COLLOQUIUM MATHEMATICUM

VOL. 135 2014 NO. 2

WEIGHTED SHARP MAXIMAL FUNCTION INEQUALITIES AND BOUNDEDNESS OF A LINEAR OPERATOR ASSOCIATED TO A SINGULAR INTEGRAL OPERATOR WITH NON-SMOOTH KERNEL

BY

DAZHAO CHEN (Shaoyang)

Abstract. We establish weighted sharp maximal function inequalities for a linear operator associated to a singular integral operator with non-smooth kernel. As an application, we obtain the boundedness of a commutator on weighted Lebesgue spaces.

1. Introduction. As a development of singular integral operators (see [GR], [S]), their commutators have been well studied. In [CRW], [PE], [PT], the authors proved that the commutators generated by singular integral operators and BMO functions are bounded on $L^p(\mathbb{R}^n)$ for $1 < p < \infty$. Chanillo [C] proved a similar result when singular integral operators are replaced by fractional integral operators. In [J], [PA], the boundedness of commutators generated by singular integral operators and Lipschitz functions on Triebel–Lizorkin and $L^p(\mathbb{R}^n)$ $(1 < p < \infty)$ spaces was obtained. In [B], [HG], the boundedness of commutators generated by singular integral operators and weighted BMO and Lipschitz functions on $L^p(\mathbb{R}^n)$ $(1 < p < \infty)$ spaces was established (see also [HEW]). In [CG], Cohen and Gosselin studied generalized commutators of singular integral operators of the form (see also [DL])

$$T^b(f)(x) = \int_{\mathbb{R}^n} \frac{R_{m+1}(b; x, y)}{|x-y|^m} K(x,y) f(y)\, dy,$$

and obtained some sharp function estimates and boundedness of the commutators if $D^\alpha b \in \mathrm{BMO}(\mathbb{R}^n)$ for all α with $|\alpha| = m$. In [DM], [MA], some singular integral operators with non-smooth kernel were introduced, and the boundedness of these operators and their commutators was obtained (see [DEY], [LIU1], [LIU2], [ZL]).

Motivated by these, in this paper, we will study certain linear operators generated by singular integral operators with non-smooth kernel and

2010 *Mathematics Subject Classification*: Primary 42B20; Secondary 42B25.

Key words and phrases: singular integral operator, commutator, sharp maximal function, weighted BMO, weighted Lipschitz function.

DOI: 10.4064/cm135-2-1

weighted Lipschitz and BMO functions, that is, $D^\alpha b \in \mathrm{BMO}(w)$ or $D^\alpha b \in \mathrm{Lip}_\beta(w)$ for all α with $|\alpha| = m$.

2. Preliminaries. We will study some singular integral operators as described below (see [DM]).

DEFINITION 2.1. A family of operators D_t, $t > 0$, is said to be an *approximation to the identity* if, for every $t > 0$, D_t can be represented by a kernel $a_t(x, y)$ in the following sense:

$$D_t(f)(x) = \int_{\mathbb{R}^n} a_t(x, y) f(y)\, dy$$

for every $f \in L^p(\mathbb{R}^n)$ with $p \ge 1$, and $a_t(x, y)$ satisfies

$$|a_t(x, y)| \le h_t(x, y) = Ct^{-n/2} \rho(|x - y|^2/t),$$

where ρ is a positive, bounded and decreasing function satisfying

$$\lim_{r \to \infty} r^{n+\epsilon} \rho(r^2) = 0 \quad \text{for some } \epsilon > 0.$$

DEFINITION 2.2. A linear operator T is called a *singular integral operator with non-smooth kernel* if T is bounded on $L^2(\mathbb{R}^n)$ and associated with a kernel $K(x, y)$ such that

$$T(f)(x) = \int_{\mathbb{R}^n} K(x, y) f(y)\, dy$$

for every continuous function f with compact support, and for almost all x not in the support of f; moreover, we assume that:

(1) There exists an approximation to the identity $\{B_t, t > 0\}$ such that TB_t has kernel $k_t(x, y)$ and there exist $c_1, c_2 > 0$ so that

$$\int_{|x-y| > c_1 t^{1/2}} |K(x, y) - k_t(x, y)|\, dx \le c_2 \quad \text{for all } y \in \mathbb{R}^n.$$

(2) There exists an approximation to the identity $\{A_t, t > 0\}$ such that $A_t T$ has kernel $K_t(x, y)$ which satisfies

$$|K_t(x, y)| \le c_4 t^{-n/2} \quad \text{if } |x - y| \le c_3 t^{1/2},$$
$$|K(x, y) - K_t(x, y)| \le c_4 t^{\delta/2} |x - y|^{-n-\delta} \quad \text{if } |x - y| \ge c_3 t^{1/2},$$

for some $\delta, c_3, c_4 > 0$.

Moreover, let m be the positive integer and b be a function on $\mathbb{R}^n$. Set

$$R_{m+1}(b; x, y) = b(x) - \sum_{|\alpha| \le m} \frac{1}{\alpha!} D^\alpha b(y) (x - y)^\alpha.$$

We relate to T the linear operator defined by

$$T^b(f)(x) = \int_{\mathbb{R}^n} \frac{R_{m+1}(b;x,y)}{|x-y|^m} K(x,y) f(y)\,dy.$$

Note that the commutator $[b,T](f) = bT(f) - T(bf)$ is a particular case of T^b if $m = 0$. The linear operator T^b is a non-trivial generalization of the commutator. It is well known that commutators are of great interest in harmonic analysis and have been widely studied by many authors (see [CG], [DL]). The main purpose of this paper is to prove sharp maximal inequalities for the linear operator T^b. As an application, we obtain the weighted L^p-boundedness of T^b.

Now, let us introduce some notations. Throughout this paper, Q will denote a cube in $\mathbb{R}^n$ with sides parallel to the axes. For a non-negative integrable function ω, let $\omega(Q) = \int_Q \omega(x)\,dx$ and $\omega_Q = |Q|^{-1}\int_Q \omega(x)\,dx$.

For any locally integrable function f, the *sharp maximal function* of f is defined by

$$M^{\#}(f)(x) = \sup_{Q \ni x} \frac{1}{|Q|} \int_Q |f(y) - f_Q|\,dy.$$

It is well known (see [GR]) that

$$M^{\#}(f)(x) \approx \sup_{Q \ni x} \inf_{c \in C} \frac{1}{|Q|} \int_Q |f(y) - c|\,dy.$$

Let

$$M(f)(x) = \sup_{Q \ni x} \frac{1}{|Q|} \int_Q |f(y)|\,dy.$$

For $\eta > 0$, let $M^{\#}_\eta(f)(x) = M^{\#}(|f|^\eta)^{1/\eta}(x)$ and $M_\eta(f)(x) = M(|f|^\eta)^{1/\eta}(x)$.

For $0 < \eta < n$, $1 \le p < \infty$ and a non-negative weight function ω, set

$$M_{\eta,p,\omega}(f)(x) = \sup_{Q \ni x} \left(\frac{1}{\omega(Q)^{1-p\eta/n}} \int_Q |f(y)|^p \omega(y)\,dy \right)^{1/p},$$

$$M_\omega(f)(x) = \sup_{Q \ni x} \frac{1}{\omega(Q)} \int_Q |f(y)|\omega(y)\,dy.$$

The sharp maximal function $M_A(f)$ associated with an approximation to the identity $\{A_t, t > 0\}$ is defined by

$$M^{\#}_A(f)(x) = \sup_{x \in Q} \frac{1}{|Q|} \int_Q |f(y) - A_{t_Q}(f)(y)|\,dy,$$

where $t_Q = l(Q)^2$ and $l(Q)$ denotes the side length of Q. For $\eta > 0$, let $M^{\#}_{A,\eta}(f) = M^{\#}_A(|f|^\eta)^{1/\eta}$.

The A_p *weights* are defined by (see [GR])

$$A_p = \left\{\omega \in L^1_{\mathrm{loc}}(\mathbb{R}^n) : \sup_Q \left(\frac{1}{|Q|}\int_Q \omega(x)\,dx\right)\left(\frac{1}{|Q|}\int_Q \omega(x)^{-1/(p-1)}\,dx\right)^{p-1} < \infty\right\}$$

for $1 < p < \infty$, and

$$A_1 = \{\omega \in L^p_{\mathrm{loc}}(\mathbb{R}^n) : M(\omega)(x) \le Cw(x) \text{ a.e.}\}.$$

Given a non-negative weight function ω, and $1 \le p < \infty$, the *weighted Lebesgue space* $L^p(\mathbb{R}^n,\omega)$ is the space of functions f such that

$$\|f\|_{L^p(\omega)} = \Big(\int_{\mathbb{R}^n} |f(x)|^p \omega(x)\,dx\Big)^{1/p} < \infty.$$

Given a non-negative weight function ω, the *weighted BMO space* $\mathrm{BMO}(\omega)$ is the space of functions b such that

$$\|b\|_{\mathrm{BMO}(\omega)} = \sup_Q \frac{1}{\omega(Q)}\int_Q |b(y) - b_Q|\,dy < \infty.$$

For $0 < \beta < 1$, the *weighted Lipschitz space* $\mathrm{Lip}_\beta(\omega)$ is the space of functions b such that

$$\|b\|_{\mathrm{Lip}_\beta(\omega)} = \sup_Q \frac{1}{\omega(Q)^{\beta/n}}\left(\frac{1}{\omega(Q)}\int_Q |b(y) - b_Q|^p \omega(x)^{1-p}\,dy\right)^{1/p} < \infty.$$

Remark. (1) It is known (see [G]) that for $b \in \mathrm{Lip}_\beta(\omega)$, $\omega \in A_1$ and $x \in Q$,

$$|b_Q - b_{2^kQ}| \le Ck\|b\|_{\mathrm{Lip}_\beta(\omega)}\omega(x)\omega(2^kQ)^{\beta/n}.$$

(2) Let $b \in \mathrm{Lip}_\beta(\omega)$ and $\omega \in A_1$. By [G], we know that the spaces $\mathrm{Lip}_\beta(\omega)$ all coincide and the norms $\|b\|_{\mathrm{Lip}_\beta(\omega)}$ for different $1 \le p < \infty$ are all equivalent.

