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AN INEQUALITY FOR SPHERICAL CAUCHY DUAL TUPLES

BY

SAMEER CHAVAN (Kanpur)

Abstract. Let T be a spherical 2-expansive m-tuple and let $T^{\mathfrak{s}}$ denote its spherical Cauchy dual. If $T^{\mathfrak{s}}$ is commuting then the inequality

$$\sum_{|\beta|=k} (\beta!)^{-1} (T^{\mathfrak{s}})^{\beta} (T^{\mathfrak{s}})^{*\beta} \leq \binom{k+m-1}{k} \sum_{|\beta|=k} (\beta!)^{-1} (T^{\mathfrak{s}})^{*\beta} (T^{\mathfrak{s}})^{\beta}$$

holds for every positive integer k. In case $m = 1$, this reveals the rather curious fact that all positive integral powers of the Cauchy dual of a 2-expansive (or concave) operator are hyponormal.

1. Introduction. If $\mathbb{N}$ denotes the set of non-negative integers, let $\mathbb{N}^m$ denote the cartesian product $\mathbb{N} \times \cdots \times \mathbb{N}$ (m times). For $p \equiv (p_1, \ldots, p_m)$ in $\mathbb{N}^m$, we write $|p| := \sum_{i=1}^m p_i$.

Given a Banach space $\mathcal{X}$, a tuple $T \equiv (T_1, \ldots, T_m)$ of bounded linear operators acting on $\mathcal{X}$, and $p \in \mathbb{N}^m$, we let $T^p := (T_1^{p_1}, \ldots, T_m^{p_m})$, where $T_i^{p_i}$ denotes the product of T_i with itself p_i times.

When $T \equiv (T_1, \ldots, T_m)$ is a tuple of bounded linear operators acting on a Hilbert space $\mathcal{H}$, we let $T^* := (T_1^*, \ldots, T_m^*)$.

Let $\mathcal{H}$ denote a complex separable Hilbert space and let $B(\mathcal{H})$ denote the C^*-algebra of bounded linear operators on $\mathcal{H}$. Let T be an m-tuple of (possibly non-commuting) bounded linear operators on $\mathcal{H}$. The *spherical generating* 1*-tuple* Q_s associated with T is given by

$$Q_s(X) := \sum_{i=1}^m T_i^* X T_i \quad (X \in B(\mathcal{H}))$$

(see [5] for the definition of the *generating m-tuples*). More generally, for m-tuples A and B, consider the so-called *elementary operator*

$$E_{A,B}(X) = \sum_{i=1}^m A_i X B_i \quad (X \in B(\mathcal{H})).$$

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If A and B are commuting m-tuples then it is easy to see that

$$E^k_{A,B}(X) = \sum_{|\beta|=k} \frac{k!}{\beta!} A^\beta X B^\beta. \tag{1.1}$$

Suppose T is *jointly left-invertible*, that is, $Q_s(I)$ is invertible. We refer to the m-tuple $T^{\mathfrak{s}} := (T^{\mathfrak{s}}_1, \ldots, T^{\mathfrak{s}}_m)$ as the *spherical Cauchy dual tuple* of T, where $T^{\mathfrak{s}}_i := T_i(Q_s(I))^{-1}$ $(i = 1, \ldots, m)$.

Remark 1.1. If P_s denotes the spherical generating 1-tuple associated with $T^{\mathfrak{s}}$ then $P_s(I) = Q_s(I)^{-1}$.

We say that T is a *spherical* 2*-expansion* if

$$I - 2Q_s(I) + Q_s^2(I) \leq 0. \tag{1.2}$$

We say that T is a *spherical* 2*-isometry* if equality occurs in (1.2).

The Drury–Arveson 2-shift $M_{z,2}$ is an important example of a spherical 2-isometry [7, Theorem 4.2]. Recall that $M_{z,2}$ is the 2-tuple of multiplication by the coordinate functions z_1, z_2 on the reproducing kernel Hilbert space associated with the positive-definite kernel

$$\frac{1}{1 - z_1\bar{w}_1 - z_2\bar{w}_2} \quad ((z_1, z_2) \in \mathbb{B}_2),$$

where $\mathbb{B}_2$ denotes the open unit ball in $\mathbb{C}^2$.

Every spherical 2-expansion T is a *spherical expansion*, that is, $Q_s(I) \geq I$ [5, Proposition 4.1(i)]. In particular, the spherical Cauchy dual tuple $T^{\mathfrak{s}}$ of a spherical 2-expansion T is well-defined. Let P_s denote the spherical generating 1-tuple associated with $T^{\mathfrak{s}}$. If T is a spherical expansion then, by Remark 1.1, $T^{\mathfrak{s}}$ is a *spherical contraction*, that is, $P_s(I) \leq I$. In all the above notions, we skip the prefix "spherical" in case $m = 1$. Also, in this case, we retain Shimorin's original notation T' for the Cauchy dual operator. When dealing with the Cauchy duals, it is tempting to mention the papers [9], [10] concerning Kaufman's transformation $T \mapsto T(I - T^*T)^{-1/2}$, which maps strict contractions reversibly onto closed densely defined operators.

