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A CHARACTERIZATION OF SOME SUBSETS OF S-ESSENTIAL SPECTRA OF A MULTIVALUED LINEAR OPERATOR

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Abstract. We characterize some S-essential spectra of a closed linear relation in terms of certain linear relations of semi-Fredholm type.

1. Introduction. Let S and T be bounded operators. Then the map $P(\lambda) := \lambda S - T$, $\lambda \in \mathbb{C}$, is called a *linear bundle*. It is known that many problems of mathematical physics (for example, quantum theory, transport theory etc.) reduce to the study of essential spectra of $\lambda S - T$. To make such investigation, the spectral theory of Fredholm type linear relations can play an important role since this class of operators is unstable under taking closure, inverse and conjugate. This is not the case if we consider the more general case of multivalued linear operators. Therefore, it seems interesting to study S-essential spectra in the context of multivalued linear operators.

Let X, Y and Z be Banach spaces and let $B(X,Y)$ denote the classes of all bounded operators from X into Y. A *linear relation* or a *multivalued linear operator* $T : X \to Y$ is a mapping from the subspace $\mathcal{D}(T)$ of X, called the *domain* of T, into the collection of nonempty subsets of Y such that $T(\alpha_1 x_1 + \alpha_2 x_2) = \alpha_1 T(x_1) + \alpha_2 T(x_2)$ for all nonzero scalars α_1, α_2 and $x_1, x_2 \in \mathcal{D}(T)$. If T maps all points of its domain to singletons, then T is said to be *single valued* or simply an *operator*. We denote the class of all linear relations from X to Y be $L\mathcal{R}(X,Y)$ and we write $L\mathcal{R}(X) := L\mathcal{R}(X,X)$.

A linear relation $T \in L\mathcal{R}(X,Y)$ is uniquely determined by its graph, $G(T)$, which is defined by

$$G(T) = \{(x,y) \in X \times Y : x \in \mathcal{D}(T) \text{ and } y \in Tx\},$$

so that we can identify T with $G(T)$. The inverse of T is the linear relation T^{-1} defined by $G(T^{-1}) = \{(y,x) : (x,y) \in G(T)\}$. We note that T is single valued if and only if the subspace $T(0)$ is $\{0\}$.

The subspace $T^{-1}(0)$ denoted by $N(T)$ is called the *null space* of T, and T is called *injective* if $N(T) = \{0\}$. The *range* of T is the subspace $T(\mathcal{D}(T))$,

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and T is said to be *surjective* if $R(T) = Y$. When T is injective and surjective we say that it is *bijective*.

For $T, S \in L\mathcal{R}(X, Y)$, we define $T + S$ by

$$G(T + S) := \{(x, y) : y = u + v \text{ with } (x, u) \in G(T),\ (x, v) \in G(S)\}.$$

For $S \in L\mathcal{R}(X, Y)$ and $R \in L\mathcal{R}(Y, Z)$, the product RS is defined by

$$G(RS) := \{(x, z) \in X \times Z : (x, y) \in G(S),\ (y, z) \in G(R) \text{ for some } y \in Y\}.$$

Let M be a subspace of X. Then $T_{|M \cap \mathcal{D}(T)} := T_{|M}$ is defined by $G(T_{|M}) := \{(x, y) \in G(T) : x \in M\}$. We write $S \subset T$ if $G(S) \subset G(T)$ and we say that T is an *extension* of S if $T|_{\mathcal{D}(S)} = S$.

If $\lambda \in \mathbb{K}$ then λT is defined by $G(\lambda T) := \{(x, \lambda y) : (x, y) \in G(T)\}$, and if $X = Y$ the linear relation $\lambda - T$ is given by $G(\lambda - T) := \{(x, \lambda x - y) : (x, y) \in G(T)\}$.

The quotient map from Y onto $Y/\overline{T(0)}$ is denoted by Q_T. It is easy to see that $Q_T T$ is single valued so that we can define $\|Tx\| := \|Q_T Tx\|$ for $x \in \mathcal{D}(T)$ and $\|T\| := \|Q_T T\|$. We say that T is *continuous* if $\|T\| < \infty$, *bounded* if it is continuous with $\mathcal{D}(T) = X$, and *open* if T^{-1} is continuous, or equivalently if the minimum modulus $\gamma(T)$ is a positive number, where $\gamma(T) = \sup\{\lambda \geq 0 : \lambda d(x, N(T)) \leq \|Tx\|,\ x \in \mathcal{D}(T)\}$.

We say that T is *closed* if its graph is a closed subspace, and *closable* if $\overline{T}$ is an extension of T where the *closure* $\overline{T}$ of T is defined by $G(\overline{T}) := \overline{G(T)}$. We denote the class of all bounded linear relations from X to Y by $B\mathcal{R}(X, Y)$; $C\mathcal{R}(X, Y)$ is the set of all closed linear relations from X to Y; and $\mathcal{KR}(X, Y)$ will denote the class of all compact linear relations from X to Y, where $T \in L\mathcal{R}(X, Y)$ is called *compact* if $\overline{Q_T T B_X}$ is compact; here B_X is the unit ball of X.

If M and N are subspaces of X and of the dual space X' respectively, then

$$M^{\perp} := \{x' \in X' : x'(x) = 0 \text{ for all } x \in M\},$$
$$N^{\top} := \{x \in X : x'(x) = 0 \text{ for all } x' \in N\}.$$

The *conjugate* of $T \in L\mathcal{R}(X, Y)$ is the linear relation T' defined by

$$G(T') := G(-T^{-1})^{\perp} \subset Y' \times X',$$

so that

$$(y', x') \in G(T') \quad \text{if and only if} \quad y'(y) = x'(x) \text{ for all } (x, y) \in G(T).$$

Let $T \in L\mathcal{R}(X, Y)$ be closed. We say that T is *bounded below* if it is injective and open; *left invertible* if T is injective and $R(T)$ is topologically complemented in Y; and *right invertible* if it is surjective and $N(T)$ is topologically complemented in X.