We give some preliminary lemmas.

Lemma 2.3 (see [GR, p. 485]). *Let $0 < p < q < \infty$. For any function $f \ge 0$ define, with $1/r = 1/p - 1/q$,*

$$\|f\|_{WL^q} = \sup_{\lambda>0} \lambda|\{x \in \mathbb{R}^n : f(x) > \lambda\}|^{1/q},$$

$$N_{p,q}(f) = \sup_Q \|f\chi_Q\|_{L^p}/\|\chi_Q\|_{L^r},$$

where the sup is taken over all measurable sets Q with $0 < |Q| < \infty$. Then

$$\|f\|_{WL^q} \le N_{p,q}(f) \le (q/(q-p))^{1/p}\|f\|_{WL^q}.$$

Lemma 2.4 (see [DM], [MA]). *Let T be a singular integral operator as in Definition 2.2. Then T is bounded on $L^p(\mathbb{R}^n,\omega)$ for $\omega \in A_p$ with $1 < p < \infty$, and weakly (L^1, L^1) bounded.*

LEMMA 2.5 (see [B]). *Let* $b \in \mathrm{BMO}(\omega)$. *Then*

$$|b_Q - b_{2^jQ}| \le Cj\|b\|_{\mathrm{BMO}(\omega)}\omega_{Q_j},$$

where $\omega_{Q_j} = \max_{1\le i\le j} |2^iQ|^{-1}\int_{2^iQ}\omega(x)\,dx$.

LEMMA 2.6 (see [B]). *Let* $\omega \in A_p$ *with* $1 < p < \infty$. *Then there exists* $\varepsilon > 0$ *such that* $\omega^{-r/p} \in A_r$ *for any* $p' \le r \le p' + \varepsilon$.

LEMMA 2.7 (see [B]). *Let* $b \in \mathrm{BMO}(\omega)$ *with* $\omega = (\mu\nu^{-1})^{1/p}$, $\mu, \nu \in A_p$ *and* $p > 1$. *Then there exists* $\varepsilon > 0$ *such that for* $p' \le r \le p' + \varepsilon$,

$$\int_Q |b(x) - b_Q|^r \mu(x)^{-r/p}\,dx \le C\|b\|^r_{\mathrm{BMO}(\omega)}\int_Q \nu(x)^{-r/p}\,dx.$$

LEMMA 2.8 (see [B]). *Let* $\omega \in A_p$ *with* $1 < p < \infty$. *Then there exists* $0 < \delta < 1$ *such that* $\omega^{1-r'/p} \in A_{p/r'}(d\mu)$ *for any* $p' < r < p'(1+\delta)$, *where* $d\mu = \omega^{r'/p}\,dx$.

LEMMA 2.9 (see [B]). *Let* $\mu, \nu \in A_p$ *and* $\omega = (\mu\nu^{-1})^{1/p}$ *with* $1 < p < \infty$. *Then there exists* $1 < q < p$ *such that*

$$\omega_Q(\nu_Q)^{1/q}\bigg(\frac{1}{|Q|}\int_Q \omega(x)^{-q'}\nu(x)^{-q'/q}\,dx\bigg)^{1/q'} \le C.$$

LEMMA 2.10 (see [C], [GR]). *Let* $0 \le \eta < n$, $1 \le s < p < n/\eta$, $1/q = 1/p - \eta/n$ *and* $\omega \in A_1$. *Then*

$$\|M_{\eta,s,\omega}(f)\|_{L^q(\omega)} \le C\|f\|_{L^p(\omega)}.$$

LEMMA 2.11 (see [DM], [MA]). *Let* $\{A_t, t > 0\}$ *be an approximation to the identity. For any* $\gamma > 0$, *there exists a constant* $C > 0$ *independent of* γ *such that*

$$|\{x \in \mathbb{R}^n : M(f)(x) > D\lambda,\ M_A^{\#}(f)(x) \le \gamma\lambda\}| \le C\gamma|\{x \in \mathbb{R}^n : M(f)(x) > \lambda\}|$$

for $\lambda > 0$, *where* D *is a fixed constant which only depends on* n. *Thus, for* $f \in L^p(\mathbb{R}^n)$, $1 < p < \infty$, $0 < \eta < \infty$ *and* $\omega \in A_1$,

$$\|M_\eta(f)\|_{L^p(\omega)} \le C\|M^{\#}_{A,\eta}(f)\|_{L^p(\omega)}.$$

LEMMA 2.12 (see [CG]). *Let* b *be a function on* $\mathbb{R}^n$ *with* $D^\alpha b \in L^s(\mathbb{R}^n)$ *for all* α *with* $|\alpha| = m$ *and any* $s > n$. *Then*

$$|R_m(b; x, y)| \le C|x-y|^m \sum_{|\alpha|=m}\bigg(\frac{1}{|\tilde{Q}(x,y)|}\int_{\tilde{Q}(x,y)} |D^\alpha b(z)|^s\,dz\bigg)^{1/s},$$

where $\tilde{Q}$ *is the cube centered at* x *and having side length* $5\sqrt{n}|x-y|$.

LEMMA 2.13. *Let $\{A_t, t > 0\}$ be an approximation to the identity, $\omega \in A_1$, $0 < \beta < 1$, $1 < r < \infty$ and $b \in \mathrm{Lip}_\beta(\omega)$. Then for every $f \in L^p(\omega)$, $p > 1$ and $\tilde{x} \in \mathbb{R}^n$,*

$$\sup_{Q \ni \tilde{x}} \frac{1}{|Q|} \int_Q |A_{t_Q}((b - b_Q)f)(y)|\, dy \le C\|b\|_{\mathrm{Lip}_\beta(\omega)} \omega(\tilde{x}) M_{\beta,\omega,r}(f)(\tilde{x}).$$

Proof. We write, for any cube Q with $\tilde{x} \in Q$,

$$\begin{aligned}\frac{1}{|Q|} \int_Q |A_{t_Q}((b - b_Q)f)(x)|\, dx &\le \frac{1}{|Q|} \int_Q \int_{\mathbb{R}^n} h_{t_Q}(x,y)|(b(y) - b_Q)f(y)|\, dy\, dx \\ &\le \frac{1}{|Q|} \int_Q \int_Q h_{t_Q}(x,y)|(b(y) - b_Q)f(y)|\, dy\, dx \\ &\quad + \sum_{k=0}^{\infty} \frac{1}{|Q|} \int_Q \int_{2^{k+1}Q \setminus 2^k Q} h_{t_Q}(x,y)|(b(y) - b_Q)f(y)|\, dy\, dx \\ &= I + II.\end{aligned}$$

We have, by Hölder's inequality,

$$\begin{aligned} I &\le \frac{C}{|Q|\,|Q|} \int_Q \int_Q |(b(y) - b_Q)f(y)|\, dy\, dx \\ &\le \frac{C}{|Q|} \int_Q |b(y) - b_Q|\omega(y)^{-1/r}|f(y)|\omega(y)^{1/r}\, dy \\ &\le \frac{C}{|Q|} \Big(\int_Q |b(y) - b_Q|^{r'} \omega(y)^{1-r'}\, dy\Big)^{1/r'} \Big(\int_Q |f(y)|^r \omega(y)\, dy\Big)^{1/r} \\ &\le \frac{C}{|Q|} \|b\|_{\mathrm{Lip}_\beta(\omega)} \omega(Q)^{\beta/n + 1/r'} \Big(\int_Q |f(y)|^r \omega(y)\, dy\Big)^{1/r} \\ &\le C\|b\|_{\mathrm{Lip}_\beta(\omega)} \frac{\omega(Q)}{|Q|} M_{\beta,r,\omega}(f)(\tilde{x}) \\ &\le C\|b\|_{\mathrm{Lip}_\beta(\omega)} \omega(\tilde{x}) M_{\beta,r,\omega}(f)(\tilde{x}).\end{aligned}$$

