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QUIVER BIALGEBRAS AND MONOIDAL CATEGORIES

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Abstract. We study bialgebra structures on quiver coalgebras and monoidal structures on the categories of locally nilpotent and locally finite quiver representations. It is shown that the path coalgebra of an arbitrary quiver admits natural bialgebra structures. This endows the category of locally nilpotent and locally finite representations of an arbitrary quiver with natural monoidal structures from bialgebras. We also obtain theorems of Gabriel type for pointed bialgebras and hereditary finite pointed monoidal categories.

1. Introduction. This paper is devoted to the study of natural bialgebra structures on the path coalgebra of an arbitrary quiver and monoidal structures on the category of its locally nilpotent and locally finite representations. A further purpose is to establish a quiver setting for general pointed bialgebras and pointed monoidal categories.

Our original motivation is to extend the Hopf quiver theory [4, 7, 8, 12, 13, 25, 31] to the setting of generalized Hopf structures. As bialgebras are a fundamental generalization of Hopf algebras, we naturally start our study from this case. The basic problem is to determine what kind of quivers can give rise to bialgebra structures on their associated path algebras or coalgebras.

It turns out that the path coalgebra of an arbitrary quiver admits natural bialgebra structures (see Theorem 3.2). This seems a bit surprising at first sight by comparison with the Hopf case given in [8], where Cibils and Rosso showed that the path coalgebra of a quiver Q admits a Hopf algebra structure if and only if Q is a Hopf quiver, which is very special. Bialgebra structures on general pointed coalgebras are also considered via quivers thanks to the Gabriel type theorem for coalgebras (see [3, 5]). Similar to the Hopf case obtained in [31], we give a Gabriel type theorem for general pointed bialgebras (see Proposition 3.3).

Another motivation comes from finite monoidal categories which are a natural generalization of finite tensor categories [10]. To the best of our knowledge, not much is known about the construction and classification of finite monoidal categories which are not tensor categories, i.e., rigid monoidal

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categories [9]. By taking advantage of the well-developed quiver representation theory, the quiver presentation of a pointed bialgebra B can help us to investigate the monoidal category of right B-comodules. Accordingly, some classification results of pointed monoidal categories are obtained (see Proposition 4.1 and Corollary 4.2).

Bialgebra structures on the path coalgebra of a quiver Q induce monoidal structures on the category $\mathrm{Rep}^{\mathrm{lnlf}}(Q)$ of locally nilpotent and locally finite representations of Q. These monoidal structures are also expected to be useful for the studying of the category $\mathrm{Rep}^{\mathrm{lnlf}}(Q)$ itself. For example, the tensor product of quiver representations naturally leads to the Clebsch–Gordan problem, i.e., the decomposition of the tensor product of any two representations into indecomposable summands, and the computation of the representation ring of $\mathrm{Rep}^{\mathrm{lnlf}}(Q)$, etc. Note that the tensor product given here is different from the vertex-wise and arrow-wise tensor product used in [14, 15, 18, 19], which in general is not from the bialgebra, and therefore should provide different information for the categories of quiver representations. This interesting problem is the third motivation and will be treated in the future.

We remark that a standard dual process will give rise to natural bialgebra structures on the path algebra and monoidal structures on the category of representations of a finite quiver. We prefer the path coalgebraic approach as it is more convenient for exposition and, more importantly, allows infinite quivers.

Throughout, we work over a field $\Bbbk$. Vector spaces, algebras, coalgebras, bialgebras, linear mappings, and unadorned $\otimes$ are over $\Bbbk$. The readers are referred to [24, 30] for general information about coalgebras and bialgebras, and to [1, 28, 29] for quivers and their applications to (co)algebras and representation theory.

2. Quivers, representations and path coalgebras. As preparation, in this section we recall some basic notions and facts about quivers, representations and path coalgebras.

A quiver is a directed graph. More precisely, a *quiver* is a quadruple $Q = (Q_0, Q_1, s, t)$, where Q_0 is the set of vertices, Q_1 is the set of arrows, and $s, t : Q_1 \to Q_0$ are two maps assigning to each arrow respectively the source and the target. Note that in this paper the sets Q_0 and Q_1 are allowed to be infinite. If Q_0 and Q_1 are finite, then we say Q is a *finite quiver*. For $a \in Q_1$, we write $a : s(a) \to t(a)$. A vertex is, by convention, said to be a trivial path of length 0. We also write $s(g) = g = t(g)$ for each $g \in Q_0$. The length of an arrow is set to be 1. In general, a non-trivial *path* of length n (≥ 1) is a concatenation of arrows of the form $p = a_n \cdots a_1$ with $s(a_{i+1}) = t(a_i)$ for $i = 1, \dots, n-1$. By Q_n we denote the set of paths of length n. A quiver is said to be *acyclic* if it has no cyclic paths, i.e. non-trivial paths with identical starting and ending vertices.