Let X be a complex Banach space and let $S, T \in C\mathcal{R}(X)$ be such that S is continuous, T is closed with $S(0) \subset T(0)$ and $\mathcal{D}(T) \subset \mathcal{D}(S)$. We define the S-*resolvent set* of T by

$$\rho_S(T) := \{\lambda \in \mathbb{C} : \lambda S - T \text{ is bijective}\}$$

and the S-*spectrum* of T by $\sigma_S(T) := \mathbb{C} \setminus \rho_S(T)$. We define the following S-essential spectra of T:

$$\begin{aligned}
\sigma_{\mathrm{e},S}(T) &:= \bigcap_{K \in \mathfrak{K}_T(X)} \sigma_S(T+K),\\
\sigma_{\mathrm{eap},S}(T) &:= \bigcap_{K \in \mathfrak{K}_T(X)} \sigma_{\mathrm{ap},S}(T+K),\\
\sigma_{\mathrm{e}\delta,S}(T) &:= \bigcap_{K \in \mathfrak{K}_T(X)} \sigma_{\delta,S}(T+K),\\
\sigma_{\mathrm{el},S}(T) &:= \bigcap_{K \in \mathfrak{K}_T(X)} \sigma_{\mathrm{l},S}(T+K),\\
\sigma_{\mathrm{er},S}(T) &:= \bigcap_{K \in \mathfrak{K}_T(X)} \sigma_{\mathrm{r},S}(T+K),
\end{aligned}$$

where

$$\begin{aligned}
\mathfrak{K}_T(X) &:= \{K \in \mathcal{KR}(X) : \mathcal{D}(T) \subset \mathcal{D}(K),\ K(0) \subset T(0)\},\\
\sigma_{\mathrm{ap},S}(T) &:= \{\lambda \in \mathbb{C} : \lambda S - T \text{ is not bounded below}\},\\
\sigma_{\delta,S}(T) &:= \{\lambda \in \mathbb{C} : \lambda S - T \text{ is not surjective}\},\\
\sigma_{\mathrm{l},S}(T) &:= \{\lambda \in \mathbb{C} : \lambda S - T \text{ is not left invertible}\},\\
\sigma_{\mathrm{r},S}(T) &:= \{\lambda \in \mathbb{C} : \lambda S - T \text{ is not right invertible}\}.
\end{aligned}$$

Note that if $S = I$ (the identity operator on X) and T is a bounded operator on X, then $\sigma_{\mathrm{e},I}(T)$ is the Schechter essential spectrum of T (see for instance [8, 9, 13]), $\sigma_{\mathrm{eap},I}(T)$ was introduced by V. Rakočević [11] and is called the essential approximate point spectrum, and $\sigma_{\mathrm{e}\delta,I}(T)$ is the essential defect spectrum of T, introduced by C. Schmoeger [14]. The classification of Fredholm operators is connected to studies of spectra of operators. H. Weyl [15] showed that the limit points of the spectrum (i.e., all points of the spectrum, except isolated eigenvalues of finite multiplicity) of a bounded symmetric transformation on a Hilbert space are invariant under perturbation by compact symmetric operators. In the modern theory of linear operators, essential spectra are subsets of the spectrum which are stable under perturbation by small and relatively compact operators. When S and T are bounded operators, the subsets $\sigma_{\mathrm{e},S}(T)$, $\sigma_{\mathrm{e}\delta,S}(T)$, $\sigma_{\mathrm{el},S}(T)$ and $\sigma_{\mathrm{er},S}(T)$ are introduced in [1], where the authors characterize such S-essential spectra in terms of semi-Fredholm operators. Recently, T. Álvarez, A. Ammar and A. Jeribi [5] have given a detailed treatment of some subsets of essential spectra of a closed multivalued linear operator.

For $T \in L\mathcal{R}(X,Y)$ we write

$$\alpha(T) := \dim N(T), \quad \beta(T) := \dim Y/R(T), \quad \overline{\beta}(T) := \dim Y/\overline{R(T)},$$

and the *index* of T is the quantity $i(T) := \alpha(T) - \beta(T)$ provided $\alpha(T)$ and $\beta(T)$ are not both infinite. The sets of *upper semi-Fredholm linear relations* and of *lower semi-Fredholm linear relations* from X into Y are defined respectively by

$$\Phi_+(X,Y) := \{T \in C\mathcal{R}(X,Y) : \alpha(T) < \infty \text{ and } R(T) \text{ is closed in } Y\},$$
$$\Phi_-(X,Y) := \{T \in C\mathcal{R}(X,Y) : \beta(T) < \infty \text{ and } R(T) \text{ is closed in } Y\}.$$

Moreover, we set $\Phi_\pm(X,Y) := \Phi_+(X,Y) \cup \Phi_-(X,Y)$ and call $\Phi(X,Y) := \Phi_+(X,Y) \cap \Phi_-(X,Y)$ the set of *Fredholm linear relations* from X into Y. The sets of *left Fredholm linear relations* and of *right Fredholm linear relations* from X into Y are defined respectively by

$$\Phi_{\mathrm{l}}(X,Y) := \{A \in \Phi_+(X,Y) : R(A) \text{ is topologically complemented in } Y\},$$
$$\Phi_{\mathrm{r}}(X,Y) := \{A \in \Phi_-(X,Y) : N(A) \text{ is topologically complemented in } X\}.$$

The purpose of this paper is to extend the results of [1] mentioned above to the case of multivalued linear operators. In Section 2 we gather some notations and results concerning Φ_+ and Φ_- linear relations, connected with Section 3. The main results of Section 3 are Theorems 3.1–3.3. Theorem 3.1 gives a characterization of $\sigma_{\mathrm{eap},S}(T)$, $\sigma_{\mathrm{e}\delta,S}(T)$ and $\sigma_{\mathrm{e},S}(T)$ by means of Φ_+, Φ_- and $\Phi_+ \cap \Phi_-$ linear relations respectively. Theorems 3.2 and 3.3 characterize the subsets $\sigma_{\mathrm{el},S}(T)$ and $\sigma_{\mathrm{er},S}(T)$ in the terms of Φ_{l} and Φ_{r} linear relations, respectively. Finally, in Section 4, we apply the results obtained in the previous section to investigate S-essential spectra of some class of multivalued linear operators. More precisely, we utilize some spectral results for a one-dimensional transport single valued operator with vacuum boundary conditions (see for instance [2, 8, 9]) to describe S-essential spectra of a certain matrix of linear relations.

2. Auxiliary results. Let T be a closed linear relation on a Banach space X. For $x \in \mathcal{D}(T)$ the graph norm of x is defined by

$$\|x\|_T := \|x\| + \|Tx\|.$$

It follows from the closedness of T that $\mathcal{D}(T)$ endowed with the norm $\|\cdot\|_T$ is a Banach space. Let $X_T := (\mathcal{D}(T), \|\cdot\|_T)$.

Definition 2.1. Let X be a Banach space and $T \in L\mathcal{R}(X)$. The graph operator $G_T \in L\mathcal{R}(X_T, X)$ is defined by

$$\mathcal{D}(G_T) = X_T \quad \text{and} \quad G_T x = x \quad \text{for } x \in X_T.$$

LEMMA 2.1 ([6, Proposition II.5.3], [3, Lemma 5.3]). *Let $T \in L\mathcal{R}(X,Y)$. Then:*

(i) *T^{-1} is closed if and only T is closed if and only if $Q_T T$ is closed and $T(0)$ is closed. In particular, $N(T)$ is closed if T is closed.*

(ii) *Assume that $T \in L\mathcal{R}(X,Y)$ is closed. Then:*

  (ii$_1$) *$R(T)$ is closed if and only if so is $R(Q_T T)$.*

  (ii$_2$) *$T \in \Phi_+(X,Y)$ if and only if $Q_T T \in \Phi_+(X, Y/T(0))$ if and only if $T' \in \Phi_-(Y', X')$. In that case $i(T) = -i(T')$.*

