear functional. A *polarization algebra* subordinate to λ is a maximal subalgebra of $\mathfrak{n}_{\mathbb{C}}$ satisfying the following conditions. Firstly, it is isotropic for the bilinear form B_λ defined as $B_\lambda(X,Y) = \lambda[X,Y]$. In other words, it is a maximal subalgebra $\mathfrak{p}$ such that $\lambda([\mathfrak{p},\mathfrak{p}]) = 0$. Secondly, $\mathfrak{p} + \overline{\mathfrak{p}}$ is a subalgebra of $\mathfrak{n}_{\mathbb{C}}$. We will denote a polarization subalgebra subordinate to λ by $\mathfrak{p}(\lambda)$. A polarization is said to be *real* if $\mathfrak{p}(\lambda) = \overline{\mathfrak{p}(\lambda)}$. Also, we say that the polarization $\mathfrak{p}(\lambda)$ is *positive at* λ if $i\lambda[X,\overline{X}] \geq 0$ for all $X \in \mathfrak{p}(\lambda)$.

Let $\mathbf{e}^\circ$ be the set of jump indices corresponding to the linear functionals in Λ_ν, and let $\mathbf{e}^\circ = \mathbf{d}^\circ/2$. Referring to Lemma 3.5 in [2], for any given linear functional λ, a polarization subalgebra subordinate to λ is given by $\mathfrak{p}(\lambda) = \mathfrak{h}_{\mathbf{d}^\circ}(\lambda)$. See the formula below (3.1). Unfortunately, in general the polarization obtained as $\mathfrak{h}_{\mathbf{d}^\circ}(\lambda)$ is not real and we must proceed by holomorphic induction in order to construct irreducible representations of N. For the interested reader, a very short introduction to holomorphic induction is available on page 78 of the book [4].

The following discussion can also be found in [15, p. 124]. Given $\lambda \in \Lambda_\nu$, let π_λ be an irreducible representation of N acting in the Hilbert space $\mathcal{H}_\lambda$ and realized via holomorphic induction. Let $\mathcal{X}$ be the domain of $\mathcal{H}_\lambda$ on which the irreducible representation π_λ is acting. It is well-known that $\mathcal{X}$ can be identified with $\mathfrak{n}/\mathfrak{e} \times \mathfrak{e}/\mathfrak{d}$, where

$$\mathfrak{d} = \mathfrak{n} \cap \mathfrak{p}(\lambda), \qquad \mathfrak{e} = (\mathfrak{p}(\lambda) + \overline{\mathfrak{p}(\lambda)}) \cap \mathfrak{n},$$

and $\mathfrak{p}(\lambda)$ is an H-invariant positive polarization inside $\mathfrak{n}_{\mathbb{C}}$. Finally, $\mathcal{H}_\lambda = L^2(\mathfrak{n}/\mathfrak{e}) \otimes \operatorname{Hol}(\mathfrak{e}/\mathfrak{d})$ with $\operatorname{Hol}(\mathfrak{e}/\mathfrak{d})$ denoting the holomorphic functions which are square integrable with respect to some Gaussian function. It is worth mentioning that, if the polarization $\mathfrak{p}(\lambda)$ is real, then $\mathcal{H}_\lambda = L^2(\mathfrak{n}/\mathfrak{e})$, $\mathcal{X} = \mathfrak{n}/\mathfrak{e}$, and holomorphic induction here is just regular induction.

The choice of how we realize the irreducible representations of N really depends on the action of the dilation group H on N. For example, if the group $N \rtimes H$ is completely solvable, there is no need to consider the complexification of $\mathfrak{n}$ since a positive polarization always exists for exponential solvable Lie groups. From now on, we will assume that a convenient choice for a positive polarization subalgebra has been made for each $\lambda \in \Lambda_\nu$, and we denote by $\mathcal{H}_\lambda$ the Hilbert space on which we realize the corresponding irreducible representation π_λ, and $\mathcal{X}$ is a domain on which we realize the action of π_λ. We fix an H-quasi-invariant measure on $\mathcal{X}$, which we denote by dn, and we define

$$\delta_{\mathcal{X}}(h) = \frac{d(h^{-1} \cdot n)}{dn}.$$

Furthermore, let $C(h, \lambda) : \mathcal{H}_\lambda \to \mathcal{H}_{h\cdot\lambda}$ be defined by

$$C(h, \lambda) f(x) = |\delta_{\mathcal{X}}(h)|^{-1/2} f(h^{-1} \cdot x)$$

so that $\pi_\lambda(h^{-1}\cdot n)C(h, \lambda) = C(h, \lambda)\pi_{h\cdot\lambda}(n)$ for all $n \in N$. We write Δ for the modular function of G where $\Delta(h) = \det(\mathrm{Ad}(h)^{-1})$, and set $\delta(h) = \Delta(h)^{-1}$.