Let Q be a quiver and $\Bbbk Q$ the associated path space which is the $\Bbbk$-span of its paths. There is a natural coalgebra structure on $\Bbbk Q$ with comultiplication as split of paths. Namely, for a trivial path g, set $\Delta(g) = g \otimes g$ and $\varepsilon(g) = 1$; for a non-trivial path $p = a_n \cdots a_1$, set

$$\Delta(p) = t(a_n) \otimes p + \sum_{i=1}^{n-1} a_n \cdots a_{i+1} \otimes a_i \cdots a_1 + p \otimes s(a_1)$$

and $\varepsilon(p) = 0$. This is the *path coalgebra* of the quiver Q.

There exists on $\Bbbk Q$ an intuitive length gradation $\Bbbk Q = \bigoplus_{n \geq 0} \Bbbk Q_n$, compatible with the comultiplication Δ. It is clear that the path coalgebra $\Bbbk Q$ is pointed, and the set $G(\Bbbk Q)$ of group-like elements is Q_0. Moreover, the coradical filtration of $\Bbbk Q$ is

$$\Bbbk Q_0 \subseteq \Bbbk Q_0 \oplus \Bbbk Q_1 \subseteq \Bbbk Q_0 \oplus \Bbbk Q_1 \oplus \Bbbk Q_2 \subseteq \cdots$$

therefore it is coradically graded in the sense of Chin–Musson [5].

The path coalgebra is exactly the dual notion of the path algebra of a quiver, which is certainly more familiar. Dual to the freeness of path algebras, path coalgebras are cofree. Precisely, for an arbitrary quiver Q, the vector space $\Bbbk Q_0$ is a subcoalgebra of $\Bbbk Q$, and over the vector space $\Bbbk Q_1$ there is an induced $\Bbbk Q_0$-bicomodule structure via

$$\delta_L(a) = t(a) \otimes a, \qquad \delta_R(a) = a \otimes s(a)$$

for each $a \in Q_1$; the path coalgebra has another presentation as the so-called cotensor coalgebra (see, e.g., [31, 32])

$$\mathrm{CoT}_{\Bbbk Q_0}(\Bbbk Q_1) = \Bbbk Q_0 \oplus \Bbbk Q_1 \oplus \Bbbk Q_1 \,\square\, \Bbbk Q_1 \oplus \cdots$$

and hence enjoys the following

Universal Mapping Property. Let $f : C \to \Bbbk Q$ be a coalgebra map, $\pi_n : \Bbbk Q \to \Bbbk Q_n$ the canonical projection and set $f_n := \pi_n \circ f : C \to \Bbbk Q_n$ for each $n \geq 0$. Then f_0 is a coalgebra map; f_1 is a $\Bbbk Q_0$-bicomodule map, where the $\Bbbk Q_0$-bicomodule structure on C is induced by f_0; and for each $n \geq 2$, f_n can be written as $f_1^{\otimes n} \circ \Delta_C^{(n-1)}$, where $\Delta_C^{(n-1)}$ is the $(n-1)$-iterated action of the comultiplication of C. Conversely, given a coalgebra map $f_0 : C \to \Bbbk Q_0$ and a $\Bbbk Q_0$-bicomodule map $f_1 : C \to \Bbbk Q_1$, set $f_n = f_1^{\otimes n} \circ \Delta_C^{(n-1)}$ for each $n \geq 2$. Then as long as $f := \sum_{n \geq 0} f_n$ is well-defined, it is the unique coalgebra map $f : C \to \Bbbk Q$ such that $f_0 = \pi_0 \circ f$ and $f_1 = \pi_1 \circ f$.

Let Q be a quiver. A *representation* of Q is a collection

$$V = (V_g, V_a)_{g \in Q_0,\, a \in Q_1}$$

consisting of a vector space V_g for each vertex g and a linear map $V_a : V_{s(a)} \to V_{t(a)}$ for each arrow a. A *morphism of representations*, $\phi : V \to W$, is a collection $\phi = (\phi_g)_{g\in Q_0}$ of linear maps $\phi_g : V_g \to W_g$ for each vertex g such that $W_a\phi_{s(a)} = \phi_{t(a)}V_a$ for each arrow a. The category of representations of Q is denoted by $\mathrm{Rep}(Q)$. Given a representation V of Q and a path p, we define V_p as follows. If p is trivial, say $p = g \in Q_0$, then put $V_p = \mathrm{Id}_{V_g}$. For a non-trivial path $p = a_n \cdots a_2a_1$, put $V_p = V_{a_n} \cdots V_{a_2}V_{a_1}$. A representation V of Q is said to be *locally nilpotent* if for all $g \in Q_0$ and all $x \in V_g$, there exist at most finitely many paths p with source g satisfying $V_p(x) \neq 0$. A representation is called *locally finite* if it is a directed union of finite-dimensional representations. We denote by $\mathrm{Rep}^{\mathrm{lnlf}}(Q)$ the full subcategory of $\mathrm{Rep}(Q)$ consisting of all locally nilpotent and locally finite representations. It is well-known that the category of right $\Bbbk Q$-comodules is equivalent to $\mathrm{Rep}^{\mathrm{lnlf}}(Q)$ (see [20]).