  (ii$_3$) *$T \in \Phi_-(X,Y)$ if and only if $Q_T T \in \Phi_-(X, Y/T(0))$ if and only if $T' \in \Phi_+(Y', X')$. In that case $i(T) = -i(T')$.*

LEMMA 2.2. *Let $T \in L\mathcal{R}(X,Y)$ be closed and let $S \in L\mathcal{R}(X,Y)$ be continuous such that $S(0) \subset T(0)$ and $\mathcal{D}(T) \subset \mathcal{D}(S)$. Then:*

(i) *S' is continuous, $S'(0) \subset T'(0)$ and $\mathcal{D}(T') \subset \mathcal{D}(S')$.*

(ii) *$T + S$ is closed.*

(iii) *$T + S - S = T$.*

(iv) *If $T \in \Phi_+(X,Y)$ and $S \in \mathcal{KR}(X,Y)$ then $T+S \in \Phi_+(X,Y)$ and $i(T+S) = i(T)$.*

(v) *If $T \in \Phi_-(X,Y)$ and $S \in \mathcal{KR}(X,Y)$ then $T+S \in \Phi_-(X,Y)$ and $i(T+S) = i(T)$.*

Proof. (i) By [6, Corollary III.1.13], S' is continuous. Further, $S'(0) = \mathcal{D}(S)^\perp \subset \mathcal{D}(T)^\perp = T'(0)$ [6, Proposition III.1.4] and $\mathcal{D}(T') \subset T(0)^\perp \subset S(0)^\perp \subset \mathcal{D}(S')$ [6, Propositions III.1.4 and III.4.6]. Hence (i) holds.

(ii) is proved in [4, Lemma 14].

(iii) Since $S(0) \subset T(0)$ we have $(T+S-S)(0) = T(0)$, and since $\mathcal{D}(T) \subset \mathcal{D}(S)$ we have $\mathcal{D}(T-S+S) = \mathcal{D}(T)$. Further, for $y \in (T-S+S)x$, we have $x \in \mathcal{D}(T)$ and $y \in Tx + S(0) \subset Tx + T(0) = T(x)$, so that $T+S-S \subset T$. Now, the use of [6, Exercise V.2.14] allows us to conclude that $T+S-S = T$.

(iv) Let us consider two cases:

CASE 1: *S and T are operators.* In that case the result follows from [7, Theorem V.2.1].

CASE 2: *S and T are linear relations.* Since $S(0) \subset T(0)$, it is clear that $Q_T = Q_{T+S}$. Further, by Lemma 2.1, $Q_T T \in \Phi_+(X, Y/T(0))$, and by [6, Lemma IV.5.2], $Q_T S = Q_{T(0)/\overline{S(0)}} Q_S S$. It follows from Case 1 applied to $Q_T T$ and $Q_S S$ that $Q_{T+S}(T+S) = Q_T T + Q_T S \in \Phi_+(X, Y/T(0))$ and $i(Q_{T+S}(T+S)) = i(Q_T T)$. The conclusion now follows immediately from Lemma 2.1.

(v) Using [6, Proposition V.5.3] we infer that S' is compact, and by [6, Proposition III.1.5], $(T+S)' = T' + S'$. These properties combined with

Lemma 2.1 and statements (i) and (ii) lead to $T+S \in \Phi_-(X,Y)$ and $i(T+S)=i(T)$. ■

LEMMA 2.3. *Let $S \in L\mathcal{R}(Y,Z)$ and $T \in \Phi_-(X,Y)$ be bounded operators such that $N(S)$ and $N(T)$ are topologically complemented in Y and X respectively. Then $N(ST)$ is topologically complemented in X.*

Proof. Let Y_1 and X_1 be closed subspaces of Y and X respectively such that $Y = N(S) \oplus Y_1$ and $X = N(T) \oplus X_1$. Since $R(T)$ is a closed finite-codimensional subspace, [7, Lemma IV.2.8] yields finite-dimensional subspaces $N \subset N(S)$ and $Y_2 \subset Y_1$ such that

$$N(S) = (N(S) \cap R(T)) \oplus N \quad \text{and} \quad Y_1 = (Y_1 \cap R(T)) \oplus Y_2.$$

Therefore, $(N(S) \cap R(T)) \oplus (Y_1 \cap R(T))$ is a finite-codimensional subspace of $R(T)$, so there is a finite-dimensional subspace $Y_3 \subset R(T)$ such that

$$R(T) = \big((N(S) \cap R(T)) \oplus (Y_1 \cap R(S))\big) + Y_3,$$
$$\{0\} = \big((N(S) \cap R(T)) \oplus (Y_1 \cap R(S))\big) \cap Y_3.$$

Next, we define $T_1 := T_{|X_1}$, $X_3 := T_1^{-1}(N(S) \cap R(T))$ and $X_4 := T_1^{-1}(Y_3 \oplus (R(T) \cap Y_1))$. Then it is easy to see that $X = N(T) \oplus X_3 \oplus X_4$, and since $TN(ST) = R(T) \cap N(S)$ we deduce that $N(ST) = N(T) \oplus X_3$ and thus $X = N(ST) \oplus X_4$. ■

3. Main results. Throughout this section, X will denote a complex Banach space and $S, T \in L\mathcal{R}(X)$ are such that S is continuous, T is closed, $S(0) \subset T(0)$ and $\mathcal{D}(T) \subset \mathcal{D}(S)$, unless otherwise stated. The purpose of this section is to characterize the sets $\sigma_{\mathrm{eap},S}(T)$, $\sigma_{\mathrm{e}\delta,S}(T)$ and $\sigma_{\mathrm{e},S}(T)$ by means of Φ_+, Φ_- and $\Phi_+ \cap \Phi_-$ linear relations respectively.

The results of the next theorem were established in [1, Theorem 2.1] for the case when S and T are bounded operators.

THEOREM 3.1. *We have*

(i) $\lambda \notin \sigma_{\mathrm{eap},S}(T)$ *if and only if* $\lambda S - T \in \Phi_+(X)$ *and* $i(\lambda S - T) \le 0$.
(ii) $\lambda \notin \sigma_{\mathrm{e}\delta,S}(T)$ *if and only if* $\lambda S - T \in \Phi_-(X)$ *and* $i(\lambda S - T) \ge 0$.
(iii) $\lambda \notin \sigma_{\mathrm{e},S}(T)$ *if and only if* $\lambda S - T \in \Phi(X)$ *and* $i(\lambda S - T) = 0$.

Proof. We first note that $\lambda S - T$ is closed by Lemma 2.2(ii).

(i) Let $\lambda \notin \sigma_{\mathrm{eap},S}(T)$. Then there is $K \in \mathfrak{K}_T(X)$ such that $\lambda S - T - K$ is injective and open, and as $\lambda S - T - K$ is closed we see that $\lambda S - T - K \in \Phi_+(X)$ with $i(\lambda S - T - K) \le 0$. Thus again by Lemma 2.2(iii) we deduce that $\lambda S - T = \lambda S - T - K + K \in \Phi_+(X)$ and $i(\lambda S - T) = i(\lambda S - T - K) \le 0$, as desired.