Proposition 24. *Let $\phi \in \mathbf{F}(L^2(N))$. Then*

$$\widehat{\tau}_\lambda(n)(\mathbf{F}\phi)(\lambda) = \pi_\lambda(n)(\mathbf{F}\phi)(\lambda),$$
$$\widehat{\tau}_\lambda(h)(\mathbf{F}\phi)(\lambda) = |\delta(h)|^{1/2} C(h, h^{-1} \cdot \lambda)(\mathbf{F}\phi)(h^{-1} \cdot \lambda)C(h, h^{-1} \cdot \lambda)^{-1}.$$

The proof is elementary, so we omit it. Now, we will describe how to obtain almost all of the irreducible representations of G via an application of the Mackey Machine.

Lemma 25. *If there exists some non-zero linear $\lambda \in \Lambda_\nu$, and a non-trivial subgroup $K \leq H$ fixing λ, then K must fix all elements in Λ_ν.*

Proof. Recall the definition of Λ_ν:

$$\Lambda_\nu = \{f \in \Lambda : f(Z_j) \neq 0,\ j \in \{1, \ldots, n\} \setminus \mathbf{e}^\circ\}.$$

Suppose there exists a linear functional $f \in \Lambda_\nu$ and $h \neq 1$ such that $h \cdot f = f$. Since the action of h is a diagonal action, it must be the case that $\mathrm{ad} \log h(Z_j) = 0$ for all $j \in \{1, \ldots, n\} \setminus \mathbf{e}^\circ$. Thus for any $f \in \Lambda_\nu$, we have

$$K = \{h \in H : \mathrm{ad} \log h(Z_j) = 0 \text{ for } j \in \{1, \ldots, n\} \setminus \mathbf{e}^\circ\}. \ \blacksquare$$

Lemma 26. *Let π_λ be an irreducible representation of N corresponding to a linear functional $\lambda \in \Lambda_\nu$ via Kirillov's map, and let K be the stabilizer subgroup of the coadjoint action of H on Λ_ν. We denote by $\widetilde{\pi}_\lambda$ the extension of π_λ, which is an irreducible representation of $N \rtimes K$ acting in $\mathcal{H}_\lambda = L^2(\mathfrak{n}/\mathfrak{e}) \otimes \mathrm{Hol}(\mathfrak{e}/\mathfrak{d})$ such that $\gamma_\lambda(\cdot)$ is the restriction of $C(\lambda, \cdot)$ to K. More precisely, such an extension is defined by $\widetilde{\pi}_\lambda(n, k)\phi(x) = \pi_\lambda(n)\gamma_\lambda(h)\phi(x)$. Furthermore, let $\{\chi_\sigma : \sigma \in \mathfrak{k}^*\} = \widehat{K}$, and recall that Σ° is a cross-section for the coadjoint orbits of H in Λ_ν. The set*

$$\{\mathrm{Ind}_{NK}^{NH}(\widetilde{\pi}_\lambda \otimes \chi_\sigma) : (\lambda, \sigma) \in \Sigma^\circ \times \mathfrak{k}^*\}$$

exhausts almost all of the irreducible representations of G *which will appear in the Plancherel transform of* G, *and if* L *denotes the left regular representation of* G, *we have*

$$L \simeq \int_{\Sigma^\circ \times \mathfrak{k}^*}^{\oplus} \mathrm{Ind}_{NK}^{NH}(\widetilde{\pi}_\lambda \otimes \chi_\sigma) \otimes 1_{L^2(H/K,\mathcal{H}_\lambda)}\, d\mu(\lambda,\sigma)$$

and $d\mu(\lambda,\sigma)$ *is absolutely continuous with respect to the natural Lebesgue measure on* $\Sigma^\circ \times \mathfrak{k}^*$.

The claims in Lemma 26 summarize some standard facts in the analysis of exponential Lie groups. We refer the reader to Theorem 10.2 in [12], where the general case of group extensions is presented, and to [6], which specializes to the class of groups considered in the present paper.

Lemma 27. *For any* $\lambda \in \Lambda_\nu$, *let* $K = \mathrm{Stab}_G(\lambda)$ *be such that* $K \neq \{1\}$. *There exists a non-trivial representation of* K *inside the symplectic group* $\mathrm{Sp}(\mathfrak{n}/\mathfrak{n}(\lambda))$, *where* $\mathfrak{n}(\lambda)$ *is the null-space of the matrix* $(\lambda[Z_i, Z_j])_{1\le i,j\le n}$.

