COLLOQUIUM MATHEMATICUM

VOL. 135 2014 NO. 2

ON HEREDITARY ARTINIAN RINGS AND THE PURE SEMISIMPLICITY CONJECTURE: RIGID TILTING MODULES AND A WEAK CONJECTURE

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Abstract. A weak form of the pure semisimplicity conjecture is introduced and characterized through properties of matrices over division rings. The step from this weak conjecture to the full pure semisimplicity conjecture would be covered by proving that there do not exist counterexamples to the conjecture in a particular class of rings, which is also studied.

1. Introduction. A ring R is *left* (resp., *right*) *pure semisimple* when every left (resp., right) R-module is a direct sum of indecomposable submodules. The ring R is of *finite representation* type if it is left artinian and there exist only finitely many indecomposable finitely presented left R-modules, up to isomorphism. A ring is of finite representation type if and only if it is left and right pure semisimple. The *pure semisimplicity conjecture* (which we shall abbreviate as pssC) states that every left pure semisimple ring is of finite representation type.

The conjecture has been proved under certain additional hypotheses [6, 21, 31, 32] but remains undecided. It is known [21] that to prove the conjecture it suffices to show that all left pure semisimple rings of matrices of the form

$$R_B = \begin{bmatrix} F & 0 \\ B & G \end{bmatrix}, \tag{1}$$

where F, G are division rings and B is a G-F-bimodule, have finite representation type.

Simson [34] showed that the pssC would be disproved if the following linear algebra problem had a positive solution: find a division ring embedding $F \leq G$ such that the right dimension of G over F is infinite, while the left dimension of ${}_FG$ and of all the successive left dual vector spaces $G^* = \mathrm{Hom}_F(G, F)$, $G^{**} = \mathrm{Hom}_G(G^*, G), \dots$ is constantly 2. However, the existence of such an example is not necessary for the conjecture to be false,

2010 *Mathematics Subject Classification*: 16G60, 16D90.

Key words and phrases: pure semisimple ring, pure semisimplicity conjecture, finite representation type, tilting module.

DOI: 10.4064/cm135-2-6

and Simson [35] has also identified other conditions on rings of the form (1) that would make them counterexamples to the pssC: these are the potential counterexamples constructed by Simson in the hope that some of them could be shown to exist, thus solving the conjecture in the negative. In fact, he studied potential counterexamples R such that all indecomposable left R-modules are either preinjective or preprojective, i.e., there exist only two Auslander–Reiten components of indecomposable left R-modules.

Even the non-existence of all these potential counterexamples would not automatically imply the truth of the conjecture. A necessary and sufficient condition for the conjecture to hold has also been given by Simson [33, Proposition 4.2] by means of the so-called *generalized Artin problems*. These ask for the existence of G-F-bimodules B with a certain condition on the left dimension of the sequence of left dual modules $B^*, B^{**}, \ldots$, and such that the corresponding ring $R_B = \left[\begin{smallmatrix} F & 0 \\ B & G \end{smallmatrix}\right]$ is left pure semisimple.

It is clear that it would be crucial to have a usable characterization of those rings of matrices $R_B = \left[\begin{smallmatrix} F & 0 \\ B & G \end{smallmatrix}\right]$ which are left pure semisimple. Much is known about the properties and distribution of indecomposable modules over left pure semisimple rings [2, 3, 15–18, 34–36], but in order to have good characterizations of rings of the form (1) that are pure semisimple it is of interest to proceed in the other way, by identifying properties of such rings that could imply pure semisimplicity. Accordingly, we try to work in the direction of finding properties of rings of the form (1) which ensure the validity of conditions known to hold in the pure semisimple case; we do this mainly in Sections 2 and 3. In this sense, tilting modules (in particular, what we call rigid tilting modules) are ubiquitous among pure semisimple rings (see [17, Theorem 3.9(d)]) and thus play a major role in our study. In fact, these tilting modules do the job of the reflection functors used by Simson to relate the sequence of the left dimensions of the bimodules $B, B^*, \ldots$ to the dimensions of the vector spaces defining the preinjective modules. But tilting modules yield relations of this same type for all the indecomposable finitely presented modules, and not only for the preinjective ones.

It turns out that such a tilting module has an endomorphism ring which is again a ring of the form (1) and has the same basic properties as the original ring. Keeping in mind this identification, we can divide the class of rings of the form $R_B = \left[\begin{smallmatrix} F & 0 \\ B & G \end{smallmatrix}\right]$ into two. The first one is the class of those rings which (up to the identification pointed to above) come from a division ring extension $F \subseteq G$, i.e., rings $R_B = \left[\begin{smallmatrix} F & 0 \\ B & G \end{smallmatrix}\right]$ with $B = G$. For these rings, we may find a characterization of their pure semisimplicity in linear algebra terms, namely, in terms of the behaviour of their matrices. The second class of pure semisimple rings lends itself to a detailed study, since it is a very special class which can be accurately described in terms of the defining bimodule B.

Our aim in this paper is to further analyze the conjecture, by splitting it into two parts, according to the above classification of pure semisimple rings of the form (1): one of these parts, which we call the *weak pure semisimplicity conjecture* (wpssC), depends on the characterization of the first class named above, and hence the wpssC is equivalent to a problem on matrices over division rings, so that it is a pure linear algebra problem. The second part of the conjecture postulates the non-existence of certain potential counterexamples with a very particular structure, and these we call *sporadic* (*potential*) *counterexamples*. We hope that this division will be useful to researchers, by isolating the part of the conjecture that is equivalent to a problem on matrices over division rings, and identifying the second part of the conjecture as a question on the existence of certain rings of the form (1), the *sporadic pure semisimple rings*. In a subsequent paper, we shall describe the structure of all potential counterexamples to the pssC that are sporadic and have only finitely many Auslander–Reiten components of indecomposable modules. In so doing, we will show how these potential counterexamples are related to (finite) dimension sequences, in the sense of Dowbor–Ringel–Simson [12].

For general concepts and terminology from ring and representation theory we refer to [1, 5, 7]. For tilting modules as they are used here, the reference is [9]. For results on pure semisimple rings and pure semisimple Grothendieck categories, and the history of the pure semisimplicity conjecture, see [6, 19–22, 24, 28–36]. For several other notions and notations ($\mathrm{add}(\mathcal{C})$, $\mathrm{Add}(\mathcal{C})$, R-Mod, R-mod, R-ind, Mod-R, preinjective or preprojective modules, strong preinjective partition) we refer to [16] (original sources are [8, 23]).