Conversely, assume that $\lambda S - T \in \Phi_+(X)$ and $i(\lambda S - T) \le 0$. It is clear that $\lambda \notin \sigma_{\mathrm{eap},S}(T)$ if $\lambda S - T$ is injective.

Suppose that $1 \leq n := \alpha(\lambda S - T)$ and let $\{x_1, \dots, x_n\}$ be a basis of $N(\lambda S - T)$. Choose $x_1', \dots, x_n' \in X'$ such that $x_i'(x_j) = \delta_{ij}$ and choose $y_1, \dots, y_n \in X$ such that $[y_1], \dots, [y_n] \in X/R(\lambda S - T)$ are linearly independent, where $\delta_{ij} = 0$ if $i \neq j$ and $\delta_{ij} = 1$ if $i = j$. Define an operator F by

$$F : X \ni x \mapsto Fx := \sum_{i=1}^{n} x_i'(x) y_i \in X.$$

Then, reasoning as in the proof of [12, Theorem 9.1(i)], we find that F is an everywhere defined operator,

$$\|Fx\| \leq \|x\| \Big(\sum_{i=1}^{n} \|x_i'\| \, \|y_i\| \Big),$$

$\dim R(F) \leq n$ and $\lambda S - T - F$ is injective. So $F \in \mathfrak{K}_T(X)$ and thus the use of Lemma 2.2(iv) allows us to conclude that $\lambda \notin \sigma_{\mathrm{eap},S}(T)$.

(ii) Assume that $\lambda \notin \sigma_{\mathrm{e}\delta,S}(T)$. Then arguing as in part (i) we infer from Lemma 2.2(v) that $\lambda S - T \in \Phi_-(X)$ with $i(\lambda S - T) \geq 0$.

Conversely, assume that $\lambda S - T \in \Phi_-(X)$ and $i(\lambda S - T) \geq 0$. Clearly $\lambda \notin \sigma_{\mathrm{e}\delta,S}(T)$ if $\lambda S - T$ is surjective. Suppose now that $1 \leq n = \beta(\lambda S - T)$ and let $x_1, \dots, x_n$ be linearly independent elements of $N(\lambda S - T)$. Choose linear functionals $x_1', \dots, x_n'$ such that $x_i'(x_j) = \delta_{ij}$ and choose $y_1, \dots, y_n \in X$ such that the corresponding cosets $[y_1], \dots, [y_n] \in X/R(\lambda S - T)$ form a basis of $X/R(\lambda S - T)$. Define an operator F by

$$F : X \ni x \mapsto Fx := \sum_{i=1}^{n} x_i'(x) y_i \in X.$$

Then, arguing as in the proof of [12, Theorem 9.1(ii)] we find that $\dim R(F) \leq n$ and $\lambda S - T - F$ is surjective. Clearly F is continuous. The use of Lemma 2.2 allow us to conclude that $\lambda \notin \sigma_{\mathrm{e}\delta,S}(T)$.

(iii) This assertion follows immediately from (i) and (ii). ∎

Proposition 3.1. *Let $T \in L\mathcal{R}(X, Y)$ be closed such that $T(0)$ is topologically complemented in Y. The following properties are equivalent:*

(i) $T \in \Phi_l(X, Y)$.

(ii) *There exist $A \in B(Y, X)$ and a bounded finite rank projection $F \in B(X)$ such that $N(A)$ is topologically complemented in Y, $R(A)$ is a closed subspace contained in $\mathcal{D}(T)$, $R(F) \subset \mathcal{D}(T)$ and $AT = (I - F)_{|\mathcal{D}(T)}$.*

(iii) *There exist $A \in B(Y, X)$ and a continuous operator B in X such that $\mathcal{D}(T) \subset \mathcal{D}(B)$, $I - B \in \Phi(X)$, $R(B) \subset \mathcal{D}(T)$ and $AT = (I - B)_{|\mathcal{D}(T)}$.*

Proof. (i)⇒(ii). Let us consider two cases:

Case 1: *T is an operator.* Since $T \in \Phi_+(X,Y)$ there is a closed finite-codimensional subspace M of X for which $N(T) \oplus M = X$ so that $N(T) \oplus (M \cap \mathcal{D}(T)) = \mathcal{D}(T)$, and since $R(T)$ is topologically complemented in Y there exists a closed subspace N of Y such that $R(T) \oplus N = Y$. Let P denote the bounded projection of Y onto $R(T)$ and let $A := (T_{|M})^{-1}P$. Then

$$\begin{aligned}\mathcal{D}(A) &= \{y \in \mathcal{D}(P) : Py \cap \mathcal{D}((T_{|M})^{-1}) \neq \emptyset\} \\ &= \{y \in Y : Py \in R(T_{|M}) = R(P)\} = Y.\end{aligned}$$

A is continuous since $(T_{|M})^{-1}$ is continuous by the Open Mapping Theorem for operators, and

$$\begin{aligned}&R(A) = (T_{|M})^{-1}PY = (T_{|M})^{-1}T_{|M}(M \cap \mathcal{D}(T)) = M \cap \mathcal{D}(T), \\ &N(A) = P^{-1}N((T_{|M})^{-1}) = N(P) = N.\end{aligned}$$

Let F be the bounded projection of X onto $N(T)$. Then it is obvious that $R(F) \subset \mathcal{D}(T)$. For $x \in \mathcal{D}(T)$ we write $x = x_1 + x_2$ where $x_1 \in N(T)$ and $x_2 \in M \cap \mathcal{D}(T)$. Then

$$ATx = (T_{|M})^{-1}P(T_{|M})x_2 = x_2 = (I - F)x.$$

Therefore (i)⇒(ii) whenever T is an operator.

Case 2: *T is a linear relation.* Then by Lemma 2.1, $Q_T T$ is in $\Phi_+(X, Y/T(0))$ with $R(Q_T T) = R(T)/T(0)$, and since $R(T) \oplus N = Y$ for some closed subspace N of Y and $N(Q_T) = T(0) \subset R(T)$ we find that $R(Q_T T) \oplus Q_T N = Y/T(0)$. It follows from Case 1 applied to $Q_T T$ that there exist $A \in B(Y/T(0), X)$ and a bounded finite rank projection $F \in B(X)$ such that $R(F) \subset \mathcal{D}(Q_T T) = \mathcal{D}(T)$ $N(A)$ is topologically complemented in $Y/T(0)$, $R(A)$ is a closed subspace contained in $\mathcal{D}(Q_T T)$ and $(AQ_T)T = (I - F)_{|\mathcal{D}(T)}$.

Hence, it only remains to prove that $N(AQ_T)$ is topologically complemented in Y. To see this, we note that Q_T is a bounded lower semi-Fredholm operator from Y onto $Y/T(0)$, and $N(Q_T) = T(0)$ is topologically complemented in Y by hypothesis, so Lemma 2.3 shows that $N(AQ_T)$ is topologically complemented in Y, as required.

(ii)⇒(iii). This is clear.