3. Quiver bialgebras. In this section we show that the path coalgebra of an arbitrary quiver can be endowed with natural bialgebra structures. A Gabriel type theorem for pointed bialgebras is also given. Some examples are presented.

We start with the definition of bialgebra bimodules. Let B be a bialgebra. A *B-bialgebra bimodule* is a vector space M which is a B-bimodule and simultaneously a B-bicomodule such that the B-bicomodule structure maps are B-bimodule maps, or equivalently, the B-bimodule structure maps are B-bicomodule maps.

LEMMA 3.1. *Let Q be a quiver. The associated path coalgebra $\Bbbk Q$ admits a bialgebra structure if and only if Q_0 has a monoid structure and $\Bbbk Q_1$ can be given a $\Bbbk Q_0$-bialgebra bimodule structure. Moreover, the set of graded bialgebra structures on the path coalgebra $\Bbbk Q$ is in one-to-one correspondence with the set of pairs (S, M) in which S is a monoid structure on Q_0 and M is a $\Bbbk S$-bialgebra bimodule structure on $\Bbbk Q_1$.*

Proof. Assume first that the path coalgebra $\Bbbk Q$ admits a bialgebra structure. By considering its graded version induced by the coradical filtration (see, e.g., [24, 26]), we can assume further that the bialgebra structure on $\Bbbk Q$ is coradically graded. Note that the identity 1 is group-like, so 1 lies in Q_0, which is the set of group-like elements of $\Bbbk Q$. For any $g, h \in Q_0$, we have $\Delta(gh) = \Delta(g)\Delta(h) = gh \otimes gh$ and $\varepsilon(gh) = \varepsilon(g)\varepsilon(h) = 1$, therefore $gh \in Q_0$. Hence the restriction of the multiplication of $\Bbbk Q$ to Q_0 gives rise to a monoid structure. The $\Bbbk Q_0$-bicomodule structure on $\Bbbk Q_1$ is given as in Subsection 2.3. The multiplication of $\Bbbk Q_0$ provides a bimodule structure on $\Bbbk Q_1$. Finally, note that the axioms for bialgebras guarantee that the $\Bbbk Q_1$ so defined is a $\Bbbk Q_0$-bialgebra bimodule.

Conversely, assume that Q_0 can be endowed with a monoid structure and the vector space $\Bbbk Q_1$ has a $\Bbbk Q_0$-bialgebra bimodule structure. By the method of Nichols [26], these data can be used to construct a graded bialgebra structure on the path coalgebra $\Bbbk Q$ by the universal mapping property. Nichols' construction was applied to the quiver setting for Hopf algebras in [7, 8, 12, 13]. For completeness we include the construction below.

The cotensor coalgebra $\mathrm{CoT}_{\Bbbk Q_0}(\Bbbk Q_1)$ is exactly the path coalgebra $\Bbbk Q$. The $\Bbbk Q_0$-bimodule structure on $\Bbbk Q_1$ can be extended to a multiplication on $\Bbbk Q$ via the universal mapping property of $\Bbbk Q$. Let $M_0 : \Bbbk Q \otimes \Bbbk Q \to \Bbbk Q_0$ be the composition of the canonical projection $\pi_0 \otimes \pi_0 : \Bbbk Q \otimes \Bbbk Q \to \Bbbk Q_0 \otimes \Bbbk Q_0$ and the multiplication of the monoid algebra $\Bbbk Q_0$, and $M_1 : \Bbbk Q \otimes \Bbbk Q \to \Bbbk Q_1$ the composition of the canonical projection

$$\pi_0 \otimes \pi_1 \oplus \pi_1 \otimes \pi_0 : \Bbbk Q \otimes \Bbbk Q \to \Bbbk Q_0 \otimes \Bbbk Q_1 \oplus \Bbbk Q_1 \otimes \Bbbk Q_0$$

and the sum of the left and right module actions. Then it is clear that M_0 is a coalgebra map and M_1 is a $\Bbbk Q_0$-bicomodule map. Let $M_n = M_1^{\otimes n} \circ \Delta_2^{(n-1)}$, where Δ_2 is the coproduct of the tensor product coalgebra $\Bbbk Q \otimes \Bbbk Q$. For any path p of length n, it is easy to see that $M_l(p) = 0$ if $l \neq n$. Therefore $M = \sum_{n \geq 0} M_n$ is a well-defined coalgebra map and moreover respects the length gradation. The associativity for M can be deduced from the associativity of the bimodule action without difficulty by a standard application of the universal mapping property as before. The unit map is obvious. Hence we have defined an associative algebra structure and we obtain a graded bialgebra structure on $\Bbbk Q$.

The one-to-one correspondence in the statement is obvious. ■

Now we state our first main result.