For II, notice that if $x \in Q$ and $y \in 2^{k+1}Q \setminus 2^k Q$, then $|x - y| \ge 2^{k-1} t_Q$ and $h_{t_Q}(x,y) \le Cs(2^{2(k-1)})/|Q|$, so

$$\begin{aligned} II &\le C \sum_{k=0}^{\infty} s(2^{2(k-1)}) \frac{1}{|Q|\,|Q|} \int_Q \int_{2^{k+1}Q} |(b(y) - b_Q)f(y)|\, dy\, dx \\ &\le C \sum_{k=0}^{\infty} 2^{kn} s(2^{2(k-1)}) \frac{1}{|2^{k+1}Q|} \int_{2^{k+1}Q} |(b(y) - b_{2^{k+1}Q}) + (b_{2^{k+1}Q} - b_Q)|\, |f(y)|\, dy\end{aligned}$$

$$\leq C\sum_{k=0}^{\infty}2^{kn}s(2^{2(k-1)})|2^{k+1}Q|^{-1}\Big(\int\limits_{2^{k+1}Q}|b(y)-b_{2^{k+1}Q}|^{r'}\omega(y)^{1-r'}\,dy\Big)^{1/r'}$$
$$\times\Big(\int\limits_{2^{k+1}Q}|f(y)|^{r}\omega(y)\,dy\Big)^{1/r}$$
$$+\,C\sum_{k=0}^{\infty}2^{kn}s(2^{2(k-1)})|2^{k+1}Q|^{-1}k\|b\|_{\mathrm{Lip}_\beta(\omega)}\omega(\tilde{x})\omega(2^{k+1}Q)^{\beta/n}$$
$$\times\Big(\int\limits_{2^{k+1}Q}|f(y)|^{r}\omega(y)\,dy\Big)^{1/r}\Big(\frac{1}{|2^{k+1}Q|}\int\limits_{2^{k+1}Q}\omega(y)^{-1/(r-1)}\,dy\Big)^{(r-1)/r}$$
$$\times\Big(\frac{1}{|2^{k+1}Q|}\int\limits_{2^{k+1}Q}\omega(y)\,dy\Big)^{1/r}|2^{k+1}Q|\omega(2^{k+1}Q)^{-1/r}$$
$$\leq C\|b\|_{\mathrm{Lip}_\beta(\omega)}\sum_{k=0}^{\infty}k2^{kn}s(2^{2(k-1)})\Big(\frac{\omega(2^{k+1}Q)}{|2^{k+1}Q|}+\omega(\tilde{x})\Big)$$
$$\times\Big(\frac{1}{\omega(2^{k+1}Q)^{1-r\beta/n}}\int\limits_{2^{k+1}Q}|f(y)|^{r}\omega(y)\,dy\Big)^{1/r}$$
$$\leq C\|b\|_{\mathrm{Lip}_\beta(\omega)}\sum_{k=0}^{\infty}k2^{kn}s(2^{2(k-1)})\omega(\tilde{x})M_{\beta,r,\omega}(f)(\tilde{x})$$
$$\leq C\|b\|_{\mathrm{Lip}_\beta(\omega)}\omega(\tilde{x})M_{\beta,r,\omega}(f)(\tilde{x}),$$

where the last inequality follows from

$$\sum_{k=1}^{\infty}k2^{(k-1)n}s(2^{2(k-1)})\leq C\sum_{k=1}^{\infty}k2^{-(k-1)\varepsilon}<\infty$$

for some $\varepsilon>0$. This completes the proof. ∎

3. Theorems and proofs

Theorem 3.1. *Let T be a singular integral operator with non-smooth kernel as in Definition 2.2, $1<p<\infty$, $\mu,\nu\in A_p$, $\omega=(\mu\nu^{-1})^{1/p}$, $0<\eta<1$ and $D^\alpha b\in\mathrm{BMO}(\omega)$ for all α with $|\alpha|=m$. Then there exists a constant $C>0$, $\varepsilon>0$, $0<\delta<1$, $1<q<p$ and $p'<r<\min(p'+\varepsilon,p'(1+\delta))$ such that, for any $f\in C_0^\infty(\mathbb{R}^n)$ and $\tilde{x}\in\mathbb{R}^n$,*

$$M^{\#}_{A,\eta}(T^b(f))(\tilde{x})\leq C\sum_{|\alpha|=m}\|D^\alpha b\|_{\mathrm{BMO}(\omega)}$$
$$\times\big([M_\nu(|\omega T(f)|^q)(\tilde{x})]^{1/q}+[M_{\nu^{r'/p}}(|\omega f|^{r'})(\tilde{x})]^{1/r'}+[M_\nu(|\omega f|^q)(\tilde{x})]^{1/q}\big).$$

Proof. It suffices to prove that for $f \in C_0^\infty(\mathbb{R}^n)$ and some constant C,

$$\left(\frac{1}{|Q|}\int_Q |T^b(f)(x) - A_{t_Q}(T^b(f))(x)|^\eta\,dx\right)^{1/\eta}$$
$$\le C\sum_{|\alpha|=m}\|D^\alpha b\|_{\mathrm{BMO}(\omega)}\big([M_\nu(|\omega T(f)|^q)(\tilde{x})]^{1/q}$$
$$+ [M_{\nu^{r'/p}}(|\omega f|^{r'})(\tilde{x})]^{1/r'} + [M_\nu(|\omega f|^q)(\tilde{x})]^{1/q}\big),$$

where $t_Q = d^2$ and d denotes the side length of Q. Fix a cube $Q = Q(x_0, d)$ and $\tilde{x} \in Q$. Let $\tilde{Q} = 5\sqrt{n}\,Q$ and $\tilde{b}(x) = b(x) - \sum_{|\alpha|=m}(1/\alpha!)(D^\alpha b)_{\tilde{Q}}x^\alpha$. Then $R_m(b;x,y) = R_m(\tilde{b};x,y)$ and $D^\alpha\tilde{b} = D^\alpha b - (D^\alpha b)_{\tilde{Q}}$ for $|\alpha| = m$. We write, for $f_1 = f\chi_{\tilde{Q}}$ and $f_2 = f\chi_{\mathbb{R}^n\setminus\tilde{Q}}$,

$$T^b(f)(x) = \int_{\mathbb{R}^n}\frac{R_m(\tilde{b};x,y)}{|x-y|^m}K(x,y)f_1(y)\,dy$$
$$-\sum_{|\alpha|=m}\frac{1}{\alpha!}\int_{\mathbb{R}^n}\frac{(x-y)^\alpha D^\alpha\tilde{b}(y)}{|x-y|^m}K(x,y)f_1(y)\,dy$$
$$+\int_{\mathbb{R}^n}\frac{R_{m+1}(\tilde{b};x,y)}{|x-y|^m}K(x,y)f_2(y)\,dy$$
$$= T\left(\frac{R_m(\tilde{b};x,\cdot)}{|x-\cdot|^m}f_1\right) - T\left(\sum_{|\alpha|=m}\frac{1}{\alpha!}\frac{(x-\cdot)^\alpha D^\alpha\tilde{b}}{|x-\cdot|^m}f_1\right) + T^{\tilde{b}}(f_2)(x)$$

and

$$A_{t_Q}T^b(f)(x) = \int_{\mathbb{R}^n}\frac{R_m(\tilde{b}_j;x,y)}{|x-y|^m}K_t(x,y)f_1(y)\,dy$$
$$-\sum_{|\alpha|=m}\frac{1}{\alpha!}\int_{\mathbb{R}^n}\frac{(x-y)^\alpha D^\alpha\tilde{b}(y)}{|x-y|^m}K_t(x,y)f_1(y)\,dy$$
$$+\int_{\mathbb{R}^n}\frac{R_{m+1}(\tilde{b};x,y)}{|x-y|^m}K_t(x,y)f_2(y)\,dy$$
$$= A_{t_Q}T\left(\frac{R_m(\tilde{b};x,\cdot)}{|x-\cdot|^m}f_1\right)$$
$$- A_{t_Q}T\left(\sum_{|\alpha|=m}\frac{1}{\alpha!}\frac{(x-\cdot)^\alpha D^\alpha\tilde{b}}{|x-\cdot|^m}f_1\right) + A_{t_Q}T^{\tilde{b}}(f_2)(x).$$

Then

$$\Big(\frac{1}{|Q|}\int_Q |T^b(f)(x)-A_{t_Q}T^b(f)(x)|^\eta\,dx\Big)^{1/\eta}$$
$$\le C\Big(\frac{1}{|Q|}\int_Q \Big|T\Big(\frac{R_m(\tilde b;x,\cdot)}{|x-\cdot|^m}f_1\Big)\Big|^\eta\,dx\Big)^{1/\eta}$$
$$+C\Big(\frac{1}{|Q|}\int_Q \Big|T\Big(\sum_{|\alpha|=m}\frac{1}{\alpha!}\frac{(x-\cdot)^\alpha D^\alpha\tilde b}{|x-\cdot|^m}f_1\Big)\Big|^\eta\,dx\Big)^{1/\eta}$$
$$+C\Big(\frac{1}{|Q|}\int_Q \Big|A_{t_Q}T\Big(\frac{R_m(\tilde b;x,\cdot)}{|x-\cdot|^m}f_1\Big)\Big|^\eta\,dx\Big)^{1/\eta}$$
$$+C\Big(\frac{1}{|Q|}\int_Q \Big|A_{t_Q}T\Big(\sum_{|\alpha|=m}\frac{1}{\alpha!}\frac{(x-\cdot)^\alpha D^\alpha\tilde b}{|x-\cdot|^m}f_1\Big)\Big|^\eta\,dx\Big)^{1/\eta}$$
$$+C\Big(\frac{1}{|Q|}\int_Q |T^{\tilde b}(f_2)(x)-A_{t_Q}T^{\tilde b}(f_2)(x)|^\eta\,dx\Big)^{1/\eta}$$
$$=I_1+I_2+I_3+I_4+I_5.$$