(iii)⇒(i). Let us consider various cases:

Case 1: *T is a densely defined operator.* In that case the result was proved by Müller-Horrig [10, Theorem 1.1].

Case 2: *T is an operator.* Let A and B satisfy the conditions in (iii). Then it is clear that $ATG_T = (I - B)_{|X_T}$. Since $TG_T \in B(X_T, Y)$ and X_T is complete, it follows from Case 1 applied to TG_T that $TG_T \in \Phi_l(X_T, Y)$ and hence $T \in \Phi_l(X, Y)$, as desired.

CASE 3: *T is a linear relation.* Let A and B be as in (iii). Recalling that a linear relation S is an operator if and only if $S(0) = \{0\}$ (see [6, Corollary I.2.9]) we have $AQ_T^{-1}(0) = AT(0) = (I - B)_{|\mathcal{D}(T)} = \{0\}$, so that AQ_T^{-1} is an operator, and clearly, it is everywhere defined and continuous, and thus $AQ_T^{-1} \in B(Y/T(0), X)$. On the other hand

$$Q_T^{-1}Q_T T(\mathcal{D}(T)) = \big(T(\mathcal{D}(T)) \cap \mathcal{D}(Q_T)\big) + N(Q_T)$$

(see [6, Proposition I.3.1]), which implies that $(AQ_T^{-1})Q_T T = AT$. Thus, it follows from what has been shown in Case 2 that $Q_T T \in \Phi_l(X, Y/T(0))$, that is, $Q_T T \in \Phi_+(X, Y/T(0))$ and $R(Q_T T)$ is topologically complemented in $Y/T(0)$.

By Lemma 2.1, $T \in \Phi_+(X, Y/T(0))$, so it only remains to see that $R(T)$ is topologically complemented in Y. To prove this, we note that $Y/T(0) = R(Q_T T) \oplus Q_T U$ for some closed subspace U of X containing $T(0)$. Now [6, Lemma I.6.8] shows that $Y = R(T) + U$ and $T(0) = R(T) \cap U$, and since by hypothesis $T(0) \oplus V = Y$ for some closed subspace V of Y, it follows that $Y = R(T) \oplus (V \cap U)$, so $R(T)$ is topologically complemented in Y, as desired. ∎

REMARK 3.1. We note that Proposition 3.1 was presented in [3, Corollary 3.24], but the proof there was incorrect.

As a consequence of Proposition 3.1 we obtain the following useful result: on the stability of Φ_l-linear relations under compact perturbation, as well as on the behaviour of the index under perturbation.

PROPOSITION 3.2. *Let $T \in \Phi_l(X)$ with $T(0)$ topologically complemented in X and let $K \in \mathfrak{K}_T(X)$. Then $T + K \in \Phi_l(X)$ and $i(T + K) = i(T)$.*

Proof. Let A and B be as in Proposition 3.1(iii). Then

$$A(T + K) = (I - (B - AK))_{|\mathcal{D}(T)}.$$

Further, $AK(0) \subset AT(0) = \{0\}$, so that AK is an operator and clearly it is compact. Applying the implication (iii)⇒(i) of Proposition 3.1 we get $T + K \in \Phi_l(X)$, and the equality $i(T + K) = i(T)$ follows immediately from Lemma 2.2. ∎

We are now in a position to state the second main result of this paper.

THEOREM 3.2. *Assume that $T(0)$ is topologically complemented in X. Then*

$$\lambda \notin \sigma_{\mathrm{el},S}(T) \quad \textit{if and only if} \quad \lambda S - T \in \Phi_l(X) \ \textit{and}\ i(\lambda S - T) \le 0.$$

Proof. Assume that $\lambda \notin \sigma_{\mathrm{el},S}(T)$. Then there exists $K \in \mathfrak{K}_T(X)$ for which $\lambda S - T - K$ is left invertible. Since $(\lambda S - T - K)(0) = T(0)$ (as $S(0)$ and $K(0)$ are contained in $T(0)$) we infer from Lemmata 2.1 and 2.2 together with Proposition 3.1 that $\lambda S - T \in \Phi_l(X)$ and $i(\lambda S - T) \le 0$.

Conversely, assume that $\lambda S - T \in \Phi_{\mathrm{l}}(X)$ and $i(\lambda S - T) \leq 0$. Then $\lambda S - T \in \Phi_{+}(X)$ with $i(\lambda S - T) \leq 0$ and thus by Theorem 3.1, $\lambda S - T$ can be expressed in the form $\lambda S - T = U + K$ where $K \in \mathfrak{K}_{\lambda S - T}(X) = \mathfrak{K}_T(X)$ (as $(\lambda S - T)(0) = T(0)$ and $\mathcal{D}(\lambda S - T) = \mathcal{D}(T)$) and $U \in L\mathcal{R}(X)$ is bounded below.

It only remains to show that $R(\lambda S - T - K)$ is topologically complemented in X. To prove this, it is enough to note that $\lambda S - T \in \Phi_{\mathrm{l}}(X)$ with $(\lambda S - T)(0)$ topologically complemented in X and $K \in \mathfrak{K}_{\lambda S - T}(X)$, which implies by Proposition 3.1 that $\lambda S - T - K \in \Phi_{\mathrm{l}}(X)$, in particular $R(\lambda S - T - K)$ is topologically complemented in X as required. ∎

In order to establish the characterization of the S-essential spectrum $\sigma_{\mathrm{er},S}(T)$ (Theorem 3.3 below) we recall the following characterization of Φ_{r} linear relations, proved in [3, Remark 3.23 and Corollary 3.25].

Proposition 3.3. *Let $T \in L\mathcal{R}(X, Y)$ be closed. The following properties are equivalent:*

(i) $T \in \Phi_{\mathrm{r}}(X, Y)$.
(ii) *There exist $B \in B(Y, X)$ and $K \in B(Y)$ such that $R(B) \subset \mathcal{D}(T)$, BT and KT are continuous operators, $I - K \in \Phi(X, Y)$ and $TB = I - K + TB - TB$.*

Theorem 3.3. *Assume that $T'(0)$ is topologically complemented in X'. Then*

$$\lambda \notin \sigma_{\mathrm{er},S}(T) \quad \text{if and only if} \quad \lambda S - T \in \Phi_{\mathrm{r}}(X) \text{ and } i(\lambda S - T) \geq 0.$$

Proof. Assume that $\lambda \notin \sigma_{\mathrm{er},S}(T)$. Then there exists $K \in \mathfrak{K}_T(X)$ for which $\lambda S - T - K$ is a right invertible linear relation in X. By [6, Propositions III.1.4, III.1.5 and III.4.6],

$$(\lambda S - T - K)' = (\lambda S - T)' - K' = \lambda S' - T' - K'$$

is bounded below with $R(\lambda S' - T' - K')$ topologically complemented in X' and since $K' \in \mathfrak{K}_{T'}(X')$ by Lemma 2.2(i), Theorem 3.2 implies that $(\lambda S - T)' \in \Phi_{\mathrm{l}}(X')$ and $i(\lambda S - T)' \leq 0$. Finally, the use of Lemma 2.1 shows that $\lambda S - T \in \Phi_{\mathrm{r}}(X)$ and $i(\lambda S - T) \geq 0$.