Proof. It is well-known that $\mathfrak{n}/\mathfrak{n}(\lambda)$ has a smooth symplectic structure since the bilinear form $B_\lambda(X,Y) = \lambda[X,Y]$ is a non-degenerate, skew-symmetric 2-form on $\mathfrak{n}/\mathfrak{n}(\lambda)$. Let $h \in K$; since $h\cdot\lambda = \lambda$, the bilinear form $B_\lambda(X,Y)$ is K-invariant. In other words, for any $h \in K$, $B_\lambda(h\cdot X, h\cdot Y) = B_\lambda(X,Y)$. Thus, there is a natural matrix representation β of K such that $\beta(K)$ is a closed subgroup of the symplectic group $\mathrm{Sp}(\mathfrak{n}/\mathfrak{n}(\lambda))$. Identifying $\mathfrak{n}/\mathfrak{n}(\lambda)$ with a complementary subspace of $\mathfrak{n}(\lambda)$ in $\mathfrak{n}$, which we denote by $\mathcal{B}$, this representation is nothing but the adjoint representation of K acting on $\mathcal{B}$. ■

In this paper, $Z(G)$ stands for the center of the Lie group G, and $\mathfrak{z}(\mathfrak{g})$ stands for its Lie algebra. Also, we remind the reader that $\gamma_\lambda(\cdot)$ is the restriction of the representation $C(\lambda,\cdot)$ to the group K.

Lemma 28. *Assume that* K_1 *is a subgroup of* K. *Then* $\gamma_\lambda(K_1) = \{1\}$ *if and only if* $K_1 \le Z(G)$.

Proof. Clearly if there exists a non-trivial subgroup such that $K_1 \le Z(G)$ then $\gamma_\lambda(K_1) = \{1\}$. For the other way around, let $k \in K_1$. Notice that

$$\gamma_\lambda(k)\phi(x) = |\delta_{\mathcal{X}}(h)|^{-1/2}\phi(\beta(k)^{-1}x).$$

We have already seen that $\beta(k)$ is a symplectic matrix, and at least half of its eigenvalues are 1. Since for any symplectic matrix, the multiplicity of eigenvalues 1 if they occur is even, it follows that $\beta(k)$ is the identity. Thus, k is a central element. ■

Remark 29. Let β be the finite-dimensional representation of K in $\mathrm{Sp}(\mathfrak{n}/\mathfrak{n}_\lambda)$. By the first isomorphism theorem, $\beta(K) \simeq K/(Z(G)\cap H)$.

LEMMA 30. *If there exists some* $x \in \mathcal{X}$ *with* $\phi_x : K \to \mathcal{X}$ *and* $\phi_x(k) = k \cdot x$ *such that* $\operatorname{rank}(\phi_x) = \max_{y\in\mathcal{X}}(\operatorname{rank}(\phi_y))$ *then the number of elements in a cross-section for the* K*-orbits in* $\mathcal{X}$ *is equal to* $2^{\dim \mathcal{X}}$ *if* $\operatorname{rank}(\phi_x) = \dim \mathcal{X}$, *and is infinite otherwise.*

Proof. Fix a cross-section $\mathcal{C} \simeq \mathcal{X}/K$ for $\mathcal{C} \subseteq \mathcal{X}$. For each $x \in \mathcal{C}$, let $r = \max_{x\in\mathcal{C}}(\operatorname{rank}(\phi_x))$ and $\mathcal{X}_1 = \{x \in \mathcal{X} : \operatorname{rank}(\phi_x) = r\}$. Then $\mathcal{X}_1$ is open and dense in $\mathcal{X}$. Assume that there exists some y in $\mathcal{C}$ such that $\operatorname{rank}(\phi_y) = \dim(\mathcal{X})$. If $r = \dim(\mathcal{X})$, then ϕ_y defines a submersion, which means that ϕ_y is an open map. Furthermore, $\phi_y(K)$, which is the orbit of y, is open in $\mathcal{X}_1$. From the definition of the action of K this is possible if and only if K acts with real eigenvalues, and in that case the number of orbits is simply equal to $2^{\dim \mathcal{X}}$. Now, assume that there exists no y in $\mathcal{C}$ such that $\operatorname{rank}(\phi_y) = \dim \mathcal{X}$; then the orbits in $\mathcal{X}_1$ are always meager in $\mathcal{X}_1$. So a cross-section will contain an infinite collection of points. ∎

LEMMA 31. *Let* $\gamma_\lambda(\cdot)$ *be the restriction of* $C(\lambda, \cdot)$ *to* K. *Then we have the direct integral decomposition*

$$\gamma_\lambda \simeq \int^{\oplus}_{(\mathfrak{k}/\mathfrak{h}\cap\mathfrak{z}(\mathfrak{g}))^*} \chi_{\overline{\sigma}} \otimes 1_{\mathbb{C}^{\mathbf{m}}}\, d\overline{\sigma},$$

where the multiplicity function is uniformly constant, and $\mathbf{m} : \mathfrak{k}^* \to \mathbb{N} \cup \{\infty\}$ *with* $\mathbf{m}(\sigma)$ *being equal to the number of elements in the cross-section* $\mathcal{X}/K$.