For I_1, note that $\omega\in A_1$ satisfies the reverse Hölder inequality

$$\Big(\frac{1}{|Q|}\int_Q \omega(x)^{p_0}\,dx\Big)^{1/p_0}\le\frac{C}{|Q|}\int_Q\omega(x)\,dx$$

for all cubes Q and some $1<p_0<\infty$ (see [GR]). We take $s=rp_0/(r+p_0-1)$ in Lemma 2.12. Then $1<s<r$ and $p_0=s(r-1)/(r-s)$. Hence by Lemma 2.12 and Hölder's inequality,

$$|R_m(b;x,y)|\le C|x-y|^m\sum_{|\alpha|=m}\Big(\frac{1}{|\tilde Q(x,y)|}\int_{\tilde Q(x,y)}|D^\alpha b(z)|^s\,dz\Big)^{1/s}$$
$$\le C|x-y|^m\sum_{|\alpha|=m}|\tilde Q|^{-1/s}\Big(\int_{\tilde Q(x,y)}|D^\alpha b(z)|^s\omega(z)^{s(1-r)/r}\omega(z)^{s(r-1)/r}\,dz\Big)^{1/s}$$
$$\le C|x-y|^m\sum_{|\alpha|=m}|\tilde Q|^{-1/s}\Big(\int_{\tilde Q(x,y)}|D^\alpha b(z)|^r\omega(z)^{1-r}\,dz\Big)^{1/r}$$
$$\times\Big(\int_{\tilde Q(x,y)}\omega(z)^{s(r-1)/(r-s)}\,dz\Big)^{(r-s)/rs}$$

$$
\begin{aligned}
&\leq C|x-y|^m \sum_{|\alpha|=m} |\tilde{Q}|^{-1/s} \|D^\alpha b\|_{\mathrm{BMO}(\omega)} \omega(\tilde{Q})^{1/r} |\tilde{Q}|^{(r-s)/rs} \\
&\quad \times \bigg(\frac{1}{|\tilde{Q}(x,y)|} \int\limits_{\tilde{Q}(x,y)} \omega(z)^{p_0}\, dz \bigg)^{(r-s)/rs} \\
&\leq C|x-y|^m \sum_{|\alpha|=m} \|D^\alpha b\|_{\mathrm{BMO}(\omega)} |\tilde{Q}|^{-1/q} \omega(\tilde{Q})^{1/r} |\tilde{Q}|^{1/s-1/r} \\
&\quad \times \bigg(\frac{1}{|\tilde{Q}(x,y)|} \int\limits_{\tilde{Q}(x,y)} \omega(z)\, dz \bigg)^{(r-1)/r} \\
&\leq C|x-y|^m \sum_{|\alpha|=m} \|D^\alpha b\|_{\mathrm{BMO}(\omega)} |\tilde{Q}|^{-1/q} \omega(\tilde{Q})^{1/r} |\tilde{Q}|^{1/s-1/r} \omega(\tilde{Q})^{1-1/r} |\tilde{Q}|^{1/r-1} \\
&\leq C|x-y|^m \sum_{|\alpha|=m} \|D^\alpha b\|_{\mathrm{BMO}(\omega)} \frac{\omega(\tilde{Q})}{|\tilde{Q}|}.
\end{aligned}
$$

Thus, by Lemma 2.9, we obtain

$$
\begin{aligned}
I_1 &\leq \frac{C}{|Q|} \int\limits_Q \bigg| T\bigg(\frac{R_m(\tilde{b};x,\cdot)}{|x-\cdot|^m} f_1 \bigg) \bigg|\, dx \\
&\leq C \sum_{|\alpha|=m} \|D^\alpha b\|_{\mathrm{BMO}(\omega)} \frac{\omega(\tilde{Q})}{|\tilde{Q}|} \frac{1}{|Q|} \int\limits_Q |T(f)(y)| \omega(y) \nu(y)^{1/q} \omega(y)^{-1} \nu(y)^{-1/q}\, dy \\
&\leq C \sum_{|\alpha|=m} \|D^\alpha b\|_{\mathrm{BMO}(\omega)} \omega_{\tilde{Q}} \bigg(\frac{1}{|Q|} \int\limits_Q |\omega(y) T(f)(y)|^q \nu(y)\, dy \bigg)^{1/q} \\
&\quad \times \bigg(\frac{1}{|Q|} \int\limits_Q \omega(y)^{-q'} \nu(y)^{-q'/q}\, dy \bigg)^{1/q'} \\
&\leq C \sum_{|\alpha|=m} \|D^\alpha b\|_{\mathrm{BMO}(\omega)} \omega_Q (\nu_Q)^{1/q} \bigg(\frac{1}{\nu(Q)} \int\limits_Q |\omega(y) T(f)(y)|^q \nu(y)\, dy \bigg)^{1/q} \\
&\quad \times \bigg(\frac{1}{|Q|} \int\limits_Q \omega(y)^{-q'} \nu(y)^{-q'/q}\, dy \bigg)^{1/q'} \\
&\leq C \sum_{|\alpha|=m} \|D^\alpha b\|_{\mathrm{BMO}(\omega)} [M_\nu(|\omega T(f)|^q)(\tilde{x})]^{1/q} \\
&\quad \times \omega_Q (\nu_Q)^{1/q} \bigg(\frac{1}{|Q|} \int\limits_Q \omega(y)^{-q'} \nu(y)^{-q'/q}\, dy \bigg)^{1/q'} \\
&\leq C \sum_{|\alpha|=m} \|D^\alpha b\|_{\mathrm{BMO}(\omega)} [M_\nu(|\omega T(f)|^q)(\tilde{x})]^{1/q}.
\end{aligned}
$$

For I_2, we know $\nu^{-r/p} \in A_r$ by Lemma 2.6, thus

$$\left(\frac{1}{|Q|}\int_Q \nu(x)^{-r/p}\,dx\right)^{1/r} \le C\left(\frac{1}{|Q|}\int_Q \nu(x)^{r'/p}\,dx\right)^{-1/r'}.$$

Then, by the weak (L^1, L^1) boundedness of T (see Lemma 2.4) and Kolmogorov's inequality (see Lemma 2.3), we obtain, by Lemma 2.7,

$$\begin{aligned}
I_2 &\le C\sum_{|\alpha|=m}\left(\frac{1}{|Q|}\int_Q |T(D^\alpha\tilde{b}f_1)(x)|^\eta\,dx\right)^{1/\eta}\\
&\le C\sum_{|\alpha|=m}\frac{|Q|^{1/\eta-1}}{|Q|^{1/\eta}}\,\frac{\|T(D^\alpha\tilde{b}f_1)\chi_Q\|_{L^\eta}}{\|\chi_Q\|_{L^{\eta/(1-\eta)}}}\\
&\le C\sum_{|\alpha|=m}\frac{1}{|Q|}\|T(D^\alpha\tilde{b}f_1)\|_{WL^1} \le C\sum_{|\alpha|=m}\frac{1}{|Q|}\int_{\mathbb{R}^n}|D^\alpha\tilde{b}(x)f_1(x)|\,dx\\
&= C\sum_{|\alpha|=m}\frac{1}{|Q|}\int_{\tilde{Q}}|D^\alpha b(x)-(D^\alpha b)_{\tilde{Q}}|\mu(x)^{-1/p}|f(x)|\omega(x)\nu(x)^{1/p}\,dx\\
&\le C\sum_{|\alpha|=m}\left(\frac{1}{|\tilde{Q}|}\int_{\tilde{Q}}|(D^\alpha b(x)-(D^\alpha b)_{\tilde{Q}})|^r\mu(x)^{-r/p}\,dx\right)^{1/r}\\
&\quad\times\left(\frac{1}{|\tilde{Q}|}\int_{\tilde{Q}}|f(x)|^{r'}\omega(x)^{r'}\nu(x)^{r'/p}\,dx\right)^{1/r'}\\
&\le C\sum_{|\alpha|=m}\|D^\alpha b\|_{\mathrm{BMO}(\omega)}\left(\frac{1}{|\tilde{Q}|}\int_{\tilde{Q}}\nu(x)^{-r/p}\,dx\right)^{1/r}\\
&\quad\times\left(\frac{1}{|\tilde{Q}|}\int_{\tilde{Q}}|f(x)\omega(x)|^{r'}\nu(x)^{r'/p}\,dx\right)^{1/r'}\\
&\le C\sum_{|\alpha|=m}\|D^\alpha b\|_{\mathrm{BMO}(\omega)}\left(\frac{1}{|\tilde{Q}|}\int_{\tilde{Q}}\nu(x)^{r'/p}\,dx\right)^{-1/r'}\\
&\quad\times\left(\frac{1}{|\tilde{Q}|}\int_{\tilde{Q}}|f(x)\omega(x)|^{r'}\nu(x)^{r'/p}\,dx\right)^{1/r'}\\
&\le C\sum_{|\alpha|=m}\|D^\alpha b\|_{\mathrm{BMO}(\omega)}\left(\frac{1}{\nu(\tilde{Q})^{r'/p}}\int_{\tilde{Q}}|f(x)\omega(x)|^{r'}\nu(x)^{r'/p}\,dx\right)^{1/r'}\\
&\le C\sum_{|\alpha|=m}\|D^\alpha b\|_{\mathrm{BMO}(\omega)}[M_{\nu^{r'/p}}(|\omega f|^{r'})(\tilde{x})]^{1/r'}.
\end{aligned}$$