The spherical Cauchy dual tuple $S := M^{\mathfrak{s}}_{z,2}$ of the Drury–Arveson 2-shift is commuting. Further, it admits a *normal extension* [4]. The fact that S is *jointly hyponormal* (that is, the 2×2 matrix $([S^*_j, S_i])_{1 \leq i,j \leq 2}$ of cross commutators of S is positive-definite) can also be deduced from Curto's Six Point Test [6].

We invoke the following basic fact about spherical Cauchy dual tuples, which plays an important role in the proof of the main result.

Lemma 1.2. *Let T be a spherical* 2*-expansive m-tuple of commuting bounded linear operators on $\mathcal{H}$ and let $T^{\mathfrak{s}}$ denote the spherical Cauchy dual of T. Let Q_s (resp. P_s) denote the spherical generating* 1*-tuple associated with the m-tuple T (resp. $T^{\mathfrak{s}}$). Then $P_s \circ Q_s(I) \leq I \leq Q_s \circ P_s(I)$.*

Proof. The first inequality is obtained in [5, proof of Theorem 6.6] while the second one is obtained in [4, proof of Proposition 5.2]. ■

REMARK 1.3. Since $P_s^2(I) = P_s(I)Q_s \circ P_s(I)P_s(I)$, we obtain the following inequality: $P_s^2(I) \geq P_s(I)^2$.

2. Main result. The purpose of this note is to prove the following inequality for spherical Cauchy dual tuples:

THEOREM 2.1. *Let T be a spherical 2-expansive m-tuple of commuting bounded linear Hilbert space operators. Assume that the spherical Cauchy dual $T^{\mathfrak{s}}$ of T is commuting. Let P_s (resp. R_s) denote the spherical generating 1-tuple associated with the m-tuple $T^{\mathfrak{s}}$ (resp. $(T^{\mathfrak{s}})^*$). Then*

$$R_s^k(I) \leq \binom{k+m-1}{k} P_s^k(I) \quad \text{for every positive integer } k. \tag{2.3}$$

Proof. By Remark 1.3, $P_s^2(I) \geq P_s(I)^2$. It is easy to see that

$$P_s^k(I) - P_s^{k+1}(I) \leq P_s^{k-1}(I) - P_s^k(I) \quad (k \in \mathbb{N}), \tag{2.4}$$

where $P^0(I) = I$. We prove the following by induction on $k \geq 1$:

$$P_s^k(I) + k(P_s^{k-1}(I) - P_s^k(I)) \leq I \quad (k \in \mathbb{N}). \tag{2.5}$$

The case $k = 1$ is trivial with equality in (2.5). Suppose (2.5) holds for some integer $k \geq 1$. By (2.4),

$$\begin{aligned} P_s^{k+1}(I) + (k+1)(P_s^k(I) - P_s^{k+1}(I)) &= P_s^k(I) + k(P_s^k(I) - P_s^{k+1}(I)) \\ &\leq P_s^k(I) + k(P_s^{k-1}(I) - P_s^k(I)). \end{aligned}$$

The desired conclusion is now immediate from the induction hypothesis.

Let Q_s denote the spherical generating 1-tuple associated with T. It was observed in [5, proof of Proposition 4.1(i)] that Q_s satisfies

$$Q_s^k(I) \leq I + k(Q_s(I) - I) \quad (k \in \mathbb{N}). \tag{2.6}$$

We claim that for all positive integers k, $P_s^k \circ Q_s^k(I) \leq I$. The case $k = 1$ is already recorded in Lemma 1.2. It now follows from (2.6) that

$$\begin{aligned} P_s^k \circ Q_s^k(I) \leq P_s^k(I + k(Q_s(I) - I)) &= P_s^k(I) + k(P_s^k \circ Q_s(I) - P_s^k(I)) \\ &\leq P_s^k(I) + k(P_s^{k-1}(I) - P_s^k(I)). \end{aligned}$$

By (2.5), $P_s^k \circ Q_s^k(I) \leq I$. Thus the claim stands verified.