2. Rigid tilting modules. Throughout this section, R will be a hereditary and left artinian ring, and R_B will denote the ring $\left[\begin{smallmatrix} F & 0 \\ B & G \end{smallmatrix}\right]$, where G, F are division rings and B is a G-F-bimodule. We shall assume that B is finite-dimensional as a left G-vector space, or equivalently R_B is left artinian. In particular, every finitely generated left R_B-module is a direct sum of indecomposable left R_B-modules.

It is well known (see, for instance, [7]) that each finitely generated left R_B-module determines a triple (V, W, h) where V (resp., W) is a finite-dimensional left F (resp., G)-vector space and $h : B \otimes_F V \to W$ is a G-linear map. Conversely, any finitely generated left R_B-module is determined by such a triple. Notable modules are the simple injective module E_0 given through the map $B \otimes_F F \to 0$ and the simple projective module P_0 given by $B \otimes_F 0 \to G$. There is only one other indecomposable projective module P_1, corresponding to the canonical map $B \otimes_F F \to B$; and one other indecomposable injective module E_1, corresponding to the canonical linear map $B \otimes_F B^* \to G$, with $B^* = \mathrm{Hom}_G(B, G)$. The submodules of P_1 are projective and hence R_B is hereditary. Note that the endomorphism ring of P_1 is isomorphic to F, and that E_1 is finitely generated if and only if B^* is left finite-dimensional.

With the above representation, a module homomorphism $(V_1, W_1, h_1) \to (V_2, W_2, h_2)$ can be identified with a pair of linear maps $f : V_1 \to V_2$ and $g : W_1 \to W_2$ such that the corresponding diagram

$$\begin{tikzcd} B \otimes_F V_1 \arrow[r, "h_1"] \arrow[d, "1\otimes f"'] & W_1 \arrow[d, "g"] \\ B \otimes_F V_2 \arrow[r, "h_2"] & W_2 \end{tikzcd}$$

is commutative. The homomorphism is an isomorphism if and only if f, g are both isomorphisms. Kernels and cokernels can be described in the expected way [7].

Given a finitely generated module X represented by the tuple (V, W, h), we say (following Simson [34]) that the pair $(\dim(V), \dim(W))$ is the *d-vector* of X. Note that if X is indecomposable and non-injective, then $\dim(W)$ is the length of the socle of X, while $\dim(V)$ is the length of the top $X/\mathrm{rad}(X)$ of X. Accordingly, we shall denote as (t, s) the d-vector of a general finitely generated module X. For instance, the d-vectors of the indecomposable modules E_0, E_1 (provided E_1 is finitely generated), P_0, P_1 are, respectively, $(1, 0), (d^*, 1), (0, 1), (1, d)$, where $d = \mathrm{l.dim}(B)$ and $d^* = \mathrm{l.dim}(B^*)$.

We will use tilting modules in the sense of Colby and Fuller [9]. Since our rings R will be left artinian and hereditary, a tilting module is a finitely generated left R-module W such that $\mathrm{Ext}^1_R(W, W) = 0$ and there is a short exact sequence $0 \to R \to W_1 \to W_2 \to 0$ where $W_1, W_2 \in \mathrm{add}(W)$. In connection with the tilting module W (with endomorphism ring S), the following functors are of interest:

$$\begin{aligned} H &= \mathrm{Hom}_R(W, -), \quad & H' &= \mathrm{Ext}^1_R(W, -) : R\text{-Mod} \to S\text{-Mod}, \\ G &= W \otimes_S -, & G' &= \mathrm{Tor}^S_1(W, -) : S\text{-Mod} \to R\text{-Mod}. \end{aligned}$$

When W is a tilting module, the pair $(\mathcal{T}, \mathcal{F})$ with $\mathcal{T} = \mathrm{Ker}(H')$ and $\mathcal{F} = \mathrm{Ker}(H)$ is a torsion theory of R-Mod; and, according to the tilting theorem [9, 1.4], we have $G' \circ H = 0$, $H' \circ G = 0$ and the functors H, G induce an equivalence between the subcategory $\mathcal{T}$ of R-Mod and a subcategory $\mathcal{Y}$ of S-Mod. Similarly, H' and G' induce an equivalence between the subcategory $\mathcal{F}$ of R-Mod and a subcategory $\mathcal{X}$ of S-Mod. Furthermore $(\mathcal{X}, \mathcal{Y})$ is a splitting torsion theory of S-Mod. By [37, Lemma 1.4], when the endomorphism ring S is left artinian, the above equivalences restrict to equivalences between the finitely presented modules of each of the categories $\mathcal{T}$, $\mathcal{Y}$, $\mathcal{F}$, $\mathcal{X}$.

When R is the ring $R_B = \left[\begin{smallmatrix} F & 0 \\ B & G \end{smallmatrix}\right]$, a basic tilting module (see [37, p. 6059] or [17, paragraph before Theorem 2.14]) is the direct sum of two non-isomorphic indecomposable modules, by [37, Theorem 1.5].

In view of the above equivalences, we have $\mathrm{Hom}_S(H(M_1), H(M_2)) \cong \mathrm{Hom}_R(M_1, M_2)$ (as bimodules) and the analogous property holds (with H' instead of H) for the modules in $\mathcal{F}$; and $\mathrm{Ext}^1_S(H(M_1), H(M_2)) \cong \mathrm{Ext}^1_R(M_1, M_2)$, with the same again true for $\mathcal{F}$. The following lemmas identify the other Hom or Ext groups.