THEOREM 3.2. *Let Q be a quiver. The associated path coalgebra $\Bbbk Q$ always admits bialgebra structures.*

Proof. By Lemma 3.1, it is enough to provide a monoid structure on Q_0 and a $\Bbbk Q_0$-bialgebra bimodule on $\Bbbk Q_1$. First, we fix a $\Bbbk Q_0$-bicomodule structure on $\Bbbk Q_1$ as in Subsection 2.3. If Q_0 has only one element, then let it be the unit group and take the trivial $\Bbbk Q_0$-bimodule structure on $\Bbbk Q_1$. Obviously, this defines a necessary bialgebra bimodule.

Now we assume Q_0 contains at least two elements. Take any $e \in Q_0$, and set it to be the identity, i.e., let $eg = g = ge$ for all $g \in Q_0$. Take any $z \in Q_0$ other than e, and make it a "zero" element, that is, let $gz = z = zg$ for any $g \in Q_0$. For any $g, h \in Q_0 - \{e, z\}$, set $gh = z$. Here, $g = h$ is allowed. One can verify without difficulty that this endows Q_0 with a monoid structure. For the $\Bbbk Q_0$-bimodule structure on $\Bbbk Q_1$, define

$$e.a = a = a.e, \qquad f.a = 0 = a.f$$

for all $a \in Q_1$ and all $f \in Q_0 - \{e\}$. Clearly, the bicomodule structure maps are bimodule maps and hence we have obtained a $\Bbbk Q_0$-bialgebra bimodule structure on $\Bbbk Q_1$. ■

Note that in the proof of the previous theorem we provide only a very "trivial" example of a graded bialgebra structure for the path coalgebra. The classification of all graded bialgebra structures on $\Bbbk Q$, or equivalently the classification of all suitable monoid structures on Q_0 and $\Bbbk Q_0$-bialgebra bimodule structures on $\Bbbk Q_1$, is still not clear and is an interesting problem of quiver combinatorics.

However, if Q_0 can be given a group structure and further $\Bbbk Q_1$ can be given a corresponding bialgebra bimodule structure over $\Bbbk Q_0$, then the situation is much clearer. Indeed, in this case $\Bbbk Q_0$ becomes a Hopf algebra and $\Bbbk Q_1$ becomes a $\Bbbk Q_0$-Hopf bimodule [26]. Now the fundamental theorem on Hopf modules of Sweedler [30, Theorem 4.1.1] can be applied, and the category of $\Bbbk Q_0$-Hopf bimodules is proved to be equivalent to the direct product of the representation categories of a class of subgroups of Q_0 by Cibils and Rosso [7]. Note that a quiver Q with Q_0 having a group structure and $\Bbbk Q_1$ having a $\Bbbk Q_0$-Hopf bimodule structure is far from arbitrary. Such quivers are called *covering quivers* in [13] and *Hopf quivers* in [8]. It would be of interest to generalize the classification of Hopf bimodules over groups to that of bialgebra bimodules over monoids.

For later use, we record the multiplication formula of paths, given by Rosso's quantum shuffle product [27], for quiver bialgebras. Given a quiver Q, take a suitable monoid structure on Q_0 and a $\Bbbk Q_0$-bialgebra bimodule structure on $\Bbbk Q_1$. Let p be a path of length l. An *n-thin split* of p is a sequence $(p_1, \ldots, p_n)$ of vertices and arrows such that the concatenation $p_n \cdots p_1$ is exactly p. These n-thin splits of p are in one-to-one correspondence with the n-sequences of $(n-l)$ 0's and l 1's. Denote the set of such sequences by D_l^n. Clearly $|D_l^n| = \binom{n}{l}$. For $d = (d_1, \ldots, d_n) \in D_l^n$, the corresponding n-thin split is written as $dp = ((dp)_1, \ldots, (dp)_n)$, where $(dp)_i$ is a vertex if $d_i = 0$ and an arrow if $d_i = 1$.

Let $\alpha = a_m \cdots a_1$ and $\beta = b_n \cdots b_1$ be a pair of paths of length m and n respectively. Let $d \in D_m^{m+n}$, and let $\bar{d} \in D_n^{m+n}$ be the complement sequence which is obtained from d by replacing each 0 by 1 and vice versa. Define an element in $\Bbbk Q_1^{\square m+n-1}$, or equivalently in $\Bbbk Q_{m+n}$, by

$$(\alpha \cdot \beta)_d = [(d\alpha)_{m+n}.(\bar{d}\beta)_{m+n}] \cdots [(d\alpha)_1.(\bar{d}\beta)_1],$$

where $[(d\alpha)_i.(\bar{d}\beta)_i]$ is understood as the $\Bbbk Q_0$-bimodule action on $\Bbbk Q_1$, and the terms in different brackets are put together by the cotensor product, or equivalently concatenation. In this notation, the formula for the product of

α and β is

$$\alpha \cdot \beta = \sum_{d \in D_m^{m+n}} (\alpha \cdot \beta)_d.$$

Now we give a Gabriel type theorem for general pointed bialgebras, which is in the same vein as the results of Chin and Montgomery [5] for pointed coalgebras and of Van Oystaeyen and Zhang [31] for pointed Hopf algebras. Let B be a pointed bialgebra and $\{B_i\}_{i\geq 0}$ its coradical filtration. Let $\operatorname{gr} B = B_0 \oplus B_1/B_0 \oplus B_2/B_1 \oplus \cdots$ be the associated coradically graded bialgebra induced by the coradical filtration.