Proof. Recall that $\gamma_\lambda(h)f(x) = |\delta_{\mathcal{X}}(h)|^{-1/2} f(h^{-1} \cdot x)$ and let $\mathbf{m}$ be the number of elements in a cross-section for the K-orbits in $\mathcal{X}$. If $K = \{1\}$ then clearly each point in $\mathcal{X}$ is its own orbit and $\mathbf{m} = \infty$. If K acts on some invariant open subset of $\mathcal{X}$ by spirals, then the cross-section will contain an infinite number of elements. Let $\mathcal{X}_1$ be as defined in Lemma 30. We have the natural diffeomorphism $\alpha : \mathcal{X}_1/K \times K/(H \cap Z(G)) \to \mathcal{X}_1$ such that $\alpha(x, \overline{k}) = \overline{k} \cdot x$. Thus, $\mathcal{X}_1$ becomes a total space with base space $\mathcal{X}_1/K$ and fibers $K/(H \cap Z(G)) \cdot x$ such that

$$\mathcal{X}_1 = \bigcup_{x\in\mathcal{X}_1/K} \big(K/(H \cap Z(G)) \cdot x\big).$$

First, for each x in the cross-section $\mathcal{X}_1/K$, identify $K/(H \cap Z(G)) \cdot x$ with $K/(H \cap Z(G))$, and the Hilbert space

$$\mathcal{H}_\lambda \simeq \big(L^2\big(K/(H \cap Z(G))\big)\big)^{\mathbf{m}} \simeq L^2\big(K/(H \cap Z(G))\big) \otimes \mathbb{C}^{\mathbf{m}}.$$

In fact, for each linear functional λ, the representation γ_λ can be modeled as being quasi-equivalent to the left regular representation on $K/(H \cap Z(G))$. Let ϕ be a function in $\mathcal{H}_\lambda$ and for each $x \in \mathcal{X}_1/K$ define ϕ_x as the restriction of the function ϕ to the orbit of x. It is easy to see that the action of $\gamma_\lambda(\cdot)$ becomes just a left translation acting on ϕ_x for each $x \in \mathcal{X}_1/K$. Since

$K/(H\cap Z(G))$ is a commutative Lie group, we can decompose its left regular representation by using its group Fourier transform. Letting $(\mathfrak{k}/\mathfrak{h}\cap\mathfrak{z}(\mathfrak{g}))^*$ be the unitary dual of the group $K/(H\cap Z(G))$, we obtain a decomposition of the representation γ_λ into its irreducible components as follows:

$$\gamma_\lambda \simeq \int_{(\mathfrak{k}/\mathfrak{h}\cap\mathfrak{z}(\mathfrak{g}))^*}^{\oplus} \chi_{\overline{\sigma}} \otimes 1_{\mathbb{C}^{\mathbf{m}}}\, d\overline{\sigma},$$

where $\chi_{\overline{\sigma}}$ are characters defined on $Z(G)\cap H$, and $\int_{(\mathfrak{k}/\mathfrak{h}\cap\mathfrak{z}(\mathfrak{g}))^*}^{\oplus} \chi_{\overline{\sigma}} \otimes 1_{\mathbb{C}^{\mathbf{m}}}\, d\overline{\sigma}$ is modeled as acting in the Hilbert space $\int_{(\mathfrak{k}/\mathfrak{h}\cap\mathfrak{z}(\mathfrak{g}))^*}^{\oplus} \mathbb{C}\otimes 1_{\mathbb{C}^{\mathbf{m}}}\, d\sigma$. ∎

LEMMA 32. *Let $\Lambda_\nu \to \Sigma^\circ \simeq \Lambda_\nu/H$ be the quotient map induced by the action of H. The push-forward of the Lebesgue measure on Λ_ν via the quotient map is equivalent to the Lebesgue measure on $\Sigma^\circ \simeq \Lambda_\nu/H$.*

Proof. This lemma follows from the following facts: the quotient map is a submersion everywhere, and the push-forward of a Lebesgue measure via a submersion is equivalent to a Lebesgue measure on the image set. ∎

Now, we will compute an explicit decomposition of the Plancherel measure on Λ_ν under the action of the dilation group H. We first recall the more general theorem on disintegration of Borel measures.

LEMMA 33. *Let G be a locally compact group. Let X be a left Borel G-space and μ a quasi-invariant σ-finite positive Borel measure on X. Assume that there is a μ-null set X_0 such that X_0 is G-invariant and $X-X_0$ is standard. Then for all $x\in X-X_0$, the orbit $G\cdot x$ is Borel isomorphic to G/G_x under the natural mapping, and there is a quasi-invariant measure μ_x concentrated on the orbit $G\cdot x$ such that, for all $f\in L^1(X,\mu)$,*

$$\int_X f(x)\,d\mu(x) = \int_{(X-X_0)/G}\int_{G/G_x} f(g\cdot x)\,d\mu_x(gG_x)\,d\overline{\mu}(x),$$

where G_x is the stability group at x.

We refer the interested reader to [12] for a proof of the above lemma.