Lemma 2.1. *Let R be a hereditary and left artinian ring, let W be a tilting module and use the notation above. Suppose that X, Y are indecomposable finitely presented left R-modules such that $X \in \mathcal{T}$ and $Y \in \mathcal{F}$, and let S be the endomorphism ring of W. Then there is an isomorphism $\mathrm{Hom}_S(H(X), H'(Y)) \cong \mathrm{Ext}^1_R(X, Y)$ of left modules over the endomorphism ring of X.*

Proof. By construction, every injective module belongs to $\mathcal{T}$. Since Y is torsionfree, it cannot be injective, and thus we have a non-split short exact sequence in R-mod, $0 \to Y \to U \to U' \to 0$, where U, U' are injective modules. Consequently, they are torsion modules, and we get another short exact sequence of left S-modules

$$0 \to \mathrm{Hom}_R(W, U) = H(U) \to H(U') \to \mathrm{Ext}^1_R(W, Y) = H'(Y) \to 0$$

because $\mathrm{Hom}_R(W, Y) = 0$. If we now apply the functor $\mathrm{Hom}_S(H(X), -)$ to this sequence and bear in mind that $\mathrm{Ext}^1_S(H(X), H(U)) \cong \mathrm{Ext}^1_R(X, U) = 0$, we get the exact sequence

$$\begin{aligned} 0 \to \mathrm{Hom}_S(H(X), H(U)) \to \mathrm{Hom}_S(H(X), H(U')) \\ \to \mathrm{Hom}_S(H(X), H'(Y)) \to 0, \end{aligned}$$

which, by the canonical isomorphisms $\mathrm{Hom}_S(H(X), H(Z)) \cong \mathrm{Hom}_R(X, Z)$, shows that $\mathrm{Hom}_S(H(X), H'(Y))$ is isomorphic to the cokernel of the homomorphism $\mathrm{Hom}_R(X, U) \to \mathrm{Hom}_R(X, U')$ induced by the given $U \to U'$. But if we start from the initial sequence, we get the exact sequence of induced homomorphisms

$$0 \to \mathrm{Hom}_R(X, Y) \to \mathrm{Hom}_R(X, U) \to \mathrm{Hom}_R(X, U') \to \mathrm{Ext}^1_R(X, Y) \to 0,$$

and this proves that $\mathrm{Hom}_S(H(X), H'(Y)) \cong \mathrm{Ext}^1_R(X, Y)$. ∎

Lemma 2.2. *Let R be a hereditary and left artinian ring and let W be a tilting module with endomorphism ring S. Assume that S is left artinian and hereditary. Suppose that X and Y are indecomposable finitely presented left R-modules such that $X \in \mathcal{F}$ and $Y \in \mathcal{T}$. Then there is an isomorphism $\mathrm{Ext}^1_S(H'(X), H(Y)) \cong \mathrm{Hom}_R(X, Y)$ of right modules over the endomorphism ring of Y.*

Proof. By the tilting theorem [9, 1.4], we know that $G' \circ H'$ is equivalent to the identity functor on the modules of $\mathcal{F}$. Therefore, we have a canonical isomorphism $X \cong G'(H'(X))$. Moreover, $H'(X)$ is not projective, because $S = H(W) \in \mathcal{Y}$ while $H'(X) \in \mathcal{X}$. Since S is hereditary, there is a non-split short exact sequence of left S-modules

$$0 \to P' \to P \to H'(X) \to 0$$

where P', P are projective. By tensoring, this gives an exact sequence

$$\begin{aligned} 0 \to \operatorname{Tor}_1^S(W, H'(X)) = G'(H'(X)) \to W \otimes_S P' \\ \to W \otimes_S P \to W \otimes_S H'(X) \to 0 \end{aligned}$$

because $\operatorname{Tor}_1^S(W, P) = G'(P) = G'(H(W'))$ with $W' \in \operatorname{add}(W)$, as P is projective. Thus $\operatorname{Tor}_1^S(W, P) = 0$ because $G' \circ H = 0$ [9, 1.4]. On the other hand, $W \otimes_S H'(X) = G(H'(X)) = 0$, again by the tilting theorem. So, we have the short exact sequence in R-Mod

$$0 \to G'(H'(X)) \cong X \to G(P') \to G(P) \to 0,$$

and $G(P') \to G(P)$ is induced by the monomorphism $P' \to P$.

If we now apply $\operatorname{Hom}_R(-, Y)$, we get another short exact sequence of right $\operatorname{End}_R(Y)$-modules

$$0 \to \operatorname{Hom}_R(G(P), Y) \to \operatorname{Hom}_R(G(P'), Y) \to \operatorname{Hom}_R(X, Y) \to 0$$

as $\operatorname{Ext}^1_R(G(P), Y) \cong \operatorname{Ext}^1_R(G(H(W')), G(H(Y))) \cong \operatorname{Ext}^1_R(W', Y) = 0$, because $G \circ H$ is naturally equivalent to the identity functor on torsion modules and $Y \in \mathcal{T}$. Thus, $\operatorname{Hom}_R(X, Y)$ is isomorphic to the cokernel of the induced homomorphism $\operatorname{Hom}_R(G(P), Y) \to \operatorname{Hom}_R(G(P'), Y)$.

But, from our starting short exact sequence of S-modules, we obtain, by applying $\operatorname{Hom}_S(-, H(Y))$, the following exact sequence in R-Mod:

$$0 \to \operatorname{Hom}_S(P, H(Y)) \to \operatorname{Hom}_S(P', H(Y)) \to \operatorname{Ext}^1_S(H'(X), H(Y)) \to 0$$

since $\operatorname{Hom}_S(H'(X), H(Y)) = 0$. Now, $G(P) = W \otimes_S P$ and $H(Y) = \operatorname{Hom}_R(W, Y)$, so that there is a natural isomorphism $\operatorname{Hom}_R(G(P), Y) \cong \operatorname{Hom}_S(P, H(Y))$, and similarly for P'. This shows that $\operatorname{Ext}^1_S(H'(X), H(Y)) \cong \operatorname{Hom}_R(X, Y)$, as was to be seen. ∎

We will make frequent use of results in [17]. For the convenience of the reader, we collect some of them in a separate proposition.

Proposition 2.3 ([17, Proposition 2.1, Theorems 3.1 and 3.9]). *Let R_B be a ring of the form* (1).