Before we obtain the desired estimate, let us note some combinatorial identities. Consider the elementary operator $E_{T^*,T^{\mathfrak{s}}}$ (see the discussion prior to (1.1)), and observe that $E_{T^*,T^{\mathfrak{s}}}(I) = I$. It follows that

$$\sum_{|\alpha|=k} \frac{k!}{\alpha!}(T^*)^\alpha (T^{\mathfrak{s}})^\alpha = E_{T^*,T^{\mathfrak{s}}}^k(I) = I \quad (k \in \mathbb{N}).$$

It is now clear that $c_{\alpha\beta} := \sqrt{k!/\alpha!}\,\sqrt{k!/\beta!}\,(T^\alpha (T^{\mathfrak{s}})^\beta)^*$ satisfies

$$\sqrt{k!/\beta!}\,(T^{\mathfrak{s}})^{*\beta} = \sum_{|\alpha|=k} c_{\alpha\beta}\sqrt{k!/\alpha!}\,(T^{\mathfrak{s}})^\alpha \quad (\beta \in \mathbb{N}^m). \tag{2.7}$$

Let $l = \binom{k+m-1}{k}$ and let $\mathcal{H}^{(l)}$ be the orthogonal direct sum of l copies of $\mathcal{H}$. For the $l \times l$ $B(\mathcal{H})$-valued matrix $[c_{\alpha\beta}] := [c_{\alpha\beta}]_{|\alpha|=k,\,|\beta|=k}$, we define the linear operator $\Phi : \mathcal{H}^{(l)} \to \mathcal{H}^{(l)}$ by

$$\Phi(X) = [c_{\alpha\beta}]X \quad (X \in \mathcal{H}^{(l)}).$$

Note that $P_s^k \circ Q_s^k(I) \le I$ is equivalent to

$$\sum_{|\alpha|=k}\sum_{|\beta|=k} c_{\alpha\beta}(c_{\alpha\beta})^* \le I,$$

which holds if and only if

$$\Big\| \sum_{|\alpha|=k}\sum_{|\beta|=k} c_{\alpha\beta}h_{\alpha,\beta}\Big\|^2 \le \sum_{|\alpha|=k}\sum_{|\beta|=k} \|h_{\alpha,\beta}\|^2 \quad (h_{\alpha,\beta} \in \mathcal{H}).$$

For the last equivalence, see [1, Remark 3.2]. It is now easy to see that for every $X \in \mathcal{H}^{(l)}$,

$$\|\Phi(X)\| \le \big|\{\alpha \in \mathbb{N}^m : |\alpha| = k\}\big|^{1/2}\|X\| = \sqrt{l}\,\|X\|.$$

For $h \in \mathcal{H}$, let $X = (\sqrt{k!/\alpha!}\ (T^{\mathfrak{s}})^\alpha h)_{|\alpha|=k}$. By (2.7),

$$\Phi(X) = (\sqrt{k!/\beta!}\,(T^{\mathfrak{s}})^{*\beta} h)_{|\beta|=k}.$$

In particular,

$$\|(\sqrt{k!/\beta!}\,(T^{\mathfrak{s}})^{*\beta} h)_{|\beta|=k}\| \le \sqrt{l}\,\|(\sqrt{k!/\alpha!}\,(T^{\mathfrak{s}})^\alpha h)_{|\alpha|=k}\|.$$

It follows that

$$\sum_{|\beta|=k} \frac{k!}{\beta!}\|(T^{\mathfrak{s}})^{*\beta}h\|^2 \le l \sum_{|\alpha|=k} \frac{k!}{\alpha!}\|(T^{\mathfrak{s}})^\alpha h\|^2.$$

Since h was arbitrary, the desired inequality follows. ∎

The special case $m = 1$, $k = 1$ of Theorem 2.1 was independently obtained by Shimorin [13] and the author [3]. For m arbitrary and $k = 1$, Theorem 2.1 recovers [5, Corollary 6.8]. To see this, note that the commutativity of $T^{\mathfrak{s}}$ is not required for the deduction of (2.3) in case $k = 1$. Unfortunately, for $m \ge 2$, there is one shortcoming of Theorem 2.1: it is not clear whether or not equality holds in (2.3) for some spherical 2-expansive m-tuple T.

QUESTION 2.2. *Let P_s, R_s be as in the statement of Theorem* 2.1. *What is the smallest positive number $\gamma_{m,k}$ such that the inequality*

$$R^k_s(I) \leq \gamma_{m,k} P^k_s(I)$$

holds for all spherical 2*-expansive m-tuples T with commuting $T^{\mathfrak{s}}$?*

Thus Theorem 2.1 says that $\gamma_{m,k}$ is at most $\binom{k+m-1}{k}$. Let us see what happens if we relax the commutativity of the spherical Cauchy dual in Question 2.2 for the case $k = 1$. To see that, we borrow the following example from unpublished notes of Prof. Stefan Richter.