PROPOSITION 34 (Disintegration of the Plancherel measure). *Under the action of H the Plancherel measure on Λ_ν is decomposed into a measure on the cross-section Σ° and a family of measures on each orbit such that, for any non-negative measurable function $F\in L^1(\Lambda_\nu)$, we have*

$$\int_{\Lambda_\nu} F(f)|\mathbf{Pf}(f)|\,df = \int_{\Sigma^\circ}\int_{H/K} F(\overline{h}\cdot\sigma)\,d\omega_\sigma(\overline{h})\,|\mathbf{Pf}(\sigma)|\,d\sigma,$$

where for each $\sigma\in\Sigma^\circ$, $d\omega_\sigma(\overline{h}) = \Delta(\overline{h})d\overline{h}$, $d\overline{h}$ is the natural Haar measure on H/K, $d\sigma$ is the Lebesgue measure on Σ° with $\overline{h}=hK$, and Δ is the modular function of the group H/K.

The proof is obtained via some elementary computations involving changing variables. It is quite trivial. Thus, we omit it.

THEOREM 35. *The quasiregular representation is unitarily equivalent to the following direct integral decomposition:*

$$\int_{\Sigma^\circ}^{\oplus}\Big(\int_{(\mathfrak{k}/\mathfrak{z}(\mathfrak{g})\cap\mathfrak{h})^*}^{\oplus} \mathrm{Ind}_{NK}^{NH}(\widehat{\pi}_\lambda \otimes \chi_{\overline{\sigma}}) \otimes 1_{\mathbb{C}^{\mathbf{m}}}\, d\overline{\sigma}\Big)|\mathbf{Pf}(\lambda)|\, d\lambda,$$

with multiplicity function $\mathbf{m}$ *equal to* $2^{\dim \mathcal{X}}$ *if* $\mathrm{rank}(\phi_x) = \dim \mathcal{X}$, *and infinite otherwise.*

Theorem 35 follows from Lemmas 32 and 31, and [14, Theorem 7.1].

PROPOSITION 36. *The quasiregular representation* $\tau = \mathrm{Ind}_H^G 1$ *is contained in the left regular representation if and only if* $\dim(Z(G)\cap H) = 0$.

Proof. Assume that $Z(G)\cap H$ is not equal to the trivial group $\{1\}$. We have proved that $\gamma_\lambda \simeq \int_{(\mathfrak{k}/(\mathfrak{z}(\mathfrak{g})\cap\mathfrak{h}))^*}^{\oplus} \chi_\sigma \otimes 1_{\mathbb{C}^{\mathbf{m}(\sigma)}}\, d\overline{\sigma}$. By Lemma 32 and [15, Theorem 3.1], we have

$$\tau \simeq \int_{\Sigma^\circ}^{\oplus} \int_{(\mathfrak{k}/(\mathfrak{z}(\mathfrak{g})\cap\mathfrak{h}))^*}^{\oplus} \mathrm{Ind}_{NK}^{NH}(\widetilde{\pi}_\lambda \otimes \chi_\sigma) \otimes 1_{\mathbb{C}^{\mathbf{m}(\lambda,\sigma)}}\, d\overline{\sigma}\, d\lambda.$$

The measure $d\overline{\sigma}$ belongs to the Lebesgue class measure on $(\mathfrak{k}/\mathfrak{z}(\mathfrak{g})\cap\mathfrak{h})^*$, which we identify with $\mathbb{R}^{\dim(\mathfrak{k}/(\mathfrak{z}(\mathfrak{g})\cap\mathfrak{h}))}$. The Plancherel measure of the group G is supported on $\Sigma^\circ \times \mathfrak{k}^*$ and belongs to the Lebesgue class measure $d\lambda d\sigma$ such that $d\sigma$ is the Lebesgue measure on $\mathfrak{k}^* = \mathbb{R}^{\dim(\mathfrak{k})}$. Clearly, if $\dim(Z(G)\cap H) > 0$, then $\mathbb{R}^{\dim(\mathfrak{k}/(\mathfrak{z}(\mathfrak{g})\cap\mathfrak{h}))}$ is meager in $\mathbb{R}^{\dim(\mathfrak{k})}$. Thus, the measure occurring in the decomposition of the quasiregular representation and the measure occurring in the decomposition of the left regular representation are mutually singular if and only if $\dim(Z(G)\cap H) > 0$. Finally, we have

$$\begin{aligned} L &\simeq \int_{\Sigma^\circ}^{\oplus}\int_{\mathfrak{k}^*}^{\oplus} \mathrm{Ind}_{NK}^{NH}(\widetilde{\pi}_\lambda \otimes \chi_\sigma) \otimes 1_{L^2(H/K,\mathcal{H}_\lambda)}\ d\sigma\, d\lambda \\ &\simeq \int_{\Sigma^\circ}^{\oplus}\int_{\mathfrak{k}^*}^{\oplus} \mathrm{Ind}_{NK}^{NH}(\widetilde{\pi}_\lambda \otimes \chi_\sigma) \otimes 1_{\mathbb{C}^\infty}\, d\sigma\, d\lambda. \end{aligned}$$

As the irreducible representations occurring in the decomposition of L have uniform infinite multiplicities, the quasiregular representation $\tau = \mathrm{Ind}_H^G 1$ is contained in the left regular representation if and only if $\dim(Z(G)\cap H) = 0$. ∎

Finally we have our main result.