For I_3, noticing that if $x \in Q$ and $y \in 2^{k+1}Q \setminus 2^kQ$, then $|x-y| \geq 2^{k-1}t_Q$ and $h_{t_Q}(x,y) \leq Cs(2^{2(k-1)})/|Q|$, similarly to the proof for I_1 we get, by Lemma 2.9,

$$\begin{aligned}
I_3 &\leq \frac{C}{|Q|} \int_Q \left| A_{t_Q} T\left(\frac{R_m(\tilde{b}; x, \cdot)}{|x - \cdot|^m} f_1 \right) \right| dx \\
&\leq \sum_{|\alpha|=m} \|D^\alpha b\|_{\mathrm{BMO}(\omega)} \frac{\omega(\tilde{Q})}{|\tilde{Q}|} \frac{1}{|Q|} \int_Q \int_Q h_{t_Q}(x,y) |T(f_1)(y)| \, dy \, dx \\
&\quad + \sum_{|\alpha|=m} \|D^\alpha b\|_{\mathrm{BMO}(\omega)} \frac{\omega(\tilde{Q})}{|\tilde{Q}|} \sum_{k=0}^{\infty} \frac{1}{|Q|} \int_Q \int_{2^{k+1}Q \setminus 2^kQ} h_{t_Q}(x,y) \\
&\qquad \times |T(f_1)(y)| \, dy \, dx \\
&\leq C \sum_{|\alpha|=m} \|D^\alpha b\|_{\mathrm{BMO}(\omega)} \frac{\omega(\tilde{Q})}{|\tilde{Q}|} \frac{1}{|Q|} \int_Q |T(f)(y)| \\
&\qquad \times \omega(y) \nu(y)^{1/q} \omega(y)^{-1} \nu(y)^{-1/q} \, dy \\
&\quad + C \sum_{|\alpha|=m} \|D^\alpha b\|_{\mathrm{BMO}(\omega)} \frac{\omega(\tilde{Q})}{|\tilde{Q}|} \sum_{k=0}^{\infty} 2^{kn} s(2^{2(k-1)}) \\
&\qquad \times \frac{1}{|2^{k+1}Q|} \int_{2^{k+1}Q} |T(f)(y)| \omega(y) \nu(y)^{1/q} \omega(y)^{-1} \nu(y)^{-1/q} \, dy \\
&\leq C \sum_{|\alpha|=m} \|D^\alpha b\|_{\mathrm{BMO}(\omega)} \left(\frac{1}{\nu(Q)} \int_Q |\omega(y) T(f)(y)|^q \nu(y) \, dy \right)^{1/q} \\
&\quad \times \omega_Q (\nu_Q)^{1/q} \left(\frac{1}{|Q|} \int_Q \omega(y)^{-q'} \nu(y)^{-q'/q} \, dy \right)^{1/q'} \\
&\quad + C \sum_{|\alpha|=m} \|D^\alpha b\|_{\mathrm{BMO}(\omega)} \sum_{k=0}^{\infty} 2^{kn} s(2^{2(k-1)}) \\
&\qquad \times \left(\frac{1}{|2^{k+1}Q|} \int_{2^{k+1}Q} |\omega(y) T(f)(y)|^q \nu(y) \, dy \right)^{1/q} \\
&\qquad \times \omega_{2^{k+1}Q} (\nu_{2^{k+1}Q})^{1/q} \left(\frac{1}{|2^k\tilde{Q}|} \int_Q \omega(y)^{-q'} \nu(y)^{-q'/q} \, dy \right)^{1/q'} \\
&\leq C \sum_{|\alpha|=m} \|D^\alpha b\|_{\mathrm{BMO}(\omega)} [M_\nu (|\omega T(f)|^q)(\tilde{x})]^{1/q}.
\end{aligned}$$

For I_4, similarly to the proofs for I_2 and I_3, we get

$$\begin{aligned}
I_4 \leq{}& \sum_{|\alpha|=m}\left(\frac{1}{|Q|}\int_Q |T(D^\alpha\tilde{b}f_1)(y)|^\eta\,dy\right)^{1/\eta}\\
&+\sum_{|\alpha|=m}\sum_{k=0}^{\infty} 2^{kn}s(2^{2(k-1)})\left(\frac{1}{|2^{k+1}Q|}\int_{2^{k+1}Q}|T(D^\alpha\tilde{b}f_1)(y)|^\eta\,dy\right)^{1/\eta}\\
\leq{}& C\sum_{|\alpha|=m}\|D^\alpha b\|_{\mathrm{BMO}(\omega)}\left(\frac{1}{\nu(\tilde{Q})^{r'/p}}\int_{\tilde{Q}}|f(x)\omega(x)|^{r'}\nu(x)^{r'/p}\,dx\right)^{1/r'}\\
&+C\sum_{|\alpha|=m}\|D^\alpha b\|_{\mathrm{BMO}(\omega)}\sum_{k=0}^{\infty}2^{kn}s(2^{2(k-1)})\\
&\qquad\times\left(\frac{1}{\nu(2^{k+1}\tilde{Q})^{r'/p}}\int_{2^{k+1}\tilde{Q}}|f(x)\omega(x)|^{r'}\nu(x)^{r'/p}\,dx\right)^{1/r'}\\
\leq{}& C\sum_{|\alpha|=m}\|D^\alpha b\|_{\mathrm{BMO}(\omega)}[M_{\nu^{r'/p}}(|\omega f|^{r'})(\tilde{x})]^{1/r'}.
\end{aligned}$$

For I_5, noting that $|x-y|\approx|x_0-y|$ for $x\in Q$ and $y\in\mathbb{R}^n\setminus Q$, similarly to the proof for I_1 we have

$$|R_m(\tilde{b};x,y)|\leq C|x-y|^m\sum_{|\alpha|=m}\|D^\alpha b\|_{\mathrm{BMO}(\omega)}\frac{\omega(2^k\tilde{Q})}{|2^k\tilde{Q}|}.$$

Thus, by the conditions on K and K_t, we get

$$\begin{aligned}
&|T^{\tilde{b}}(f_2)(x)-A_{t_Q}T^{\tilde{b}}(f_2)(x_0)|\\
&\quad\leq\int_{\mathbb{R}^n}\frac{|R_m(\tilde{b};x,y)|}{|x-y|^m}|K(x,y)-K_t(x,y)|\,|f_2(y)|\,dy\\
&\qquad+\sum_{|\alpha|=m}\frac{1}{\alpha!}\int_{\mathbb{R}^n}\frac{|D^\alpha\tilde{b}_1(y)|\,|(x-y)^{\alpha_1}|}{|x-y|^m}|K(x,y)-K_t(x,y)|\,|f_2(y)|\,dy\\
&\quad\leq\sum_{k=0}^{\infty}\sum_{|\alpha|=m}\|D^\alpha b\|_{\mathrm{BMO}(\omega)}\frac{\omega(2^{k+1}\tilde{Q})}{|2^{k+1}\tilde{Q}|}\int_{2^{k+1}\tilde{Q}\setminus 2^k\tilde{Q}}\frac{d^\delta}{|x_0-y|^{n+\delta}}|f(y)|\,dy\\
&\qquad+C\sum_{|\alpha|=m}\sum_{k=0}^{\infty}\int_{2^{k+1}\tilde{Q}}|(D^\alpha b)_{2^{k+1}\tilde{Q}}-(D^\alpha b)_{\tilde{Q}}|\frac{d^\delta}{|x_0-y|^{n+\delta}}|f(y)|\,dy\\
&\qquad+C\sum_{|\alpha|=m}\sum_{k=0}^{\infty}\int_{2^{k+1}\tilde{Q}}|D^\alpha b(y)-(D^\alpha b)_{2^{k+1}\tilde{Q}}|\frac{d^\delta}{|x_0-y|^{n+\delta}}|f(y)|\,dy
\end{aligned}$$

$$\leq C \sum_{|\alpha|=m} \|D^\alpha b\|_{\mathrm{BMO}(\omega)} \sum_{k=1}^{\infty} k 2^{-k\delta} \bigg(\frac{1}{\nu(2^k\tilde{Q})} \int_{2^k\tilde{Q}} |\omega(y) f(y)|^q \nu(y)\, dy \bigg)^{1/q}$$

$$\times\, \omega_{2^k\tilde{Q}} (\nu_{2^k\tilde{Q}})^{1/q} \bigg(\frac{1}{|2^k\tilde{Q}|} \int_{2^k\tilde{Q}} \omega(y)^{-q'} \nu(y)^{-q'/q}\, dy \bigg)^{1/q'}$$

$$+ C \sum_{k=1}^{\infty} 2^{-k\delta} \bigg(\frac{1}{|2^k\tilde{Q}|} \int_{2^k\tilde{Q}} |D^\alpha b(y) - (D^\alpha b)_{2^k\tilde{Q}}|^r \mu(y)^{-r/p}\, dy \bigg)^{1/r}$$

$$\times \bigg(\frac{1}{|2^k\tilde{Q}|} \int_{2^k\tilde{Q}} |f(y)|^{r'} \omega(y)^{r'} \nu(y)^{r'/p}\, dy \bigg)^{1/r'}$$

$$\leq C \sum_{|\alpha|=m} \|D^\alpha b\|_{\mathrm{BMO}(\omega)} \big([M_\nu(|\omega f|^q)(\tilde{x})]^{1/q} + [M_{\nu^{r'/p}}(|\omega f|^{r'})(\tilde{x})]^{1/r'} \big).$$

Thus

$$I_3 \leq C \sum_{|\alpha|=m} \|D^\alpha b\|_{\mathrm{BMO}(\omega)} \big([M_{\nu^{r'/p}}(|\omega f|^{r'})(\tilde{x})]^{1/r'} + [M_\nu(|\omega f|^q)(\tilde{x})]^{1/q} \big).$$