(a) *R_B is left pure semisimple if and only if R_B*-mod *has a strong preinjective partition.*

(b) *If R_B is left pure semisimple, then there is a well-ordering of the set R_B-ind of indecomposable left R_B-modules giving $X_0 = E_0$, $X_1, \ldots, X_{\delta+1} = P_0$ for some ordinal δ, and such that $\alpha < \beta$ if and only if $\operatorname{Hom}_{R_B}(X_\alpha, X_\beta) = 0$. In that case:*

(b.1) *Each endomorphism ring $\operatorname{End}_{R_B}(X_\alpha)$ is a division ring.*

(b.2) *For each α such that $0 \leq \alpha$ and $\alpha + 2 \leq \delta + 1$, there is an almost split sequence $0 \to X_{\alpha+2} \to X_{\alpha+1}^k \to X_\alpha \to 0$.*

(b.3) *For each $0 \leq \alpha < \delta + 1$, the module $X_\alpha \oplus X_{\alpha+1}$ is a tilting module and $\operatorname{End}_{R_B}(X_\alpha \oplus X_{\alpha+1})$ is again a left pure semisimple ring of the form (1). Conversely, if M is a basic tilting left R_B-module, then $M \cong X_\alpha \oplus X_{\alpha+1}$ for some α.*

Our first purpose is to determine when the left artinian ring $R_B = \left[\begin{smallmatrix} F & 0 \\ B & G \end{smallmatrix}\right]$ has some properties similar to those in Proposition 2.3(b). We will see that the existence of tilting modules is crucial for having these properties in a more general setting.

Definition 2.4. Let the ring $R_B = \left[\begin{smallmatrix} F & 0 \\ B & G \end{smallmatrix}\right]$ be left artinian. A basic tilting module $M \oplus N$ will be called a *rigid tilting module* if the endomorphism rings of M, N are division rings, $\operatorname{Hom}_{R_B}(M, N) = 0$ and the left dimension of $\operatorname{Hom}_{R_B}(N, M)$ (over $\operatorname{End}_{R_B}(N)$) is finite and ≥ 1.

By convention, when we say that $M \oplus N$ is a rigid tilting module, the order is such that $\operatorname{Hom}_{R_B}(M, N) = 0$ (and not the other way round). If S is the endomorphism ring of a rigid tilting module W, then S is again of the form (1), left artinian and hereditary. Consequently, the torsion theory $(\mathcal{T}, \mathcal{F})$ defined on R_B-Mod by W is splitting [4, Lemma 4.5]. Note that $P_1 \oplus P_0$ is a projective rigid tilting module. On the other hand, when R_B is left pure semisimple, every basic tilting module is rigid by Proposition 2.3. In the following result, the projective tilting module has to be excluded.

Proposition 2.5. *Let $R_B = \left[\begin{smallmatrix} F & 0 \\ B & G \end{smallmatrix}\right]$ and let $W = M \oplus N$ be a rigid tilting module with $d = \operatorname{l.dim}(\operatorname{Hom}_{R_B}(N, M)) > 0$ and such that M is not projective. Then there is an indecomposable module K and an almost split short exact sequence*

$$0 \to K \to N^d \to M \to 0.$$

Moreover, $S = \operatorname{End}_{R_B}(W)$ is left artinian and hereditary, and $\operatorname{End}_{R_B}(K) \cong \operatorname{End}_{R_B}(M)$.

Proof. The torsion theory $(\mathcal{T}, \mathcal{F})$ determined by the tilting module W is splitting because the endomorphism ring S of W is a left artinian ring of the form (1) by our hypothesis that $\operatorname{Hom}_{R_B}(N, M)$ is finite-dimensional. Hence S is hereditary and the claim follows from [4, Lemma 4.5].

With the notation above for tilting modules, we have $H(M) = P_1'$ and $H(N) = P_0'$ (respectively, the non-simple and simple projective indecomposable left S-modules). We first show that the homomorphism $h : N^d \to M$ induced by a basis $\{f_1, \ldots, f_d\}$ of $\mathrm{Hom}_{R_B}(N, M)$ is an epimorphism.

Suppose, to the contrary, that $C = \mathrm{Coker}(h) \neq 0$ and $X = \mathrm{Im}(h)$, so that there is a short exact sequence

$$0 \to X \to M \to C \to 0$$

and an epimorphism $N^d \to X$. Since $X, C \in \mathcal{T}$, we get the exact sequence in S-Mod

$$0 \to H(X) \to H(M) = P_1' \to H(C) \to H'(X) = 0$$

where $H(C) \neq 0$ and the homomorphism $H(h) : (P_0')^d \to P_1'$ factors through the monomorphism $H(X) \to P_1'$.

On the other hand, since $H(f_1), \ldots, H(f_n)$ form a basis of $\mathrm{Hom}_S(P_0', P_1')$ by equivalence, we infer that $H(h)$ is a monomorphism that gives a short exact sequence

$$0 \to (P_0')^d \to P_1' \to E_0' \to 0$$

where E_0' is the simple injective left S-module, because the d-vector of P_1' is $(1, d)$. By comparing with the previous sequence, we see that $(P_0')^d \cong H(X)$ and $E_0' \cong H(C)$. Thus the simple injective S-module belongs to $\mathcal{Y}$. This means that $\mathcal{X}$ is trivial and so $\mathcal{F}$ is trivial too, and all left R_B-modules are generated by M and N. Hence M and N are projective modules, contrary to the hypothesis about M.

This shows that $h : N^d \to M$ is an epimorphism. Thus we have a short exact sequence

$$0 \to K \to N^d \xrightarrow{h} M \to 0$$

with $K = \mathrm{Ker}(h)$, and h is not split. We claim that $K \in \mathcal{F}$. Since $(\mathcal{T}, \mathcal{F})$ is splitting, it will suffice to check that no indecomposable direct summand K_0 of K is torsion. So, assume that some K_0 is in $\mathcal{T}$. This entails that $\mathrm{Ext}^1_{R_B}(M, K_0) = 0$ and, by the above short exact sequence, K_0 is isomorphic to a direct summand of N^d, hence is isomorphic to N. If we delete this summand K_0, we obtain a factorization of $h : N^d \to M$ through $h_0 : N^{d-1} \to M$. But then every homomorphism $N \to M$ could be factored through h_0, which contradicts the hypothesis that d is the left dimension of $\mathrm{Hom}_{R_B}(N, M)$. This shows that $K \in \mathcal{F}$, and thus $H(K) = 0$.

By applying the functor $\mathrm{Hom}_{R_B}(W, -)$ to the epimorphism h, we get a short exact sequence of left S-modules

$$0 \to H(N)^d = (P_0')^d \to H(M) = P_1' \to H'(K) \to 0.$$

But it has been observed in the first part of this proof that $H(h)$ gives the exact sequence $0 \to (P_0')^d \to P_1' \to E_0' \to 0$, and hence $H'(K) \cong E_0'$ and $H'(K)$ is indecomposable. By equivalence, K is indecomposable.