THEOREM 37. *Assume that* $G = N \rtimes H$ *is unimodular. Then* τ *is never admissible. Assume that* G *is non-unimodular. Then* τ *is admissible if and only if* $\dim(Z(G)\cap H) = 0$.

Proof. First, assume that G is unimodular. Clearly if

$$\dim(Z(G)\cap H)=0$$

then τ will be contained in the left regular representation. However, G being unimodular, it is known (see [10]) that any subrepresentation of the left regular representation is admissible if and only if

$$\int_{\Sigma}\mathbf{m}(\lambda,\sigma)\,d\mu(\lambda,\sigma)<\infty. \tag{4.1}$$

But that is not possible because the multiplicity is constant a.e., $\mathbf{m}(\lambda,\sigma) = \mathbf{m}$ and

$$\int_{\Sigma}\mathbf{m}(\lambda,\sigma)\,d\mu(\lambda,\sigma)=\int_{\Sigma}\mathbf{m}\,d\mu(\lambda,\sigma)=\mathbf{m}\cdot\mu(\Sigma).$$

If $\mathbf{m}$ is infinite, then clearly the integral will diverge. Now assume that $\mathbf{m}$ is finite. Then there exists a non-trivial $k\in\mathfrak{k}$ such that $\Sigma=\Sigma^{\circ}\times\mathfrak{k}^{*}$ and, using Currey's measure ([6]), up to multiplication by a constant,

$$d\mu(\lambda,\sigma)=|\mathbf{Pf_e}(\lambda,\sigma)|\,d\lambda\,d\sigma$$

where $\mathbf{Pf_e}(\lambda,\sigma)=\det((\lambda,\sigma)[Z_{i_r},Z_{j_s}])_{1\le r,s\le\mathbf{d}}$. It is clear from the definition of the action of H that the function $\mathbf{Pf_e}(\lambda,\sigma)$ is really a function of λ. Thus, we just write $\mathbf{Pf_e}(\lambda,\sigma)=\mathbf{Pf_e}(\lambda)$ and

$$\int_{\Sigma}\mathbf{m}(\lambda,\sigma)\,d\mu(\lambda,\sigma)=\mathbf{m}\int_{\Sigma^{\circ}}\int_{\mathfrak{k}^{*}}|\mathbf{Pf_e}(\lambda)|\,d\lambda\,d\sigma=\infty.$$

If G is unimodular and $\dim(Z(G)\cap H)>0$ then τ must be disjoint from the left regular representation. Now assume that G is non-unimodular. We have two different cases. If $\dim(Z(G)\cap H)>0$ then the quasiregular representation is disjoint from the left regular representation, which automatically prevents τ from being admissible. Secondly, assume that $\dim(Z(G)\cap H)=0$. We have

$$\tau\simeq\int_{\Sigma^{\circ}}^{\oplus}\int_{\mathfrak{k}^{*}}^{\oplus}\mathrm{Ind}_{NK}^{NH}(\widetilde{\pi}_{\lambda}\otimes\chi_{\sigma})\otimes 1_{\mathbb{C}^{\mathbf{m}(\lambda,\sigma)}}\,d\sigma\,d\lambda,$$

and of course, as seen previously, the multiplicity function is uniformly constant and $\mathbf{m}(\lambda,\sigma)\le\infty$. Thus, τ is quasiequivalent to the left regular representation. G being non-unimodular, it follows that τ is admissible. ■

REMARK 38. We call the attention of the reader to the fact that the theorem above supports Conjecture 3.7 in [8], which states that a monomial representation of a unimodular exponential solvable Lie group G never has admissible vectors. The general case remains an open problem.

Based on our main theorem, we can assert the following.

REMARK 39. Let N be a nilpotent Lie group with Lie algebra $\mathfrak{n}$. Let H be given such that at least one of the basis elements of $\mathfrak{h}$ commutes with all basis elements of $\mathfrak{n}$. Then $Z(N \rtimes H) \cap H$ is clearly non-trivial, and τ cannot be admissible as a representation of G.

5. Examples. In this section, we will present several examples, and we will show how to apply our results in order to settle the admissibility of τ in each case.

EXAMPLE 40. Coming back to Example 21, clearly G is not unimodular. Since the center of the group has a non-trivial intersection with H, we can see τ is not an admissible representation.

EXAMPLE 41. Recall Example 22. Since G is non-unimodular and since the center of the group is trivial, τ is an admissible representation of G.