This completes the proof of Theorem 3.1. ∎

THEOREM 3.2. *Let T be a singular integral operator with non-smooth kernel as in Definition 2.2, $\omega \in A_1$, $0 < \eta < 1$, $1 < r < \infty$, $0 < \beta < 1$ and $D^\alpha b \in \mathrm{Lip}_\beta(\omega)$ for all α with $|\alpha| = m$. Then there exists a constant $C > 0$ such that, for any $f \in C_0^\infty(\mathbb{R}^n)$ and $\tilde{x} \in \mathbb{R}^n$,*

$$M^{\#}_{A,\eta}(T^b(f))(\tilde{x}) \leq C \sum_{|\alpha|=m} \|D^\alpha b\|_{\mathrm{Lip}_\beta(\omega)} \omega(\tilde{x}) M_{\beta,r,\omega}(f)(\tilde{x}).$$

Proof. It suffices to prove that, for $f \in C_0^\infty(\mathbb{R}^n)$ and some constant C,

$$\bigg(\frac{1}{|Q|} \int_Q |T^b(f)(x) - A_{t_Q}(T^b(f))(x)|^\eta\, dx \bigg)^{1/\eta} \leq C \sum_{|\alpha|=m} \|D^\alpha b\|_{\mathrm{Lip}_\beta(\omega)} \omega(\tilde{x}) M_{\beta,r,\omega}(f)(\tilde{x}),$$

where $t_Q = d^2$ and d denotes the side length of Q. Fix a cube $Q = Q(x_0, d)$ and $\tilde{x} \in Q$. Similarly to the proof of Theorem 3.1, we have, for $f_1 = f\chi_{\tilde{Q}}$ and $f_2 = f\chi_{\mathbb{R}^n \setminus \tilde{Q}}$,

$$\bigg(\frac{1}{|Q|} \int_Q |T^b(f)(x) - A_{t_Q} T^b(f)(x)|^\eta\, dx \bigg)^{1/\eta} \leq \bigg(\frac{1}{|Q|} \int_Q \bigg| T\bigg(\frac{R_m(\tilde{b}; x, \cdot)}{|x - \cdot|^m} f_1 \bigg) \bigg|^\eta dx \bigg)^{1/\eta}$$

$$+ \bigg(\frac{1}{|Q|} \int_Q \bigg| T\bigg(\sum_{|\alpha|=m} \frac{1}{\alpha!} \frac{(x - \cdot)^\alpha D^\alpha \tilde{b}}{|x - \cdot|^m} f_1 \bigg) \bigg|^\eta dx \bigg)^{1/\eta}$$

$$+\left(\frac{1}{|Q|}\int_Q\left|A_{t_Q}T\left(\frac{R_m(\tilde{b};x,\cdot)}{|x-\cdot|^m}f_1\right)\right|^\eta dx\right)^{1/\eta}$$

$$+\left(\frac{1}{|Q|}\int_Q\left|A_{t_Q}T\left(\sum_{|\alpha|=m}\frac{1}{\alpha!}\,\frac{(x-\cdot)^\alpha D^\alpha\tilde{b}}{|x-\cdot|^m}f_1\right)\right|^\eta dx\right)^{1/\eta}$$

$$+\left(\frac{1}{|Q|}\int_Q|T^{\tilde{b}}(f_2)(x)-A_{t_Q}T^{\tilde{b}}(f_2)(x)|^\eta\,dx\right)^{1/\eta}$$

$$=J_1+J_2+J_3+J_4+J_5.$$

For J_1 and J_2, by using the same argument as in the proof of Theorem 3.1, we get

$$\begin{aligned}
&|R_m(\tilde{b};x,y)|\\
&\quad\le C|x-y|^m\sum_{|\alpha|=m}|\tilde{Q}|^{-1/q}\Big(\int_{\tilde{Q}(x,y)}|D^\alpha\tilde{b}(z)|^q\omega(z)^{q(1-r)/r}\omega(z)^{q(r-1)/r}\,dz\Big)^{1/q}\\
&\quad\le C|x-y|^m\sum_{|\alpha|=m}|\tilde{Q}|^{-1/q}\Big(\int_{\tilde{Q}(x,y)}|D^\alpha\tilde{b}(z)|^r\omega(z)^{1-r}\,dz\Big)^{1/r}\\
&\qquad\times\Big(\int_{\tilde{Q}(x,y)}\omega(z)^{q(r-1)/(r-q)}\,dz\Big)^{(r-q)/rq}\\
&\quad\le C|x-y|^m\sum_{|\alpha|=m}|\tilde{Q}|^{-1/q}\|D^\alpha b\|_{\mathrm{Lip}_\beta(w)}\omega(\tilde{Q})^{\beta/n+1/r}|\tilde{Q}|^{(r-q)/rq}\\
&\qquad\times\left(\frac{1}{|\tilde{Q}(x,y)|}\int_{\tilde{Q}(x,y)}\omega(z)^{p_0}\,dz\right)^{(r-q)/rq}\\
&\quad\le C|x-y|^m\sum_{|\alpha|=m}\|D^\alpha b\|_{\mathrm{Lip}_\beta(\omega)}|\tilde{Q}|^{-1/q}\omega(\tilde{Q})^{\beta/n+1/r}|\tilde{Q}|^{1/q-1/r}\\
&\qquad\times\left(\frac{1}{|\tilde{Q}(x,y)|}\int_{\tilde{Q}(x,y)}\omega(z)\,dz\right)^{(r-1)/r}\\
&\quad\le C|x-y|^m\sum_{|\alpha|=m}\|D^\alpha b\|_{\mathrm{Lip}_\beta(\omega)}|\tilde{Q}|^{-1/q}\omega(\tilde{Q})^{\beta/n+1/r}|\tilde{Q}|^{1/q-1/r}\\
&\qquad\qquad\times\omega(\tilde{Q})^{1-1/r}|\tilde{Q}|^{1/r-1}\\
&\quad\le C|x-y|^m\sum_{|\alpha|=m}\|D^\alpha b\|_{\mathrm{Lip}_\beta(\omega)}\frac{\omega(\tilde{Q})^{\beta/n+1}}{|\tilde{Q}|}\\
&\quad\le C|x-y|^m\sum_{|\alpha|=m}\|D^\alpha b\|_{\mathrm{Lip}_\beta(\omega)}\omega(\tilde{Q})^{\beta/n}\omega(\tilde{x}),
\end{aligned}$$

and thus

$$J_1 \le C \sum_{|\alpha|=m} \|D^\alpha b\|_{\mathrm{Lip}_\beta(\omega)} \omega(\tilde{Q})^{\beta/n} \omega(\tilde{x}) |Q|^{-1/s} \Big(\int_{\mathbb{R}^n} |f_1(x)|^s \, dx \Big)^{1/s}$$

$$\le C \sum_{|\alpha|=m} \|D^\alpha b\|_{\mathrm{Lip}_\beta(\omega)} \omega(\tilde{Q})^{\beta/n} \omega(\tilde{x}) |Q|^{-1/s} \Big(\int_{\tilde{Q}} |f(x)|^r \omega(x) \, dx \Big)^{1/r}$$
$$\times \Big(\int_{\tilde{Q}} \omega(x)^{-s/(r-s)} \, dx \Big)^{(r-s)/rs}$$

$$\le C \sum_{|\alpha|=m} \|D^\alpha b\|_{\mathrm{Lip}_\beta(\omega)} \omega(\tilde{x}) |\tilde{Q}|^{-1/s} \omega(\tilde{Q})^{1/r} \Big(\frac{1}{\omega(\tilde{Q})^{1-r\beta/n}} \int_{\tilde{Q}} |f(x)|^r \omega(x) \, dx \Big)^{1/r}$$
$$\times \Big(\frac{1}{|\tilde{Q}|} \int_{\tilde{Q}} \omega(x)^{-s/(r-s)} \, dx \Big)^{(r-s)/rs} \Big(\frac{1}{|\tilde{Q}|} \int_{\tilde{Q}} \omega(x) \, dx \Big)^{1/r} |\tilde{Q}|^{1/s} \omega(\tilde{Q})^{-1/r}$$

$$\le C \sum_{|\alpha|=m} \|D^\alpha b\|_{\mathrm{Lip}_\beta(\omega)} \omega(\tilde{x}) M_{\beta,r,\omega}(f)(\tilde{x}),$$

and

$$J_2 \le C \sum_{|\alpha|=m} \frac{1}{|Q|} \int_{\tilde{Q}} |D^\alpha b(x) - (D^\alpha b)_{\tilde{Q}}| \omega(x)^{-1/r} |f(x)| \omega(x)^{1/r} \, dx$$

$$\le C \sum_{|\alpha|=m} \frac{1}{|Q|} \Big(\int_{\tilde{Q}} |D^\alpha b(x) - (D^\alpha b)_{\tilde{Q}}|^{r'} \omega(x)^{1-r'} \, dx \Big)^{1/r'} \Big(\int_{\tilde{Q}} |f(x)|^r \omega(x) \, dx \Big)^{1/r}$$

$$\le C \sum_{|\alpha|=m} \frac{1}{|Q|} \|D^\alpha b\|_{\mathrm{Lip}_\beta(\omega)} \omega(\tilde{Q})^{\beta/n + 1/r'} \omega(\tilde{Q})^{1/r - \beta/n}$$
$$\times \Big(\frac{1}{\omega(\tilde{Q})^{1-r\beta/n}} \int_{\tilde{Q}} |f(x)|^r \omega(x) \, dx \Big)^{1/r}$$

$$\le C \sum_{|\alpha|=m} \|D^\alpha b\|_{\mathrm{Lip}_\beta(\omega)} \frac{\omega(\tilde{Q})}{|\tilde{Q}|} M_{\beta,r,\omega}(f)(\tilde{x})$$

$$\le C \sum_{|\alpha|=m} \|D^\alpha b\|_{\mathrm{Lip}_\beta(\omega)} \omega(\tilde{x}) M_{\beta,r,\omega}(f)(\tilde{x}).$$