We next show that $\mathrm{Hom}_{R_B}(K, Z) = 0$ for every indecomposable module Z such that $Z \ncong K$ and $Z \in \mathcal{F}$. Let $g : K \to Z$ be such that $Z \in \mathcal{F}$. Then $H'(g) : H'(K) \cong E'_0 \to H'(Z)$ is zero, or otherwise $H'(Z) \cong E'_0$, so that either $H'(g)$ is 0 or it is an isomorphism. Since H' is an equivalence, the claim follows immediately.

Back to our short exact sequence $0 \to K \xrightarrow{u} N^d \xrightarrow{h} M \to 0$, we show now that u is a left almost split map of R_B-mod. To this end, take a non-zero and non-split homomorphism $f : K \to X$ where we may assume that X is indecomposable and torsion. By completing the pushout, we obtain a short exact sequence $0 \to X \to Y \to M \to 0$ which has to be split, since $\mathrm{Ext}^1_{R_B}(M, X) = 0$. Therefore, we obtain a homomorphism $g : N^d \to X$ with $f = g \circ u$, so f factors through u.

We now define a ring homomorphism $\mathrm{End}_{R_B}(M) \to \mathrm{End}_{R_B}(K)$ in a natural way: given $\alpha : M \to M$, the composition $\alpha \circ h : N^d \to M$ is determined by d homomorphisms $N \to M$ which can be factored as $N \to N^d \xrightarrow{h} M$, because h is constructed from a generating set of $\mathrm{Hom}_{R_B}(N, M)$. So, $\alpha \circ h$ can be factored through h, giving $\beta : N^d \to N^d$ satisfying $\alpha \circ h = h \circ \beta$.

We observe that β is unique satisfying this commutativity relation, as two different β's would give a homomorphism $N^d \to K$; but $\mathrm{Hom}_{R_B}(N, K) = 0$ because N is torsion and K is torsionfree. Then this homomorphism β determines a unique $\gamma : K \to K$ such that $u \circ \gamma = \beta \circ u$. We set $\alpha \mapsto \gamma$; it is easily seen that this is an injective ring homomorphism.

We must check that it is surjective. Take an endomorphism $f : K \to K$. Since u is a left almost split map, f can be extended to some $\beta : N^d \to N^d$ so that $u \circ f = \beta \circ u$. But β clearly induces a homomorphism $\alpha : M \to M$ such that $\alpha \circ h = h \circ \beta$. By the construction of the ring homomorphism, $\alpha \mapsto f$ and hence the homomorphism is an isomorphism, and $\mathrm{End}_{R_B}(K) \cong \mathrm{End}_{R_B}(M)$.

Since $\mathrm{End}_{R_B}(K)$ is a division ring, the left almost split map u is left minimal, and $0 \to K \to N^d \to M \to 0$ is an almost split sequence. ∎

From the preceding result and [7, Proposition V.1.14], we identify the kernel K as $D(\mathrm{Tr}(M))$, where Tr denotes the usual transpose operator (see [1, p. 356]), and $D(X) = \mathrm{Hom}_E(X, C)$ is the local dual of X; here C is a minimal injective cogenerator of Mod-E, E being the endomorphism ring of X. We remark that the above proof also shows that, under the stated hypotheses, $H'(K)$ is isomorphic to the simple injective left S-module E'_0, and that if $\mathrm{Hom}_{R_B}(K, Z) \neq 0$ and $Z \in \mathcal{F}$, then Z has a direct summand isomorphic to K. Also, we have seen that $H(N)$ is isomorphic to the simple projective left S-module P'_0.

We recall that a left R-module X is *endofinite* when X is finitely generated as a right module over its endomorphism ring $E = \mathrm{End}_R(X)$. This is equivalent to $\mathrm{Hom}_R(A,X)$ being finitely generated as a right E-module for each finitely presented left R-module A.

PROPOSITION 2.6. *Let R_B and $W = M \oplus N$ be as in Proposition 2.5. Then both $K = D(\mathrm{Tr}(M))$ and N are endofinite modules.*

Proof. If $N = P_0$, then $\mathrm{Ext}^1_{R_B}(M,N) = 0$ implies that M is projective, contrary to the hypothesis, so that this is impossible. If $N = P_1$, then the only torsionfree modules (in the theory $(\mathcal{T},\mathcal{F})$ determined by the tilting module W) are the direct sums of copies of P_0 so that $K = P_0$. But then K is trivially endofinite.

Assume now that N is not projective. Observe that the projective indecomposable modules P_0, P_1 are both torsionfree, and that $\mathrm{Hom}_{R_B}(P_i,K) \cong \mathrm{Hom}_S(H'(P_i),H'(K))$, where $S = \mathrm{End}_{R_B}(W)$. But $H'(K)$ is the simple injective left S-module E'_0 by the remark following Proposition 2.5. Now, E'_0 is clearly endofinite in S-Mod and hence $\mathrm{Hom}_S(X,E'_0)$ is right finitely generated, for any finitely generated left S-module X. In particular, $\mathrm{Hom}_{R_B}(P_i,K)$ is finitely generated as a right module over $\mathrm{End}_{R_B}(K)$, and this implies that K is endofinite.

Next, let X be any finitely presented torsionfree left R_B-module and consider $\mathrm{Hom}_{R_B}(X,N)$. By Lemma 2.2, $\mathrm{Hom}_{R_B}(X,N) \cong \mathrm{Ext}^1_S(H'(X),H(N))$, and $H(N)$ is the simple projective module P'_0 over the ring $S = \mathrm{End}_{R_B}(W)$, which is of the form (1). So, it will be enough to show that over a left artinian ring $R_B = \left[\begin{smallmatrix} F & 0 \\ B & G \end{smallmatrix}\right]$, $\mathrm{Ext}^1_S(A,P_0)$ is finitely generated as a right $\mathrm{End}_S(P_0)$-module, for every finitely generated left S-module A.