EXAMPLE 42. Let G be a Lie group with Lie algebra $\mathfrak{g}$ spanned by

$$\{Z, Y, X, A_1, A_2, A_3\}$$

such that

$$\begin{array}{lll} [X,Y] = Z, & [A_1, X] = X, & [A_2, X] = X, \\ [A_3, X] = 2X, & [A_1, Y] = Y, & [A_2, Y] = -Y, \\ [A_3, Y] = -Y, & [A_1, Z] = Z, & [A_3, Z] = Z. \end{array}$$

The center of G is equal to

$$\exp\bigl(\mathbb{R}\bigl(-\tfrac{1}{2}A_1 - \tfrac{3}{2}A_2 + A_3\bigr)\bigr) < H,$$

and so τ is not admissible.

EXAMPLE 43. Let G be a Lie group with Lie algebra $\mathfrak{g}$ spanned by

$$\{Z, Y, X, W, A_1, A_2, A_3, A_4\}$$

with non-trivial Lie brackets

$$\begin{array}{ll} [X,Y] = Z, & [W, X] = Y, \\ [A_1, W] = \tfrac{1}{3}W, & [A_1, X] = \tfrac{1}{3}X, \\ [A_1, Y] = \tfrac{2}{3}Y, & [A_1, Z] = Z \\ [A_2, W] = -W, & [A_2, X] = X, \\ [A_2, Z] = Z, & [A_3, W] = \tfrac{1}{5}W, \\ [A_3, X] = \tfrac{2}{5}X, & [A_2, Y] = \tfrac{3}{5}Y, \\ [A_3, Z] = Z, & [A_4, X] = \tfrac{1}{2}X, \\ [A_4, Y] = \tfrac{1}{2}Y, & [A_4, Z] = Z. \end{array}$$

In this example the Lie algebra $\mathfrak{h}$ is spanned by the vectors A_1, A_2, A_3, A_4. The center of G is equal to

$$\exp\bigl(\mathbb{R}\bigl(-\tfrac{9}{10}A_1 - \tfrac{1}{10}A_2 - A_3\bigr)\bigr)\exp\bigl(\mathbb{R}\bigl(-\tfrac{3}{4}A_1 - \tfrac{1}{4}A_2 + A_4\bigr)\bigr) < H,$$

hence τ is not admissible.

Example 44. Suppose that $\mathfrak{g}$ is spanned by the vectors

$$U_1, U_2, Z_1, Z_2, Z_3, X_1, X_2, X_3, A$$

and $\mathfrak{h}$ is spanned by the vector A. Furthermore, assume that we have the following non-trivial Lie brackets:

$$\begin{aligned} &[X_3, X_2] = Z_1, \quad [X_3, X_1] = Z_2, \quad [X_2, X_1] = Z_3, \\ &[A, U_1 + iU_2] = (1+i)(U_1 + iU_2). \end{aligned}$$

We remark that in this example the nilradical of $\mathfrak{g}$ contains a step-2 freely generated nilpotent Lie algebra with three generators. Since G is non-unimodular, and since the center of G is trivial, we deduce that τ is admissible.

Example 45. Let N be the Heisenberg group

$$N = \left\{ \begin{pmatrix} 1 & x & y & z \\ 0 & 1 & 0 & y \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} : \begin{pmatrix} z \\ y \\ x \end{pmatrix} \in \mathbb{R}^3 \right\},$$

and the dilation group H be isomorphic to $\mathbb{R}^2$ such that

$$H = \left\{ \begin{pmatrix} e^t & 0 & 0 & 0 \\ 0 & e^{t-r} & 0 & 0 \\ 0 & 0 & e^r & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} : \begin{pmatrix} t \\ r \end{pmatrix} \in \mathbb{R}^2 \right\}.$$

The action of H on N is given as follows:

$$\begin{pmatrix} e^t & 0 & 0 & 0 \\ 0 & e^{t-r} & 0 & 0 \\ 0 & 0 & e^r & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & x & y & z \\ 0 & 1 & 0 & y \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} e^t & 0 & 0 & 0 \\ 0 & e^{t-r} & 0 & 0 \\ 0 & 0 & e^r & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}^{-1}$$
$$= \begin{pmatrix} 1 & xe^r & ye^te^{-r} & ze^t \\ 0 & 1 & 0 & ye^{t-r} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

It is easy to see that the Lie algebra of G is spanned by $\{Z, Y, X, A\}$ with non-trivial Lie brackets

$$[A_1, Z] = Z, \quad [A_1, Y] = Y, \quad [A_2, Y] = -Y, \quad [A_2, X] = X.$$

Here

$$K = \left\{ \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & e^{-r} & 0 & 0 \\ 0 & 0 & e^{r} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} : r \in \mathbb{R} \right\}$$

but the center of G is trivial. Thus, there is a non-trivial subgroup of the dilation group stabilizing the center of N and thus stabilizing almost all of elements of the unitary dual of N. The spectrum of the left regular representation of $G = N \rtimes H$ is supported on two disjoint lines, and the irreducible representations occurring in the decomposition of the left regular representation occur with infinite multiplicities. Also, the spectrum of the quasiregular representation τ is parametrized by two disjoint lines, but the irreducible representations occurring in the decomposition of τ occur twice almost everywhere. Since the group G is non-unimodular, and τ is contained in L, we find that τ is admissible.