Let $\alpha = (\alpha_1, \ldots, \alpha_m)$ be such that $|\alpha_1|^2 + \cdots + |\alpha_m|^2 = 1$. Let $V = (V_1, \ldots, V_m)$ be an m-tuple of bounded linear operators from $\mathcal{H}$ into $\mathcal{K}$ such that $\sum_{i=1}^m \bar{\alpha}_i V_i = 0$. Define the m-tuple $S_{V,\alpha} = (S_1, \ldots, S_m)$ by

$$S_i := \begin{pmatrix} \alpha_i I & V_i \\ 0 & \alpha_i I \end{pmatrix} \quad \text{on } \mathcal{H} \oplus \mathcal{K} \quad (i = 1, \ldots, m). \tag{2.8}$$

Then $S_{V,\alpha}$ is a spherical 2-isometry. This follows from

$$\sum_{i=1}^m (S_i)^* S_i = \begin{pmatrix} I & 0 \\ 0 & A \end{pmatrix}, \quad \sum_{i,j=1}^m S_j^* (S_i)^* S_i S_j = \begin{pmatrix} I & 0 \\ 0 & 2A - I \end{pmatrix},$$

where $A := I + \sum_{i=1}^m V_i^* V_i$. Note further that spherical Cauchy dual of $S_{V,\alpha}$ is given by

$$S_i^{\mathfrak{s}} = \begin{pmatrix} \alpha_i I & V_i A^{-1} \\ 0 & \alpha_i A^{-1} \end{pmatrix} \quad (i = 1, \ldots, m).$$

Let $\alpha = (1, 0)$, $V = (0, I)$, and consider the 2-tuple $S_{V,\alpha} = (S_1, S_2)$. It is easy to see that $\gamma_{2,1} \geq 5/4$. Since components of a jointly hyponormal m-tuple are hyponormal [2] and $S_2^{\mathfrak{s}}$ is not hyponormal, $S_{V,\alpha}^{\mathfrak{s}}$ is not jointly hyponormal. In particular, in dimension greater than 1, the spherical Cauchy dual tuple of a spherical 2-isometry is *not* necessarily jointly hyponormal. This answers [5, Question 6.9] in the negative. Note, however, that $S_{V,\alpha}^{\mathfrak{s}}$ is not commuting.

3. Dimension $m = 1$. We state below a special case of Theorem 2.1, which is a small but important step towards the question of subnormality of the Cauchy dual of a complete hyperexpansion [3].

THEOREM 3.1. *Let T be a* 2*-expansion and let T' denote its Cauchy dual. Then T'^k is hyponormal for every positive integer k.*

COROLLARY 3.2. *If S in $\mathcal{B}(\mathcal{H})$ satisfies*

$$\|Sx + y\|^2 \leq 2(\|x\|^2 + \|Sy\|^2) \quad (x, y \in \mathcal{H}), \tag{3.9}$$

then S^k is hyponormal for every positive integer k.

Proof. If S satisfies (3.9) then S' is 2-expansive [12, proof of Theorem 3.6]. Now the required conclusion is immediate from the identity $S = (S')'$ and the last theorem. ■

Let $T \in B(\mathcal{H})$ be left-invertible. Consider the 2-parameter family

$$\mathscr{F}_T := \{(((T^p)')^q)' : p, q \in \mathbb{N}\}$$

associated with T. Note that all operators in $\mathscr{F}_T$ are left-invertible.

COROLLARY 3.3. *Suppose T is a 2-expansion with finite-dimensional cokernel. Then all operators in $\mathscr{F}_T$ admit trace-class self-commutator.*

Proof. We first assume that $A \in \mathscr{F}_T$ is of the form $(T'^q)'$ for some positive integer q. Since any 2-expansion can be written as a direct sum of a unitary and a completely non-unitary 2-expansion, we may assume that T is completely non-unitary. By [3, Lemma 2.19], T' (and hence T'^q) is finitely multi-cyclic. By Theorem 3.1, and the Berger–Shaw Theorem [11], $A' = T'^q$ admits a trace-class self-commutator.

We now imitate the argument of [3, Proposition 2.21] to see that A has a trace-class self-commutator. Check first that

$$[A^*, A]A = -A^*A([A'^*, A']A)A^*A.$$

In particular, the operator $[A^*, A]A$, and hence $[A^*, A]AA'^*$, is trace-class. It is easy to see that

$$[A^*, A] = [A^*, A]AA'^* + [A^*, A]P_{\ker(A^*)},$$

where $P_{\ker(A^*)}$, the orthogonal projection onto $\ker(A^*)$, is a finite-rank operator. It follows that $[A^*, A]$ is a trace-class operator.

The general case follows from the fact that a positive integral power of a 2-expansion is again a 2-expansion [8, Proposition 4.2]. ■

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Sameer Chavan
Department of Mathematics
Indian Institute of Technology Kanpur
Kanpur 208016, India
E-mail: chavan@iitk.ac.in

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