To this end, we note that there is a short exact sequence $0 \to P_0^k \to P_1^r \to A \to 0$. Then we get another exact sequence over $\mathrm{End}_S(P_0)$: $0 = \mathrm{Hom}_S(P_1^r,P_0) \to \mathrm{Hom}_S(P_0^k,P_0) \to \mathrm{Ext}^1_S(A,P_0) \to 0$. Now $\mathrm{Hom}_S(P_0^k,P_0)$ is finitely generated as a right $\mathrm{End}_S(P_0)$-module, and so is $\mathrm{Ext}^1_S(A,P_0)$, by the above isomorphism. ∎

In the proof of Proposition 2.5 we showed that $\mathrm{End}_{R_B}(K) \cong \mathrm{End}_{R_B}(M)$ by constructing, from the almost split sequence $0 \to K \to N^d \to M \to 0$, injective ring homomorphisms $\mathrm{End}_{R_B}(M) \to \mathrm{End}_{R_B}(N^d)$ and $\mathrm{End}_{R_B}(K) \to \mathrm{End}_{R_B}(N^d)$ whose images coincide.

A rigid tilting module $W = M \oplus N$ determines the bimodule $B_W = \mathrm{Hom}_{R_B}(N,M)$ and the ring $R_{B_W} = \left[\begin{smallmatrix} E & 0 \\ B_W & H \end{smallmatrix}\right]$ where $E = \mathrm{End}_{R_B}(M)$ and $H = \mathrm{End}_{R_B}(N)$. If $W = {}_{R_B}R_B$, then $B_W = B$.

LEMMA 2.7. *Let $R_B = \left[\begin{smallmatrix} F & 0 \\ B & G \end{smallmatrix}\right]$ be left artinian, let $W = M \oplus N$ be a rigid tilting module such that M is not projective, and let $K = D(\mathrm{Tr}(M))$. We write $B_W^* = \mathrm{Hom}_H(B_W,H)$. Then $B_W^* \cong \mathrm{Hom}_{R_B}(K,N)$ as E-H-bimodules.*

Proof. By Proposition 2.5 there is an almost split sequence $0 \to K \to N^d \to M \to 0$. This gives an isomorphism $\mathrm{Hom}_{R_B}(N^d, N) \cong \mathrm{Hom}_{R_B}(K, N)$, as $\mathrm{Hom}_{R_B}(M, N) = \mathrm{Ext}^1_{R_B}(M, N) = 0$. It can be checked that this isomorphism is an E-H-bimodule isomorphism, where the left structure of $\mathrm{Hom}_{R_B}(N^d, N)$ comes by restriction of scalars from the ring homomorphism $E = \mathrm{End}_{R_B}(M) \to \mathrm{End}_{R_B}(N^d)$. In fact, N^d is in this way a right E-module.

There is also a natural isomorphism of bimodules $B_W \cong \mathrm{Hom}_{R_B}(N, N^d)$, due to the fact that the sequence $0 \to K \to N^d \to M \to 0$ is almost split and $\mathrm{Hom}_{R_B}(N, K) = 0$ because K belongs to $\mathcal{F}$ (here, the right structure of $\mathrm{Hom}_{R_B}(N, N^d)$ comes again from the structure of N^d). Thus $N \otimes_H B_W \cong N \otimes_H \mathrm{Hom}_{R_B}(N, N^d) \cong N \otimes_H H^d \cong N^d$. Therefore $B_W^* = \mathrm{Hom}_H(B_W, \mathrm{Hom}_{R_B}(N, N)) \cong \mathrm{Hom}_{R_B}(N \otimes_H B_W, N) \cong \mathrm{Hom}_{R_B}(N^d, N)$ and we get the bimodule isomorphism $B_W^* \cong \mathrm{Hom}_{R_B}(K, N)$. ∎

PROPOSITION 2.8. *Let* $R_B = \left[\begin{smallmatrix} F & 0 \\ B & G \end{smallmatrix}\right]$ *be left artinian, let* $W = M \oplus N$ *be a rigid tilting module such that* M *is not projective, and let* $K \cong D(\mathrm{Tr}(M))$. *Then* $N \oplus K$ *is again a tilting module. It is a rigid tilting module if and only if* B_W^* *is left finite-dimensional.*

Proof. Let $(\mathcal{T}, \mathcal{F})$ be the splitting torsion theory determined by W (see [4, Lemma 4.5]). We will show that the class of all left R_B-modules generated by $W' = N \oplus K$ coincides with the class of left R_B-modules X that are right perpendicular to W' (i.e., such that $\mathrm{Ext}^1_{R_B}(W', X) = 0$). By [10, Proposition 1.3], this proves that W' is tilting.

Since N generates M by Proposition 2.5, every module in the class $\mathcal{T}$ is generated by W'. On the other hand, if $X \in \mathcal{F}$ is generated by W', then X is K-generated, as $\mathrm{Hom}_{R_B}(N, X) = 0$. But our observation after Proposition 2.5 means that in this case X is isomorphic to a direct sum of copies of K. Therefore, the modules generated by W' are the direct sums of a module in $\mathcal{T}$ plus a direct sum of copies of K.

If X is such that $\mathrm{Ext}^1_{R_B}(N, X) = 0$, then the exactness of the sequence $0 \to K \to N^d \to M \to 0$ implies that $\mathrm{Ext}^1_{R_B}(K, X) = 0$ and thus all modules in $\mathcal{T}$ are right perpendicular to $N \oplus K = W'$. Also $\mathrm{Ext}^1_{R_B}(N, K) = \mathrm{Ext}^1_{R_B}(K, K) = 0$, according to [2, Corollary 1.4]. Therefore, all the modules generated by W' are right perpendicular to W'.

Finally, if $X \neq 0$ is any module in $\mathcal{F}$ without direct summands isomorphic to K, then $\mathrm{Hom}_{R_B}(K, X) = 0$ so that the induced sequence $0 \to \mathrm{Ext}^1_{R_B}(M, X) \to \mathrm{Ext}^1_{R_B}(N^d, X) \to \mathrm{Ext}^1_{R_B}(K, X) \to 0$ is exact. Thus if $\mathrm{Ext}^1_{R_B}(N, X) = 0$, then $\mathrm{Ext}^1_{R_B}(M, X) = 0$ and X would belong to $\mathcal{T}$. Therefore, $\mathrm{Ext}^1_{R_B}(N, X) \neq 0$ and X is not right perpendicular to W'. This completes the proof that W' is tilting.