PROPOSITION 3.3. *Let B be a pointed bialgebra and $\operatorname{gr} B$ its coradically graded version. There exist a unique quiver Q and a unique graded bialgebra structure on $\Bbbk Q$ such that $\operatorname{gr} B$ can be realized as a sub-bialgebra of $\Bbbk Q$ with $\Bbbk Q_0 \oplus \Bbbk Q_1 \subseteq \operatorname{gr} B$.*

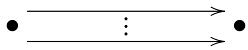
Proof. Note that $\operatorname{gr} B$ is still pointed and its coradical is B_0. Let Q_0 be the set of group-like elements of $\operatorname{gr} B$. Then clearly Q_0 is a monoid and $B_0 = \Bbbk Q_0$ as bialgebras, where the latter is the usual monoid bialgebra. Induced from the graded bialgebra structure, B_1/B_0 is a $B_0 = \Bbbk Q_0$-bialgebra bimodule. The bicomodule structure maps are denoted δ_L and δ_R. As a $\Bbbk Q_0$-bicomodule, B_1/B_0 is in fact a Q_0-bigraded space. Namely, write $M = B_1/B_0$; then

$$M = \bigoplus_{g,h \in Q_0} {}^g M^h,$$

where ${}^g M^h = \{m \in M \mid \delta_L(m) = g \otimes m,\ \delta_R(m) = m \otimes h\}$. We attach to these data a quiver Q as follows. Let the set of vertices be Q_0. For all $g, h \in Q_0$, let the number of arrows with source h and target g be equal to the $\Bbbk$-dimension of the isotypic space ${}^g M^h$. Thus we have obtained a quiver Q and moreover $\Bbbk Q_1$ has a $\Bbbk Q_0$-bialgebra bimodule structure which is identical with the B_0-bialgebra bimodule B_1/B_0. The $\Bbbk Q_0$-bialgebra bimodule $\Bbbk Q_1$ gives rise to a unique graded bialgebra structure on $\Bbbk Q$. By the universal mapping property, the coalgebra map $\operatorname{gr} B \xrightarrow{\pi_0} B_0 \simeq \Bbbk Q_0$ and the $\Bbbk Q_0$-bicomodule map $\operatorname{gr} B \xrightarrow{\pi_1} B_1/B_0 \simeq \Bbbk Q_1$ determine a unique coalgebra map $\Theta : \operatorname{gr} B \to \Bbbk Q$. Here, π_i denotes the canonical projection $\operatorname{gr} B \to B_i/B_{i-1}$. By a theorem of Heyneman and Radford (see e.g. [24, Theorem 5.3.1]), the coalgebra map Θ is injective since its restriction to the first term of the coradical filtration is injective. Again, by the universal mapping property, one can show that Θ is also an algebra map. Therefore, it is actually an embedding of bialgebras. The last condition $\Bbbk Q_0 \oplus \Bbbk Q_1 \subseteq \operatorname{gr} B$ guarantees that the quiver Q is unique. ∎

In the following we give some examples of bialgebras on the path coalgebras of quivers.

Example 3.4. Let $\mathcal{K}_n$ be the n-Kronecker quiver, i.e. a quiver of the form



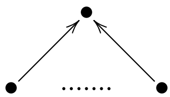
Denote the arrows as $a_1, \dots, a_n$. Let the source vertex be e and the target vertex be z as in Theorem 3.2. Then there is a bimodule structure on the space spanned by $\{a_i\}_{i=1}^n$ over the monoid $\{e, z\}$ defined by

$$e.a_i = a_i = a_i.e, \qquad z.a_i = 0 = a_i.z$$

for all i. We have the following multiplication formulas for the quiver bialgebra $\Bbbk\mathcal{K}_n$:

$$a_i \cdot a_j = 0, \qquad \forall\, 1 \le i, j \le n.$$

Example 3.5. Let $\mathcal{S}_n$ be the n-subspace quiver, i.e.

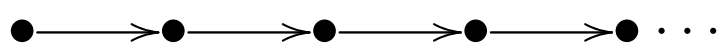


Denote the target vertex by e, the source vertices by $f_1, \dots, f_n$, and the corresponding arrows by $a_1, \dots, a_n$. Declaring e to be the identity, and f_1 to be the "zero" element, we get a monoid structure on the set of vertices as in Theorem 3.2. The bimodule structure is defined similarly:

$$e.a_i = a_i = a_i.e, \qquad f_i.a_j = 0 = a_j.f_i$$

for all $1 \le i, j \le n$. The multiplication of the quiver bialgebra $\Bbbk\mathcal{S}_n$ is similar: $a_i \cdot a_j = 0$ for all i, j.