Conversely, let $\lambda \in \mathbb{C}$ be such that $\lambda S - T \in \Phi_{\mathrm{r}}(X)$ and $i(\lambda S - T) \geq 0$. Then, applying Proposition 3.3 we have

$$(\lambda S - T)B = I - K + (\lambda S - T)B - (\lambda S - T)B \tag{3.1}$$

for some bounded operators B and K such that $R(B) \subset \mathcal{D}(T)$, $B(\lambda S - T)$ and $K(\lambda S - T)$ are continuous operators and $I - K \in \Phi(X)$. On the other hand, arguing exactly as in the proof of ($\Leftarrow$) in Theorem 3.1(ii) one sees that there exists a bounded finite rank operator F in X such that $\lambda S - T - F$ is closed and surjective. Then one finds from (3.1) that

$$(3.2)\quad (\lambda S - T - F)B = I - K - FB + (\lambda S - T - F)B - (\lambda S - T - F)B.$$

Clearly $K + FB$ is a bounded operator, $I - K - FB \in \Phi(X)$, and $B(\lambda S - T - F)$ and $(K + FB)(\lambda S - T - F)$ are continuous operators, and thus by Proposition 3.3, $N(\lambda S - T - F)$ is topologically complemented in X. This last property together with (3.2) ensures that $\lambda \notin \sigma_{\mathrm{er},S}(T)$. ∎

4. Application. Let us first make precise the functional setting of the problem. Let

$$X_1 := L_1((-a,a) \times (-1,1), dx d\xi)$$

where $0 < a < \infty$. We define the partial Sobolev space W_1 by

$$W_1 := \left\{ \psi \in X_1 : \xi \frac{\partial \psi}{\partial x} \in X_1 \right\}.$$

We define the single valued *free streaming operator* T_0 by

$$\begin{cases} T_0 : X_1 \supset \mathcal{D}(T_0) \to X_1, \\ \qquad \psi \mapsto T_0\psi(x,\xi) = -\xi \dfrac{\partial \psi}{\partial x}(x,\xi) - \sigma(\xi)\psi(x,\xi), \\ \mathcal{D}(T_0) = \{\psi \in W_1 : \psi(a,-\xi) = \psi(-a,\xi) = 0,\ \xi \in (0,1)\}, \end{cases}$$

where $\sigma \in L^\infty(-1,1)$. Moreover, we denote $\lambda^* = \liminf_{|\xi| \to 0} \sigma(\xi)$. Next, we consider the single valued *collision operator* K defined by

$$K : X_1 \to X_1, \qquad \psi \mapsto \int_{-1}^{1} \kappa(x,\xi,\xi')\psi(x,\xi')\, d\xi',$$

where $\kappa(\cdot,\cdot,\cdot)$ is a measurable function from $[-a,a] \times [-1,1] \times [-1,1]$ to $\mathbb{R}$. Observe that K acts only on the variable ξ, so x may be viewed merely as a parameter in $[-a,a]$. Hence we may consider K as a function $K : [-a,a] \ni x \mapsto K(x) \in \mathcal{Z}$ where $\mathcal{Z} := \mathcal{L}(L_1([-1,1], d\xi))$.

We make the following assumptions:

$$(\mathcal{H}) : \begin{cases} K \text{ is a measurable, i.e.,} \\ \{x \in [-a,a] : K(x) \in \mathcal{O}\} \text{ is measurable if } \mathcal{O} \subset \mathcal{Z} \text{ is open,} \\ \text{there exists a compact subset } T \subset \mathcal{Z} \text{ such that } K(x) \in T \text{ a.e.,} \\ \text{and finally } K(x) \in \mathcal{K}(L_1([-1,1], d\xi)) \text{ a.e.} \end{cases}$$

where $\mathcal{K}(L_1([-1,1], d\xi))$ denotes the set of all compact operators on $L_1([-1,1], d\xi)$.

Definition 4.1. A collision operator K is said to be *regular* if it satisfies the assumptions $(\mathcal{H})$.

Proposition 4.1 ([2, Proposition 3.1]). *If the single valued collision operator K is regular, then, for any $\lambda \in \mathbb{C}$ such that* $\operatorname{Re}\lambda > -\lambda^*$, *the operator* $(\lambda - T_0)^{-1}K$ *is weakly compact on* X_1.

Lemma 4.1 ([2, Lemma 3.1]). *If* $\|K\| < \lambda^*$, *then*

$$\sigma_{i,I}(T_0^{-1}) = \{\lambda \in \mathbb{C} : \operatorname{Re}(1/\lambda) \le -\lambda^*\}, \qquad i \in \{\mathrm{e}, \mathrm{eap}, \mathrm{e}\delta\}.$$

In the Banach space $X_1 \times X_1$ we define the linear relations

$$\mathcal{A} = \begin{pmatrix} C_0 & 0 \\ 0 & B \end{pmatrix}, \quad \mathcal{S} = \begin{pmatrix} I & K \\ 0 & I \end{pmatrix}, \quad \mathcal{N} = \begin{pmatrix} U & 0 \\ 0 & 0 \end{pmatrix},$$

where $C_0 = T_0^{-1} + B$, $B \in \mathcal{KR}(X_1)$ such that $\mathcal{D}(B) = X_1$, $B(0)$ is closed, $\dim B(0) < \infty$ and $U = \sum_{n\ge 1}[T_0^{-1}K]^n T_0^{-1}$. So, it is clear that $\mathcal{A} \in L\mathcal{R}(X_1 \times X_1)$ and $\mathcal{D}(\mathcal{A}) = X_1 \times X_1$, $\mathcal{N}$ and $\mathcal{S}$ are single valued linear operators,

$$\mathcal{S}(0) \subset \mathcal{A}(0) = \begin{pmatrix} B(0) \\ B(0) \end{pmatrix} = (\mathcal{A}+\mathcal{N})(0) \quad \text{and} \quad \mathcal{D}(S) = \mathcal{D}(\mathcal{N}) = X_1 \times X_1.$$

Remark 4.1. It is clear that if B is a single valued compact operator then for all $\lambda \in \mathbb{C} \setminus \{0\}$, $\lambda - B$ is a Fredholm operator and $i(\lambda - B) = 0$. But this is not true if B is a compact linear relation (one example can be found in [6, Example VI.3.1]). Indeed, note that B and $\lambda - B$ are bounded and closed. Moreover

$$Q_{\lambda - B}(\lambda - B) = Q_B(\lambda - B) = Q_B(\lambda) - Q_B(B).$$

Here $Q_B B$ is a bounded compact, $N(Q_B(\lambda)) = B(0)$ and $R(Q_B(\lambda)) = X/B(0)$, so that $Q_B(\lambda)$ is a Fredholm operator. Hence $Q_{\lambda-B}(\lambda - B)$ is a Fredholm operator and so by Lemma 2.1, $\lambda - B$ is a Fredholm relation and $i(\lambda - B) = i(Q_{\lambda - B}(\lambda - B)) = \dim B(0)$.