We already saw that $\mathrm{End}_{R_B}(K)$ is a division ring, and so is $\mathrm{End}_{R_B}(N)$. We also have $\mathrm{Hom}_{R_B}(N,K)=0$ because $K\in\mathcal{F}$. Therefore, $N\oplus K$ is a rigid tilting module if and only if $\mathrm{Hom}_{R_B}(K,N)$ is finite-dimensional over $\mathrm{End}_{R_B}(K)$. By Lemma 2.7, this happens if and only if B_W^* is left finite-dimensional. ∎

The preceding result lends significance to the following definition, in which we follow the terminology in [27] and [31].

Definition 2.9. Let B be a left finite-dimensional G-F-bimodule. Let us consider the sequence $B^*=\mathrm{Hom}_G(B,G)$, $B^{**}=\mathrm{Hom}_F(B^*,F),\dots,$ $B^{(k*)},\dots$ of left dualizations, which are again bimodules over the rings G, F. We say that B *has the left finite-dimension property* when each $B^{(k*)}$ is left finite-dimensional.

Proposition 2.10. *Let* $R_B=\left[\begin{smallmatrix}F&0\\B&G\end{smallmatrix}\right]$ *be left pure semisimple. Then* B *has the left finite-dimension property. More generally, if* R_B *is left pure semisimple and* $W=M\oplus N$ *is a rigid tilting module, then the associated bimodule* B_W *has the left finite-dimension property.*

Proof. If $W=M\oplus N$ is a rigid tilting module, then B_W is left finite-dimensional by definition. Moreover, if W is not projective and $K=D(\mathrm{Tr}(M))$, then $W'=N\oplus K$ is again a rigid tilting module and $B_{W'}=B_W^*$ is left finite-dimensional, by Lemma 2.7, Proposition 2.8 and [13, Corollary 3.13]. Inductively, we see that $B_W^{**}=B_{W'}^*,\dots,$ are all left finite-dimensional, hence B_W has the left finite-dimension property.

If W is the projective tilting module, then $B_W=B$ and B^* is left finite-dimensional because the ring R has a left Morita duality (see [31, Proposition 2.4]), and $W'=E_0\oplus E_1$ is obviously a rigid tilting module with $B_{W'}=B^{**}$. The first case then applies to show that B has the left finite-dimension property. ∎

We have seen in Proposition 2.3 that if R_B is left pure semisimple, then all indecomposable and non-simple left modules are totally ordered by the relation: $X<Y$ if and only if $\mathrm{Hom}_{R_B}(X,Y)=0$. We see next that, more generally, rigid tilting modules determine a similar ordering for a set of finitely presented indecomposable modules in the torsionfree class $\mathcal{F}$.

Proposition 2.11. *Let* $R_B=\left[\begin{smallmatrix}F&0\\B&G\end{smallmatrix}\right]$ *be left artinian and* $W=M\oplus N$ *a rigid tilting module such that* B_W *has the left finite-dimension property. If* $(\mathcal{T},\mathcal{F})$ *is the splitting torsion theory of* R_B-Mod *determined by* W*, then there is a (uniquely determined) sequence* $X_0,X_1,X_2,\dots$ *of finitely presented indecomposable modules, such that:*

(i) $X_0=M$ *and* $X_1=N$*;*

(ii) *for $k \geq 1$, if the set $\mathcal{S}_k$ of indecomposable finitely presented modules of $\mathcal{F}$ which are not isomorphic to any of the modules $X_2, \dots, X_k$ is not empty, then X_{k+1} is the only element in $\mathcal{S}_k$ such that $A \in \mathcal{S}_k$ and $\operatorname{Hom}_{R_B}(X_{k+1}, A) \neq 0$ imply $A = X_{k+1}$.*

If there is a smallest $k \geq 0$ such that X_k is projective, then X_{k+1} is the simple projective, $\mathcal{S}_{k+1}$ is empty and the sequence is finite. Otherwise the sequence is infinite. Moreover, for each index $k \geq 0$, $X_k \oplus X_{k+1}$ is a rigid tilting module and $X_{k+2} \cong D(\operatorname{Tr}(X_k))$.

In particular, if $R_B = \left[\begin{smallmatrix} F & 0 \\ B & G \end{smallmatrix}\right]$ is left artinian and B has the left finite-dimension property, then $W = E_0 \oplus E_1$ is a rigid tilting module, $B_W = B^{**}$ has the left finite-dimension property, and the sequence defined as above by this tilting module is formed precisely by all the preinjective indecomposable modules.

Proof. As induction hypothesis, we assume that $W_k = X_k \oplus X_{k+1}$ is a rigid tilting module and B_{W_k} has the left finite-dimension property, which is our hypothesis for $k = 0$. Furthermore, we also assume that the torsion pair $(\mathcal{T}_k, \mathcal{F}_k)$ determined by W_k is such that $\mathcal{F}_k \subseteq \mathcal{F}$ and the finitely presented indecomposable modules in $\mathcal{F}$ that do not belong to $\mathcal{F}_k$ are precisely $X_0, X_1, \dots, X_{k+1}$. We must see that, by choosing $X_{k+2} = D(\operatorname{Tr}(X_k))$, this module has the stated property, and moreover $W_{k+1} = X_{k+1} \oplus X_{k+2}$ and the torsion pair $(\mathcal{T}_{k+1}, \mathcal{F}_{k+1})$ fulfill the same conditions above.

Proposition 2.5 shows that $\operatorname{Hom}_{R_B}(X_{k+2}, A) \neq 0$ for $A \in \mathcal{F} \setminus \mathcal{S}_{k+1}$ implies $X_{k+2} = A$. Suppose, on the other hand, that $Y \in \mathcal{F}_k$ is such that $\operatorname{Hom}_{R_B}(Y, X_{k+2}) = 0$. Then $\operatorname{Ext}^1_{R_B}(X_k, Y) = 0$ by [2, Corollary 1.4]. If $S = \operatorname{End}_{R_B}(W_k)$, we deduce by Lemma 2.1 that $\operatorname{Hom}_S(H(X_k), H'(Y)) = 0$. But $H(X_k)$ is the non-simple projective indecomposable left S-module and this implies that $H'(Y) = 0$, which is a contradiction. This proves the uniqueness of X_{k+2} relative to the stated condition.