Example 3.6. Let $\mathcal{A}_\infty$ be the quiver



Index the vertices g_i, from left to right, by $\mathbb{N} = \{0, 1, 2, \dots\}$ and consider the additive monoid structure, i.e. $g_i g_j = g_{i+j}$. Denote the arrow $g_i \to g_{i+1}$ by a_i. Define a bialgebra bimodule structure on the space of arrows by

$$g_i.a_j = a_{i+j}, \qquad a_j.g_i = q^i a_{i+j}$$

for all $i, j \in \mathbb{N}$, where $q \in \Bbbk - \{0\}$ is a parameter. Assume further that q is not a root of unity. By a routine verification one can show that the axioms of bialgebra bimodules are satisfied. Let p_i^l denote the path $a_{i+l-1} \cdots a_{i+1} a_i$ if $l \ge 1$, and g_i if $l = 0$. Apparently, $\{p_i^l\}_{i,l \ge 0}$ is a basis of $\Bbbk\mathcal{A}_\infty$. Then using Subsection 3.4 and induction we obtain the multiplication formula

$$p_i^l \cdot p_j^m = q^{jl} \binom{l+m}{m}_q p_{i+j}^{l+m}$$

for all $i, j, l, m \in \mathbb{N}$. Here we use the quantum binomial coefficients as in [17]. For completeness, we recall their definition. For any $q \in \Bbbk$ and integers

$l, m \geq 0$, define

$$l_q = 1 + q + \cdots + q^{l-1}, \quad l!_q = 1_q \cdots l_q, \quad \binom{l+m}{l}_q = \frac{(l+m)!_q}{l!_q m!_q}.$$

Clearly the quiver bialgebra $\Bbbk\mathcal{A}_\infty$ is generated as an algebra by g_1 and a_0 with relation $a_0 g_1 = q g_1 a_0$. Therefore, $\Bbbk\mathcal{A}_\infty$ is exactly the quantum plane of Manin [23] and the bialgebra structure given here is identical to that in [17, p. 118].

We remark that the quivers in the previous examples admit no Hopf algebra structures, as they are certainly not Hopf quivers. For simple quivers as in Examples 3.4 and 3.5, it is not difficult to classify all the possible graded bialgebra structures on the path coalgebras. The bialgebra structure given in Example 3.6 is different from the "trivial" one as given in the proof of Theorem 3.2.

Though theoretically any graded pointed bialgebra can be obtained as a large sub-bialgebra of a quiver bialgebra, there seems to be no chance to give a general method for such construction. However, in the following we will show that a large class of pointed bialgebras whose coradical filtration has length 2 can be constructed systematically.

Proposition 3.7. *Let Q be a quiver. Assume that in Q_0 there is a sink* (*i.e., admitting only incoming arrows*) *or a source* (*i.e., admitting only outgoing arrows*). *Then there exists on $\Bbbk Q$ a bialgebra structure such that $\Bbbk Q_0 \oplus \Bbbk Q_1$ becomes its sub-bialgebra.*

Proof. By assumption, there is a sink or a source in Q_0. Let it be the identity e of the monoid on Q_0 and take the graded bialgebra structure on $\Bbbk Q$ as given in the proof of Theorem 3.2. Let $B = \Bbbk Q_0 \oplus \Bbbk Q_1$. We claim that B is a sub-bialgebra of $\Bbbk Q$. In fact, we only need to verify that multiplication of arrows is closed in B. Given arrows $a : g \to h$ and $b : u \to v$, by Subsection 3.4 their product is

$$a \cdot b = [a.v][g.b] + [h.b][a.u].$$

Clearly the term $[a.v][g.b]$ survives, i.e. is not zero, only if $v = g = e$, but this contradicts the assumption that e is a sink or a source. Similarly $[h.b][a.u] = 0$. This shows that the multiplication of $\Bbbk Q$ is indeed closed in B. ■

Remark 3.8. Coalgebras with coradical filtration having length 2 have been studied by Kosakowska and Simson [20], where a reduction to hereditary coalgebras is presented and the Gabriel quiver is discussed in terms of irreducible morphisms. The dual of such a coalgebra is an algebra of radical square zero. The class of radical square zero algebras is very important in the representation theory of Artin algebras (see e.g. [1, 2]). The dual of the previous proposition asserts that every elementary radical square zero algebra with $\mathrm{Ext}^1(S,-)=0$ or $\mathrm{Ext}^1(-,S)=0$ for some simple module S has a bialgebra structure, therefore its module category has a natural tensor product.

4. Monoidal structures over quiver representations. In this section we consider natural monoidal structures on the categories of locally nilpotent representations of quivers arising from bialgebra structures.