Lemma 4.2. *Assume that the single valued collision operator K is regular. Then:*

(i) *If* $\mathcal{A}'(0)$ *is topologically complemented in* $X_1' \times X_1'$, *we have*

$$\sigma_{\mathrm{er},S}(\mathcal{A} + \mathcal{N}) = \sigma_{\mathrm{er},S}(\mathcal{A}).$$

(ii) *If* $\mathcal{A}(0)$ *is topologically complemented in* $X_1 \times X_1$, *we have*

$$\sigma_{\mathrm{el},S}(\mathcal{A} + \mathcal{N}) = \sigma_{\mathrm{el},S}(\mathcal{A}).$$

(iii) $\sigma_{i,S}(\mathcal{A} + \mathcal{N}) = \sigma_{i,S}(\mathcal{A})$, $i \in \{\mathrm{e}, \mathrm{eap}, \mathrm{e}\delta\}$.

Proof. Let $\lambda \notin \sigma_{\mathrm{er},\mathcal{S}}(\mathcal{A})$. Then by Theorem 3.3, $\mathcal{A} - \lambda\mathcal{S} \in \Phi_{\mathrm{r}}(X_1 \times X_1)$ and $i(\mathcal{A} - \lambda\mathcal{S}) \ge 0$. Proposition 3.3 yields bounded matrix operators

$$\mathcal{A}_0 = \begin{pmatrix} A_1 & B_1 \\ C_1 & D_1 \end{pmatrix} \quad \text{and} \quad \mathcal{R} = \begin{pmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{pmatrix}$$

such that $I - \mathcal{R} \in \Phi(X_1 \times X_1)$ and

$$(\mathcal{A} - \lambda\mathcal{S})\mathcal{A}_0 = (I - \mathcal{R}) + (\mathcal{A} - \lambda\mathcal{S})\mathcal{A}_0 - (\mathcal{A} - \lambda\mathcal{S})\mathcal{A}_0.$$

Hence,

$$(\mathcal{A} - \lambda\mathcal{S} + \mathcal{N})\mathcal{A}_0 = \mathcal{P} + (\mathcal{A} - \lambda\mathcal{S})\mathcal{A}_0 - (\mathcal{A} - \lambda\mathcal{S})\mathcal{A}_0,$$

where

$$\mathcal{P} = \begin{pmatrix} I - K_{11} + UA_0 & -K_{12} + UB_0 \\ K_{21} & I - K_{22} \end{pmatrix} \in \Phi(X_1 \times X_1).$$

Therefore $(\mathcal{A} - \lambda\mathcal{S} + \mathcal{N})\mathcal{A}_0 \in \Phi(X_1 \times X_1)$. Since $\mathcal{A}_0$ is single valued, we have $\mathcal{A} - \lambda\mathcal{S} + \mathcal{N} \in \Phi(X_1 \times X_1)$. So, $\mathcal{A} + \mathcal{N} - \lambda\mathcal{S} \in \Phi_{\mathrm{r}}(X_1 \times X_1)$. On the other hand $i(\mathcal{A} - \lambda\mathcal{S} + \mathcal{N}) = i(\mathcal{A} - \lambda\mathcal{S})$. We infer that $\mathcal{A} + \mathcal{N} - \lambda\mathcal{S} \in \Phi_{\mathrm{r}}(X_1 \times X_1)$ and $i(\mathcal{A} - \lambda\mathcal{S}) \geq 0$. Thus, $\lambda \notin \sigma_{\mathrm{er},\mathcal{S}}(A)$. So, $\sigma_{\mathrm{er},\mathcal{S}}(A) \subset \sigma_{\mathrm{er},\mathcal{S}}(\mathcal{A} + \mathcal{N})$.

Conversely, suppose that $\lambda \notin \sigma_{\mathrm{er},\mathcal{S}}(\mathcal{A} + \mathcal{N})$. Then $\mathcal{A} - \lambda\mathcal{S} + \mathcal{N}$ is in $\Phi(X_1 \times X_1)$ and $i(\mathcal{A} + \mathcal{N} - \lambda\mathcal{S}) \geq 0$. Now, the linear relation $\mathcal{A} - \lambda\mathcal{S}$ can be written in the form

$$\mathcal{A} - \lambda\mathcal{S} = \mathcal{A} + \mathcal{N} - \lambda\mathcal{S} + \mathcal{N}.$$

So, using the same reasoning we have $\sigma_{\mathrm{er},\mathcal{S}}(A + \mathcal{N}) \subset \sigma_{\mathrm{er},\mathcal{S}}(\mathcal{A})$.

(ii) and (iii) may be checked in the same way. ∎

Lemma 4.3. *If* $\|K\| < \lambda^*$ *and* $T_0^{-1}B \subset BT_0^{-1}$. *Then*

(i) $$\sigma_{\mathrm{e},I}(C_0) \setminus \{0\} = \big(\{\lambda \in \mathbb{C} : \mathrm{Re}(1/\lambda) \leq -\lambda^*\} \cup \sigma_{\mathrm{e},I}(B)\big) \setminus \{0\} = \mathbb{C} \setminus \{0\},$$

(ii) $$\sigma_{\mathrm{e}\delta,I}(C_0) \setminus \{0\} = \big(\{\lambda \in \mathbb{C} : \mathrm{Re}(1/\lambda) \leq -\lambda^*\} \cup \sigma_{\mathrm{e}\delta,I}(B)\big) \setminus \{0\} = \{\lambda \in \mathbb{C} : \mathrm{Re}(1/\lambda) \leq -\lambda^*\},$$

(iii) $$\sigma_{\mathrm{eap},I}(C_0) \setminus \{0\} = \{\lambda \in \mathbb{C} : \mathrm{Re}(1/\lambda) \leq -\lambda^*\} \cup \sigma_{\mathrm{eap},I}(B) \setminus \{0\} = \mathbb{C} \setminus \{0\}.$$

Proof. Since T_0^{-1} and B are everywhere defined, it is easy to prove for $\lambda \in \mathbb{C}$ that

$$(4.1)\qquad \big(\lambda - T_0^{-1})(\lambda - B) = T_0^{-1}B + \lambda(\lambda - T_0^{-1} - B),$$

$$(4.2)\qquad (\lambda - B)(\lambda - T_0^{-1}) = BT_0^{-1} + \lambda(\lambda - T_0^{-1} - B).$$

(i) Let $\lambda \notin [\sigma_{\mathrm{e},I}(T_0^{-1}) \cup \sigma_{\mathrm{e},I}(B)] \setminus \{0\}$. Then $T_0^{-1} - \lambda \in \Phi(X_1)$, $i(T_0^{-1} - \lambda) = 0$, $B - \lambda \in \Phi(X_1)$ and $i(B - \lambda) = 0$. On the other hand,

$$\begin{aligned} i((B - \lambda)(T_0^{-1} - \lambda)) &= i(B - \lambda) + i(T_0^{-1} - \lambda) + \dim\big(X_1/[R(T_0^{-1} - \lambda) + \mathcal{D}(B - \lambda)]\big) \\ &\quad - \dim(T_0^{-1}(0) \cap N(B - \lambda)). \end{aligned}$$