Example 46. Suppose that $\mathfrak{n}$ is spanned by T_1, T_2, Z, Y, X such that $[X, Y] = Z$, and $\mathfrak{h}$ is spanned by A_1, A_2, A_3, A_4, A_5 such that

$$[A_2, X] = \tfrac{1}{2}X, \quad [A_2, Y] = \tfrac{1}{2}Y, \quad [A_2, Z] = Z, \quad [A_3, X] = X,$$
$$[A_3, Y] = -Y, \quad [A_5, X] = X, \quad [A_6, Y] = Y,$$
$$[A_3, T_1 + iT_1] = (1+i)(T_1 + iT_1),$$
$$[A_4, T_1 + iT_1] = (2+2i)(T_1 + iT_1),$$
$$[A_1, T_1 + iT_1] = (1+i)(T_1 + iT_1).$$

The center of G is given by

$$\exp\big(\mathbb{R}\big(A_1 - 2A_2 - \tfrac{1}{2}A_4\big)\big) \exp\big(\mathbb{R}\big(A_3 - \tfrac{1}{2}A_4 - A_5\big)\big) < H.$$

Thus τ is not admissible.

Acknowledgements. Thanks go to Professor Bradley Currey who introduced me to the theory of admissibility. Without his support and guidance, this work would have never been possible. I also thank my wife Lindsay and my three children Senami, Kemi and Donah for their support.

Finally, I am grateful to the anonymous referee for his careful reading, comments, corrections, and helpful suggestions.

REFERENCES

[1] S. T. Ali, J.-P. Antoine and J.-P. Gazeau, *Coherent States, Wavelets and Their Generalizations*, Springer, New York, 2000.

[2] D. Arnal, B. Currey and B. Dali, *Construction of canonical coordinates for exponential Lie groups*, Trans. Amer. Math. Soc. 361 (2009), 6283–6348.

[3] E. Cordero and A. Tabacco, *Triangular subgroups of* $\mathrm{Sp}(d,\mathbb{R})$ *and reproducing formulae*, J. Funct. Anal. 264 (2013), 2034–2058.

[4] L. Corwin and P. Greenleaf, *Representations of Nilpotent Lie Groups and Their Applications. Part 1: Basic Theory and Examples*, Cambridge Stud. Adv. Math. 18, Cambridge Univ. Press, 1990.

[5] B. N. Currey, *An explicit Plancherel formula for completely solvable Lie groups*, Michigan Math. J. 38 (1991), 75–87.

[6] B. N. Currey, *Explicit orbital parameters and the Plancherel measure for exponential Lie groups*, Pacific J. Math. 219 (2005), 97–138.

[7] B. N. Currey, *Admissibility for a class of quasiregular representations*, Canad. J. Math. 59 (2007), 917–942.

[8] B. N. Currey and V. Oussa, *Admissibility for monomial representations of exponential Lie groups*, J. Lie Theory 22 (2012), 481–487.

[9] F. De Mari and E. De Vito, *Admissible vectors for mock metaplectic representations*, Appl. Comput. Harmon. Anal. 34 (2013), 163–200.

[10] H. Führ, *Abstract Harmonic Analysis of Continuous Wavelet Transforms*, Lecture Notes in Math. 1863, Springer, Berlin, 2005.

[11] H. Führ, *Generalized Calderón conditions and regular orbit spaces*, Colloq. Math. 120 (2010), 103–126.

[12] A. Kleppner and R. L. Lipsman, *The Plancherel formula for group extensions*, Ann. Sci. École Norm. Sup. (4) 5 (1972), 459–516.

[13] R. Laugesen, N. Weaver, G. Weiss and E. Wilson, *A characterization of the higher dimensional groups associated with continuous wavelets*, J. Geom. Anal. 12 (2002), 89–102.

[14] R. Lipsman, *Orbital parameters for induced and restricted representations*, Trans. Amer. Math. Soc. 313 (1989), 433–473.

[15] R. Lipsman, *Harmonic analysis on exponential solvable homogeneous spaces: the algebraic or symmetric cases*, Pacific J. Math. 140 (1989), 117–147.

[16] H. Liu and L. Peng, *Admissible wavelets associated with the Heisenberg group*, Pacific J. Math. 180 (1997), 101–123.

[17] G. Mackey, *Unitary representations of group extensions I*, Acta Math. 99 (1958), 265–311.

Vignon Oussa
Bridgewater State University
Bridgewater, MA 02325, U.S.A.
E-mail: vignon.oussa@bridgew.edu

Received 28 December 2012;
revised 27 April 2013 (5837)