Recall that a *monoidal category* is a sextuple $(\mathcal{C},\otimes,\mathbf{1},\alpha,\lambda,\rho)$, where $\mathcal{C}$ is a category, $\otimes:\mathcal{C}\times\mathcal{C}\to\mathcal{C}$ is a functor, $\mathbf{1}$ an object, and $\alpha:\otimes\circ(\otimes\times\mathrm{Id})\to\otimes\circ(\mathrm{Id}\times\otimes)$, $\lambda:\mathbf{1}\otimes-\to\mathrm{Id}$, $\rho:-\otimes\mathbf{1}\to\mathrm{Id}$ are natural isomorphisms such that the associativity and unitarity constraints hold, or equivalently the pentagon and the triangle diagrams are commutative (see e.g. [21] for details).

Natural examples of monoidal structures are the categories of B-modules and B-comodules with B a bialgebra (see e.g. [17, 24]). Recall that, if U and V are right B-comodules and $U\otimes V$ the usual tensor product of $\Bbbk$-spaces, then the comodule structure of $U\otimes V$ is given by $u\otimes v\mapsto u_0\otimes v_0\otimes u_1v_1$, where we use the Sweedler notation $u\mapsto u_0\otimes u_1$ for comodule structure maps. The unit object is the trivial comodule $\Bbbk$ with comodule structure map $k\mapsto k\otimes 1$. On the other hand, by the reconstruction formalism, monoidal categories with fiber functors are obtained in this manner (see e.g. [9, 22]).

The monoidal categories arising from quiver bialgebras (as their right comodule categories) share a common property: their simple objects all have $\Bbbk$-dimension 1 and consist of a monoid. Inspired by this and the notion of pointed tensor categories introduced in [9], we call a $\Bbbk$-linear monoidal category *pointed* if the iso-classes of simple objects constitute a monoid (under tensor product).

From now on, the field $\Bbbk$ *is assumed to be algebraically closed and the monoidal categories under consideration are* $\Bbbk$*-linear abelian.* A monoidal category is said to be *finite* if the underlying category is equivalent to the category of finite-dimensional comodules over a finite-dimensional coalgebra. This is a natural generalization of the notion of finite tensor categories of Etingof and Ostrik [10].

Classification of finite monoidal categories is a fundamental problem. Our results in Section 3 indicate that even finite pointed monoidal categories are "over" pervasive, it is necessary to impose a proper condition before considering the classification problem. In the following, by taking advantage of the theory of quivers and their representations, we give classification results for some classes of finite pointed monoidal categories.

An abelian category is said to be *hereditary* if the extension bifunctor Ext^n vanishes in each degree $n\geq 2$. Next we consider hereditary finite pointed monoidal categories. Though the following results are direct consequences of some well-known theorems on quivers and representations, we feel it is of interest to include them here.

PROPOSITION 4.1. *A hereditary finite pointed monoidal category with a fiber functor is equivalent to* $(\mathrm{Rep}(Q), \mathcal{F})$ *with* Q *a finite acyclic quiver and* $\mathcal{F}$ *the forgetful functor from* $\mathrm{Rep}(Q)$ *to the category* $\mathrm{Vec}_{\Bbbk}$ *of vector spaces.*

Proof. By the standard reconstruction process (see e.g. [9, 22]), a finite monoidal category with fiber functor is equivalent to $(B\text{-comod}, \mathcal{F})$ in which B-comod is the category of finite-dimensional right comodules over a finite-dimensional bialgebra B, and $\mathcal{F}$ is the forgetful functor. Note that the category of right comodules over a finite-dimensional bialgebra B is pointed if and only if B is pointed. Now by the Gabriel type theorem for pointed coalgebras [5], there exists a unique quiver Q such that B is isomorphic to a large subcoalgebra, i.e. including the space spanned by the set of vertices and arrows, of the path coalgebra $\Bbbk Q$. The hereditariness of the category B-comod forces B to be hereditary as a coalgebra. This indicates that B is isomorphic to $\Bbbk Q$ as a coalgebra. Since $B \cong \Bbbk Q$ is finite-dimensional, the quiver Q must be finite and acyclic. Obviously, any representation of Q is automatically locally nilpotent, hence B-comod is equivalent to $\mathrm{Rep}(Q)$. ∎

Now we can use Gabriel's famous classification theorem [11] on quivers of finite representation type, i.e. admitting only finitely many indecomposable representations up to isomorphism, to describe hereditary pointed monoidal categories in which there are only finitely many iso-classes of indecomposable objects. Following the terminology of [16], a finite monoidal category is said to be *of finite type* if it has only finitely many iso-classes of indecomposable objects.

COROLLARY 4.2. *A hereditary pointed monoidal category of finite type with a fiber functor is of the form* $\mathrm{Rep}(Q)$ *where* Q *is a finite disjoint union of quivers of* ADE *type.*

Finally we give two examples of quiver monoidal categories. We also work out their Clebsch–Gordan formula and representation ring respectively.