For J_3 and J_4, by Lemmas 2.12 and 2.13, and similarly to the proof for J_1 and J_2, we get

$$J_3 + J_4 \le C \sum_{|\alpha|=m} \|D^\alpha b\|_{\mathrm{Lip}_\beta(\omega)} \omega(\tilde{x}) M_{\beta,r,\omega}(f)(\tilde{x}).$$

For J_5, by Lemma 2.12 and similarly to the proof of J_1, for $k \geq 0$,

$$|R_m(\tilde{b}; x, y)| \leq C|x-y|^m \sum_{|\alpha|=m} \|D^\alpha b\|_{\mathrm{Lip}_\beta(\omega)} \omega(2^k\tilde{Q})^{\beta/n}\omega(\tilde{x}),$$

thus

$$\begin{aligned}
&|T^{\tilde{b}}(f_2)(x) - A_{t_Q}T^{\tilde{b}}(f_2)(x_0)| \\
&\leq \int_{\mathbb{R}^n} \frac{|R_m(\tilde{b}; x, y)|}{|x-y|^m} |K(x,y) - K_t(x,y)|\,|f_2(y)|\,dy \\
&\quad + \sum_{|\alpha|=m} \frac{1}{\alpha!} \int_{\mathbb{R}^n} \frac{|D^\alpha \tilde{b}_1(y)||(x-y)^{\alpha_1}|}{|x-y|^m} |K(x,y) - K_t(x,y)|\,|f_2(y)|\,dy \\
&\leq \sum_{k=0}^{\infty} \sum_{|\alpha|=m} \|D^\alpha b\|_{\mathrm{Lip}_\beta(\omega)} \omega(\tilde{x})\omega(2^k\tilde{Q})^{\beta/n} \\
&\qquad \times \int_{2^{k+1}\tilde{Q}\setminus 2^k\tilde{Q}} \frac{d^\delta}{|x_0-y|^{n+\delta}} |f(y)|\omega(y)^{1/r}\omega(y)^{-1/r}\,dy \\
&\quad + C\sum_{|\alpha|=m}\sum_{k=0}^{\infty} \int_{2^{k+1}\tilde{Q}} |(D^\alpha b)_{2^{k+1}\tilde{Q}} - (D^\alpha b)_{\tilde{Q}}| \frac{d^\delta}{|x_0-y|^{n+\delta}} |f(y)| \\
&\qquad \times \omega(y)^{1/r}\omega(y)^{-1/r}\,dy \\
&\quad + C\sum_{|\alpha|=m}\sum_{k=0}^{\infty} \int_{2^{k+1}\tilde{Q}} |D^\alpha b(y) - (D^\alpha b)_{2^{k+1}\tilde{Q}}| \frac{d^\delta}{|x_0-y|^{n+\delta}} |f(y)| \\
&\qquad \times \omega(y)^{1/r}\omega(y)^{-1/r}\,dy \\
&\leq C\sum_{|\alpha|=m} \|D^\alpha b\|_{\mathrm{Lip}_\beta(\omega)} \omega(\tilde{x}) \sum_{k=1}^{\infty} k\frac{d^\delta}{(2^kd)^{n+\delta}} \omega(2^k\tilde{Q})^{\beta/n} \\
&\qquad \times \Big(\int_{2^k\tilde{Q}} |f(y)|^r\omega(y)\,dx\Big)^{1/r} \Big(\frac{1}{|2^k\tilde{Q}|}\int_{2^k\tilde{Q}} \omega(y)^{-1/(r-1)}\,dy\Big)^{(r-1)/r} \\
&\qquad \times \Big(\frac{1}{|2^k\tilde{Q}|}\int_{2^k\tilde{Q}} \omega(y)\,dy\Big)^{1/r} |2^k\tilde{Q}|\omega(2^k\tilde{Q})^{-1/r} \\
&\quad + C\sum_{|\alpha|=m}\sum_{k=1}^{\infty} \frac{d^\delta}{(2^kd)^{n+\delta}} \Big(\int_{2^k\tilde{Q}} |D^\alpha b(y) - (D^\alpha b)_{2^k\tilde{Q}}|^{r'}\omega(y)^{1-r'}\,dy\Big)^{1/r'} \\
&\qquad \times \Big(\int_{2^k\tilde{Q}} |f(y)|^r\omega(y)\,dy\Big)^{1/r}
\end{aligned}$$

$$\leq C \sum_{|\alpha|=m} \|D^\alpha b\|_{\mathrm{Lip}_\beta(\omega)} \omega(\tilde{x}) \sum_{k=1}^\infty k 2^{-k\delta} \bigg(\frac{1}{\omega(2^k \tilde{Q})^{1-r\beta/n}} \int_{2^k\tilde{Q}} |f(y)|^r \omega(y)\,dx \bigg)^{1/r}$$
$$+ C \sum_{|\alpha|=m} \|D^\alpha b\|_{\mathrm{Lip}_\beta(\omega)} \sum_{k=1}^\infty 2^{-k\delta} \frac{\omega(2^k\tilde{Q})}{|2^k\tilde{Q}|} \times \bigg(\frac{1}{\omega(2^k \tilde{Q})^{1-r\beta/n}} \int_{2^k\tilde{Q}} |f(y)|^r \omega(y)\,dx \bigg)^{1/r}$$
$$\leq C \sum_{|\alpha|=m} \|D^\alpha b\|_{\mathrm{Lip}_\beta(\omega)} \omega(\tilde{x}) M_{\beta,r,\omega}(f)(\tilde{x})$$

and

$$J_5 \leq C \sum_{|\alpha|=m} \|D^\alpha b\|_{\mathrm{Lip}_\beta(\omega)} \omega(\tilde{x}) M_{\beta,r,\omega}(f)(\tilde{x}).$$

This completes the proof of Theorem 3.2. ■

THEOREM 3.3. *Let T be a singular integral operator with non-smooth kernel as in Definition 2.2, $1 < p < \infty$, $\mu, \nu \in A_p$, $\omega = (\mu\nu^{-1})^{1/p}$ and $D^\alpha b \in \mathrm{BMO}(\omega)$ for all α with $|\alpha| = m$. Then T^b is bounded from $L^p(\mathbb{R}^n, \mu)$ to $L^p(\mathbb{R}^n, \nu)$.*

Proof. Notice $\nu^{r'/p} \in A_{r'+1-r'/p} \subset A_p$ and $\nu(x)\,dx \in A_{p/r'}(\nu(x)^{r'/p}\,dx)$ by Lemma 2.8. Thus, by Theorem 3.1, Lemmas 2.4 and 2.11, we get

$$\int_{\mathbb{R}^n} |T^b(f)(x)|^p \nu(x)\,dx \leq \int_{\mathbb{R}^n} |M_\eta(T^b(f))(x)|^p \nu(x)\,dx$$
$$\leq C \int_{\mathbb{R}^n} |M^{\#}_{A,\eta}(T^b(f))(x)|^p \nu(x)\,dx$$
$$\leq C \sum_{|\alpha|=m} \|D^\alpha b\|_{\mathrm{BMO}(\omega)} \int_{\mathbb{R}^n} \big([M_\nu(|\omega T(f)|^q)(x)]^{p/q} + [M_{\nu^{r'/p}}(|\omega f|^{r'})(x)]^{p/r'} + [M_\nu(|\omega f|^q)(x)]^{p/q} \big) \nu(x)\,dx$$
$$\leq C \sum_{|\alpha|=m} \|D^\alpha b\|_{\mathrm{BMO}(\omega)} \Big(\int_{\mathbb{R}^n} |\omega(x) f(x)|^p \nu(x)\,dx + \int_{\mathbb{R}^n} |\omega(x) T(f)(x)|^p \nu(x)\,dx \Big)$$
$$= C \sum_{|\alpha|=m} \|D^\alpha b\|_{\mathrm{BMO}(\omega)} \Big(\int_{\mathbb{R}^n} |f(x)|^p \mu(x)\,dx + \int_{\mathbb{R}^n} |T(f)(x)|^p \mu(x)\,dx \Big)$$
$$\leq C \sum_{|\alpha|=m} \|D^\alpha b\|_{\mathrm{BMO}(\omega)} \int_{\mathbb{R}^n} |f(x)|^p \mu(x)\,dx. \ ■$$

THEOREM 3.4. *Let T be a singular integral operator with non-smooth kernel as in Definition 2.2, $\omega \in A_1$, $0 < \beta < 1$, $1 < p < n/\beta$, $1/q =$*

$1/p - \beta/n$ and $D^\alpha b \in \mathrm{Lip}_\beta(\omega)$ for all α with $|\alpha| = m$. Then T^b is bounded from $L^p(\mathbb{R}^n, \omega)$ to $L^q(\mathbb{R}^n, \omega^{1-q})$.