Clearly,

$$R(T_0^{-1}-\lambda)+\mathcal{D}(B-\lambda)=X_1,$$
$$T_0^{-1}(0)\cap N(B-\lambda)=\{0\}\cap N(B-\lambda)=\{0\}.$$

Then

$$i((B-\lambda)(T_0^{-1}-\lambda))=i(B-\lambda)+i(T_0^{-1}-\lambda)=0 \tag{4.3}$$

implies that $i(BT_0^{-1}+\lambda(\lambda-T_0^{-1}-B))=0$. Moreover, $BT_0^{-1}\in\mathcal{KR}(X_1)$ and

$$BT_0^{-1}(0)=B(0)\subset\lambda(T_0^{-1}+B-\lambda)(0)=(T_0^{-1}+B)(0)=T_0^{-1}(0)+B(0).$$

So, $\lambda-T_0^{-1}-B\in\Phi(X_1)$ and $i(\lambda-T_0^{-1}-B)=0$. Therefore, $\lambda\notin\sigma_{\mathrm{e},I}(T_0^{-1}+B)$, whence

$$\sigma_{\mathrm{e},I}(C_0)\setminus\{0\}\subset[\sigma_{\mathrm{e},I}(T_0^{-1})\cup\sigma_{\mathrm{e},I}(B)]\setminus\{0\}. \tag{4.4}$$

To prove the inverse inclusion, let $\lambda\notin\sigma_{\mathrm{e},I}(T_0^{-1}+B)\setminus\{0\}$. Then $T_0^{-1}+B-\lambda\in\Phi(X_1)$ and $i(T_0^{-1}+B-\lambda)=0$. Since $T_0^{-1}B, BT_0^{-1}\in K\mathcal{R}(X_1)$ it is easy to show that $T_0^{-1}-\lambda, B-\lambda\in\Phi(X_1)$.

On the other hand, applying (4.1)–(4.3) we have

$$\begin{aligned} i[(B-\lambda)(T_0^{-1}-\lambda)] &= i(T_0^{-1}-\lambda)+i(B-\lambda) \\ &= i(T_0^{-1}+B-\lambda)=0. \end{aligned} \tag{4.5}$$

Since T_0^{-1} is bounded single valued and

$$\begin{aligned}\sigma_{\mathrm{e},I}(T_0^{-1}) &= \{\lambda\in\mathbb{C}:\mathrm{Re}(1/\lambda)\le-\lambda^*\} \\ &= \{\lambda\in\mathbb{C}:T_0^{-1}-\lambda\notin\Phi(X_1)\},\end{aligned}$$

we have $T_0^{-1}-\lambda\in\Phi(X_1)$, and we deduce that $i(T_0^{-1}-\lambda)=0$. It follows from (4.5) that $i(B-\lambda)=0$. We conclude that $\lambda\notin\sigma_{\mathrm{e},I}(T_0^{-1})\cup\sigma_{\mathrm{e},I}(B)$, hence

$$[\sigma_{\mathrm{e},I}(T_0^{-1})\cup\sigma_{\mathrm{e},I}(B)]\setminus\{0\}\subset\sigma_{\mathrm{e},I}(T_0^{-1}+B)\setminus\{0\}.$$

So,

$$[\sigma_{\mathrm{e},I}(T_0^{-1})\cup\sigma_{\mathrm{e},I}(B)]\setminus\{0\}=\sigma_{\mathrm{e},I}(T_0^{-1}+B)\setminus\{0\}.$$

On the other hand, using Theorem 3.1(iii) and Remark 4.1 we have $\sigma_{\mathrm{e},I}(B)=\mathbb{C}$, and by Lemma 4.1 we have $\sigma_{\mathrm{e},I}(T_0^{-1})=\{\lambda\in\mathbb{C}:\mathrm{Re}(1/\lambda)\le-\lambda^*\}$. So, $\sigma_{\mathrm{e},I}(C_0)=\mathbb{C}\setminus\{0\}$.

(ii) and (iii) may be checked in the same way; here $\sigma_{\mathrm{eap},I}(B)=\mathbb{C}$ and $\sigma_{\mathrm{e}\delta,I}(B)=\{0\}$. ∎

THEOREM 4.1. *If the single valued collision operator K is regular, $\|K\|<\lambda^*$ and $T_0^{-1}B\subset BT_0^{-1}$, then*

$$\sigma_{\mathrm{e}\delta,\mathcal{S}}(\mathcal{A}+\mathcal{N})\setminus\{0\}\subset\{\lambda\in\mathbb{C}:\mathrm{Re}(1/\lambda)\le-\lambda^*\}.$$

Proof. Using Lemma 4.2(iii) we have $\lambda \notin \sigma_{e\delta,\mathcal{S}}(\mathcal{A}+\mathcal{N})$ if and only if $\lambda \notin \sigma_{e\delta,\mathcal{S}}(\mathcal{A})$. On the other hand,

$$\mathcal{A}-\lambda\mathcal{S} = \begin{pmatrix} I & 0 \\ 0 & B-\lambda \end{pmatrix} \begin{pmatrix} I & K \\ 0 & I \end{pmatrix} \begin{pmatrix} C_0-\lambda & 0 \\ 0 & I \end{pmatrix}.$$

Let $\lambda \notin [\sigma_{e\delta,I}(C_0) \cup \sigma_{e\delta,I}(B)]$. Then

$$\begin{pmatrix} C_0-\lambda & 0 \\ 0 & I \end{pmatrix} \in \Phi_-(X_1 \times X_1), \qquad i\left(\begin{pmatrix} C_0-\lambda & 0 \\ 0 & I \end{pmatrix}\right) \geq 0,$$

$$\begin{pmatrix} I & 0 \\ 0 & B-\lambda \end{pmatrix} \in \Phi_-(X_1 \times X_1), \qquad i\left(\begin{pmatrix} I & 0 \\ 0 & B-\lambda \end{pmatrix}\right) \geq 0.$$

On the other hand, $\left(\begin{smallmatrix} I & K \\ 0 & I \end{smallmatrix}\right)$ is invertible single valued. So, $\mathcal{A}-\lambda\mathcal{S} \in \Phi_-(X_1 \times X_1)$ and $i(\mathcal{A}-\lambda\mathcal{S}) \geq 0$. Thus $\lambda \notin \sigma_{e\delta,\mathcal{S}}(\mathcal{A})$. Applying Lemma 4.3(ii), we obtain $\sigma_{e\delta,\mathcal{S}}(\mathcal{A}+\mathcal{N}) \setminus \{0\} \subset \{\lambda \in \mathbb{C} : \mathrm{Re}(1/\lambda) \leq -\lambda^*\}$. ∎

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