EXAMPLE 4.3. Let $\mathcal{A}_n$ be the quiver

$$\bullet \longrightarrow \bullet \longrightarrow \bullet \cdots\cdots \bullet \longrightarrow \bullet$$

with $n \geq 2$ vertices $v_1, \ldots, v_n$ and $n-1$ arrows $a_1, \ldots, a_{n-1}$ where $a_i : v_i \to v_{i+1}$. Set v_1 to be the identity, v_2 to be the zero element, take the monoid structure on $\{v_i \mid 1 \leq i \leq n\}$ as in Theorem 3.2 and consider the corresponding bialgebra structure with multiplication given by

$$v_1 \cdot a_i = a_i = a_i \cdot v_1, \quad v_j \cdot a_i = 0 = a_i \cdot v_j \quad (j \geq 2), \quad a_i \cdot a_j = 0.$$

For a pair of integers i, j satisfying $1 \le i \le j \le n$, define a representation $V(i,j)$ of $\mathcal{A}_n$ by

$$V(i,j)_{v_k} = \begin{cases} \Bbbk, & i \le k \le j, \\ 0, & \text{otherwise}, \end{cases} \qquad V(i,j)_{a_k} = \begin{cases} 1, & i \le k \le j-1, \\ 0, & \text{otherwise}. \end{cases}$$

It is well-known that the set $\{V(i,j) \mid 1 \le i \le j \le n\}$ is a complete list of indecomposable representations of $\mathcal{A}_n$. For the Clebsch–Gordan problem of the category $\operatorname{Rep}(\mathcal{A}_n)$, it is enough to consider the decomposition rule of $V(i,j) \otimes V(k,l)$ thanks to the Krull–Schmidt theorem.

Given a representation $V(i,j)$, let e_s $(i \le s \le j)$ denote a basis element of the vector space $\Bbbk$ assigned to the sth vertex. Recall that the associated comodule structure map for $V(i,j)$ is given by

$$\delta(e_s) = \sum_{x=s}^{j} e_x \otimes p_s^{x-s},$$

where p_x^y denotes the path of length y starting at x. Now for $e_s \otimes e_t \in V(i,j) \otimes V(k,l)$, we have

$$\delta(e_s \otimes e_t) = \begin{cases} \displaystyle\sum_{x=s}^{j} e_x \otimes e_t \otimes p_s^{x-s} + \sum_{y=t}^{l} e_s \otimes e_y \otimes p_t^{y-t}, & s = t = 1, \\ \displaystyle\sum_{y=t}^{l} e_s \otimes e_y \otimes p_t^{y-t}, & s = 1,\ t \ge 2, \\ \displaystyle\sum_{x=s}^{j} e_x \otimes e_t \otimes p_s^{x-s}, & s \ge 2,\ t = 1, \\ e_s \otimes e_t \otimes v_2, & s \ge 2,\ t \ge 2. \end{cases}$$

Therefore, it is clear that

$$V(i,j) \otimes V(k,l) = \begin{cases} V(i,j) \oplus V(k,l) \oplus V(2,2)^{(j-i)(l-k)+1}, & i = k = 1, \\ V(k,l) \oplus V(2,2)^{(j-i)(l-k+1)}, & i = 1,\ k \ge 2, \\ V(i,j) \oplus V(2,2)^{(j-i+1)(l-k)}, & i \ge 2,\ k = 1, \\ V(2,2)^{(j-i+1)(l-k+1)}, & i \ge 2,\ k \ge 2. \end{cases}$$

EXAMPLE 4.4. Consider the infinite quiver $\mathcal{A}_\infty$. Take the bialgebra structure on $\Bbbk\mathcal{A}_\infty$ as in Example 3.6 and keep the notations therein. For any pair of integers i, j with $0 \le i \le j$, define a representation $V(i,j)$ of $\mathcal{A}_\infty$ as in the previous example. Clearly, $\{V(i,j) \mid 0 \le i \le j\}$ is a complete set of locally nilpotent and locally finite indecomposable representations of $\mathcal{A}_\infty$. As in Example 4.3, take a basis element e_s for the vector space $\Bbbk$ in $V(i,j)$ attached to the sth vertex g_s. The corresponding comodule structure map of $V(i,j)$ is

$$\delta(e_s) = \sum_{x=s}^{j} e_x \otimes p_s^{x-s}.$$

Consider the tensor product $V(i,j) \otimes V(k,l)$. For $e_s \otimes e_t \in V(i,j) \otimes V(k,l)$, we have

$$\delta(e_s \otimes e_t) = \sum_{x=s}^{j} \sum_{y=t}^{l} \binom{x-s+y-t}{y-t}_q e_x \otimes e_y \otimes p_{s+t}^{x-s+y-t}.$$

From this, it is not hard to see that

$$V(0,1) \otimes V(0,n) = V(0,n+1) \oplus V(1,n),$$
$$V(i,j) \otimes V(1,1) = V(i+1,j+1) = V(1,1) \otimes V(i,j).$$

This implies that the representation ring of $\mathrm{Rep}^{\mathrm{lnlf}}(\mathcal{A}_\infty)$ is generated by $V(0,1)$ and $V(1,1)$ and is isomorphic to the polynomial ring in two variables $\mathbb{Z}[X,Y]$.

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