Proof. Choose $1 < r < p$ in Theorem 3.2 and notice $\omega^{1-q} \in A_1$. Then we have, by Lemmas 2.10 and 2.11,

$$\begin{aligned}\|T^b(f)\|_{L^q(\omega^{1-q})} &\le \|M_\eta(T^b(f))\|_{L^q(\omega^{1-q})} \le C\|M^{\#}_{A,\eta}(T^b(f))\|_{L^q(\omega^{1-q})}\\ &\le C\sum_{|\alpha|=m} \|D^\alpha b\|_{\mathrm{Lip}_\beta(\omega)}\|\omega M_{\beta,r,\omega}(f)\|_{L^q(\omega^{1-q})}\\ &= C\sum_{|\alpha|=m} \|D^\alpha b\|_{\mathrm{Lip}_\beta(\omega)}\|M_{\beta,r,\omega}(f)\|_{L^q(\omega)}\\ &\le C\sum_{|\alpha|=m} \|D^\alpha b\|_{\mathrm{Lip}_\beta(\omega)}\|f\|_{L^p(\omega)}. \ \blacksquare\end{aligned}$$

COROLLARY 3.5. *Let $[b,T](f) = bT(f) - T(bf)$ be the commutator generated by a singular integral operator T as in Definition 2.2 and b. Then the conclusion of Theorems 3.1–3.4 hold for $[b,T]$ in place of T^b.*

4. Applications. In this section we shall apply the theorems of this paper to the holomorphic functional calculus of linear elliptic operators. First, we review some definitions regarding holomorphic functional calculus (see [DM], [MA]). Given $0 \le \theta < \pi$, define

$$S_\theta = \{z \in \mathbb{C} : |\arg(z)| \le \theta\} \cup \{0\}$$

and denote its interior by S^0_θ. Set $\tilde{S}_\theta = S_\theta \setminus \{0\}$. A closed linear elliptic operator L on some Banach space E is said to be of *type θ* if its spectrum $\sigma(L)$ is contained in S_θ and for every $\nu \in (\theta, \pi]$, there exists a constant C_ν such that

$$|\eta|\,\|(\eta I - L)^{-1}\| \le C_\nu, \qquad \eta \notin \tilde{S}_\theta.$$

By the Hille–Yosida theorem, such an operator with $\theta < \pi/2$ is the generator of a bounded holomorphic semigroup e^{-zL} in the sector S^0_μ with $\mu = \pi/2 - \theta$. For $\nu \in (0, \pi]$, let

$$H_\infty(S^0_\mu) = \{f : S^0_\theta \to \mathbb{C} : f \text{ is holomorphic and } \|f\|_{L^\infty} < \infty\},$$

where $\|f\|_{L^\infty} = \sup\{|f(z)| : z \in S^0_\mu\}$. Set

$$\Psi(S^0_\mu) = \left\{g \in H_\infty(S^0_\mu) : \exists s, c > 0 \text{ such that } |g(z)| \le c\frac{|z|^s}{1 + |z|^{2s}}\right\}.$$

If L is of type θ and $g \in H_\infty(S^0_\mu)$, we define an operator $g(L) \in L(E)$ by

$$g(L) = -(2\pi i)^{-1}\int_\Gamma (\eta I - L)^{-1} g(\eta)\, d\eta,$$

where Γ is the contour $\{\xi = re^{\pm i\phi} : r \geq 0\}$ parameterized clockwise around S_θ with $\theta < \phi < \mu$. If, in addition, L is one-one and has dense range, then, for $f \in H_\infty(S^0_\mu)$,

$$f(L) = [h(L)]^{-1}(fh)(L),$$

where $h(z) = z(1+z)^{-2}$. By [DM], [MA], $f(L)$ is a well-defined linear operator in E for $f \in \Psi(S^0_\mu)$. The definition of $f(L)$ can even be extended to unbounded holomorphic functions f (see [DM], [MA] for details). L is said to have a *bounded holomorphic functional calculus* on the sector S_μ if

$$\|g(L)\| \leq N\|g\|_{L^\infty}$$

for some $N > 0$ and for all $g \in H_\infty(S^0_\mu)$.

Now, let L be a linear operator on $L^2(\mathbb{R}^n)$ with $\theta < \pi/2$ so that $-L$ generates a holomorphic semigroup e^{-zL}, $0 \leq |\arg(z)| < \pi/2 - \theta$. Applying [DM, Theorem 6], [MA, Theorem 7.2] and Theorems 3.1–3.4, we get

Corollary 4.1. *Assume the following conditions are satisfied:*

(i) *The holomorphic semigroup e^{-zL}, $0 \leq |\arg(z)| < \pi/2 - \theta$, is represented by kernels $a_z(x,y)$ which satisfy, for all $\nu > \theta$, an upper bound*

$$|a_z(x,y)| \leq c_\nu h_{|z|}(x,y)$$

for $x, y \in \mathbb{R}^n$ and $0 \leq |\arg(z)| < \pi/2 - \theta$, where $h_t(x,y) = Ct^{-n/2}s(|x-y|^2/t)$ and s is a positive, bounded and decreasing function satisfying

$$\lim_{r\to\infty} r^{n+\epsilon}s(r^2) = 0 \quad \textit{for some } \epsilon > 0.$$

(ii) *The operator L has a bounded holomorphic functional calculus in $L^2(\mathbb{R}^n)$, that is, for all $\nu > \theta$ and $g \in H_\infty(S^0_\mu)$, the operator $g(L)$ satisfies*

$$\|g(L)(f)\|_{L^2} \leq c_\nu \|g\|_{L^\infty}\|f\|_{L^2}.$$

We relate to the operator $g(L)$ and b the linear operator defined by

$$g(L)^b(f)(x) = \int_{\mathbb{R}^n} \frac{R_{m+1}(b;x,y)}{|x-y|^m} K(x,y)f(y)\,dy.$$

Then the conclusion of Theorems 1–4 *holds for the linear operator $g(L)^b$ in place of T^b.*

In fact, it suffices to justify that the operator $g(L)$ satisfies the conditions of Definition 2.2. From [MA], for such an operator, taking the approximation to the identity $A_t = D_t = e^{-tL}$ yields $K_t = k_t$, and using the assumption (i), it was proved in [MA, Theorem 6] that the conditions of Definition 2.2 are satisfied. Thus the operator $g(L)$ satisfies the conditions in the corresponding theorem by Theorem 7.3 of [MA].

Acknowledgments. The author would like to express his gratitude to the referee for his comments and suggestions.

REFERENCES

[B] S. Bloom, *A commutator theorem and weighted BMO*, Trans. Amer. Math. Soc. 292 (1985), 103–122.

[C] S. Chanillo, *A note on commutators*, Indiana Univ. Math. J. 31 (1982), 7–16.

[CG] J. Cohen and J. Gosselin, *A BMO estimate for multilinear singular integral operators*, Illinois J. Math. 30 (1986), 445–465.

[CRW] R. R. Coifman, R. Rochberg and G. Weiss, *Factorization theorems for Hardy spaces in several variables*, Ann. of Math. 103 (1976), 611–635.

[DEY] D. G. Deng and L. X. Yan, *Commutators of singular integral operators with non-smooth kernels*, Acta Math. Sci. 25 (2005), 137–144.

[DL] Y. Ding and S. Z. Lu, *Weighted boundedness for a class of rough multilinear operators*, Acta Math. Sinica 17 (2001), 517–526.

[DM] X. T. Duong and A. McIntosh, *Singular integral operators with non-smooth kernels on irregular domains*, Rev. Mat. Iberoamer. 15 (1999), 233–265.

[G] J. García-Cuerva, *Weighted H^p spaces*, Dissertationes Math. 162 (1979).

[GR] J. García-Cuerva and J. L. Rubio de Francia, *Weighted Norm Inequalities and Related Topics*, North-Holland Math. Stud. 16, North-Holland, Amsterdam, 1985.

[HEW] Y. X. He and Y. S. Wang, *Commutators of Marcinkiewicz integrals and weighted BMO*, Acta Math. Sinica (Chinese Ser.) 54 (2011), 513–520.

[HG] B. Hu and J. Gu, *Necessary and sufficient conditions for boundedness of some commutators with weighted Lipschitz spaces*, J. Math. Anal. Appl. 340 (2008), 598–605.

[J] S. Janson, *Mean oscillation and commutators of singular integral operators*, Ark. Mat. 16 (1978), 263–270.

[LIU1] L. Z. Liu, *Sharp function boundedness for vector-valued multilinear singular integral operators with non-smooth kernels*, J. Contemp. Math. Anal. 45 (2010), 185–196.

[LIU2] L. Z. Liu, *Multilinear singular integral operators on Triebel–Lizorkin and Lebesgue spaces*, Bull. Malaysian Math. Sci. Soc. 35 (2012), 1075–1086.

[MA] J. M. Martell, *Sharp maximal functions associated with approximations of the identity in spaces of homogeneous type and applications*, Studia Math. 161 (2004), 113–145.

[PA] M. Paluszyński, *Characterization of the Besov spaces via the commutator operator of Coifman, Rochberg and Weiss*, Indiana Univ. Math. J. 44 (1995), 1–17.

[PE] C. Pérez, *Endpoint estimate for commutators of singular integral operators*, J. Funct. Anal. 128 (1995), 163–185.

[PT] C. Pérez and R. Trujillo-González, *Sharp weighted estimates for multilinear commutators*, J. London Math. Soc. 65 (2002), 672–692.

[S] E. M. Stein, *Harmonic Analysis: Real-Variable Methods, Orthogonality, and Oscillatory Integrals*, Princeton Univ. Press, Princeton, NJ, 1993.

[ZL] X. S. Zhou and L. Z. Liu, *Weighted boundedness for multilinear singular integral operators with non-smooth kernels on Morrey space*, Rev. Real Acad. Ciencias Exactas Ser. A Mat. 104 (2010), 115–127.

Dazhao Chen
Department of Science and Information Science
Shaoyang University
Shaoyang, Hunan 422000, P.R. China
E-mail: chendazhao27@sina.com

Received 4 August 2013;
revised 22 January 2014 (5999)