Next, W_{k+1} is a rigid tilting module by Proposition 2.8 and $B_{W_{k+1}} = (B_{W_k})^*$ by Lemma 2.7, hence it has the left finite-dimension property. The fact that $\mathcal{F}_{k+1}$ has the same finitely presented indecomposable modules as $\mathcal{F}_k$ except for X_{k+2} is given in the proof of Proposition 2.8. This completes the induction.

Finally, if B has the left finite-dimension property, then so does $B^{**} \cong \operatorname{Hom}_{R_B}(E_1, E_0)$. It is clear that $W = E_0 \oplus E_1$ is a rigid tilting module and $B_W = B^{**}$, so we may apply the result to get a sequence $X_0, X_1, \dots$ of preinjective modules. On the other hand, we have just seen that if $Y \in \mathcal{F}$ is finitely presented indecomposable but is not isomorphic to any of the modules $X_0, \dots, X_k$, then $\operatorname{Hom}_{R_B}(Y, X_j) \neq 0$ for any $j \leq k$. This shows that in case the above sequence is infinite and Y is not a member of the sequence, then Y is not preinjective and thus the sequence consists of all preinjective modules. If the sequence is finite, then the number of finitely presented indecomposable modules is finite and all these modules are preinjective. ∎

3. The sequence of d-vectors. When the ring $R_B = \left[\begin{smallmatrix} F & 0 \\ B & G \end{smallmatrix}\right]$ is such that B has the left finite-dimension property, the existence of enough reflection functors has allowed Simson [33, 34] to relate the d-vectors of the preinjective left R_B-modules to the sequence of the left dimensions of the successive dualizations of the bimodule B. Our purpose in this section is to use rigid tilting modules for the study of this relationship even for modules that are not preinjective.

LEMMA 3.1. *Let $M \oplus N$ be a rigid tilting module such that M is not projective. Suppose that $(t, s), (t_\tau, s_\tau)$ are the d-vectors of M, $D(\mathrm{Tr}(M))$ respectively, and (t', s') is the d-vector of N. Then for $d = \mathrm{l.dim}(\mathrm{Hom}_{R_B}(N, M))$ we have*

$$t + t_\tau = dt', \qquad s + s_\tau = ds'.$$

Proof. This is straightforward from Proposition 2.5. ∎

We make the overall assumption that the G-F-bimodule B has the left finite-dimension property. By Proposition 2.11, the preinjective indecomposable finitely presented left R_B-modules can be written as $M_0, M_1, M_2, \ldots$ so that: $\mathrm{Hom}_{R_B}(M_k, M_j) = 0$ if $k < j$; each module $M_k \oplus M_{k+1}$ is a rigid tilting module; each endomorphism ring $\mathrm{End}_{R_B}(M_k)$ is a division ring; and $M_{k+2} \cong D(\mathrm{Tr}(M_k))$.

Moreover, we are going to see how to obtain the d-vectors (t_k, s_k) of each preinjective module X_k from the sequence of the dimensions $d_k = \mathrm{l.dim}(B^{(k+2)*})$. We add to these dimensions the numbers $d = \mathrm{l.dim}(B)$ and $d^* = \mathrm{l.dim}(B^*)$.

In order to state the result, we make a construction that is analogous to that of continued fractions. Let $a, a_1, a_2, \ldots, a_k$ be positive integers and $b \neq 0$ a rational number. We define

$$[a, b] := a - \frac{1}{b}, \qquad [a_1, \ldots, a_k] := [a_1, [a_2, \ldots, a_k]], \quad \text{if } [a_2, \ldots, a_k] \neq 0.$$

We have $d_k = \mathrm{l.dim}(B^{(k+2)*}) = \mathrm{l.dim}(\mathrm{Hom}_{R_B}(M_{k+1}, M_k))$ (see Lemma 2.7). Then, we may define recursively the sequences p_n and q_n thus:

$$p_0 = 1, \qquad q_0 = 0, \qquad p_1 = d^*, \qquad q_1 = 1$$

and

$$p_{n+2} = d_n p_{n+1} - p_n, \qquad q_{n+2} = d_n q_{n+1} - q_n.$$

In view of Lemma 3.1 and Proposition 2.11, and the values of the d-vectors for E_0, E_1, it is clear that $(p_k, q_k) = (t_k, s_k)$ is the d-vector of M_k, following the order of the preinjective modules given in Proposition 2.11.

Lemma 3.2. *Let $R_B = \left[\begin{smallmatrix} F & 0 \\ B & G \end{smallmatrix}\right]$, assume that B has the left finite-dimension property, and suppose that R_B is not of finite representation type. Then for any $k \geq 0$, the value $[d^*, d_0, d_1, \ldots, d_k]$ is defined and non-zero, and*

$$[d^*, d_0, d_1, \ldots, d_k] = \frac{p_{k+2}}{q_{k+2}}.$$

Proof. By induction, we assume that, as B^* has the left finite-dimension property and $R_{B^*} = \left[\begin{smallmatrix} G & 0 \\ B^* & F \end{smallmatrix}\right]$ is not of finite representation type, the value $[d_0, \ldots, d_k]$ is defined and non-zero, and equals u_{k+1}/v_{k+1}, where (u_j, v_j) are given as (p_n, q_n) but from the sequence for B^*. Consequently, $[d^*, d_0, \ldots, d_k]$ is defined and the equation of the statement follows in the same way as the corresponding property of continued fractions (see, e.g., [25]). Finally, $[d^*, d_0, \ldots, d_k] \neq 0$ because the d-vector (p_{k+2}, q_{k+2}) has a positive ratio. ∎

Of course, if $R_B = \left[\begin{smallmatrix} F & 0 \\ B & G \end{smallmatrix}\right]$ is of finite representation type, then there is some n such that $[d^*, d_0, \ldots, d_n] = 0$, and the above relation holds only for $k < n$. In that case, there are exactly $n + 1$ non-isomorphic indecomposable left modules.

This result relates the sequence of dimensions of the dualizations of B to the d-vectors of the sequence of indecomposable finitely presented preinjective modules (the same was done by Simson [33–35] through other means). We now collect other properties which are also easily proven in an analogous way to the corresponding properties of continued fractions.

Lemma 3.3. *With the notation and hypotheses of Lemma 3.2, the following properties hold for each $k \geq 0$:*

(1) $t_k s_{k+1} - t_{k+1} s_k = 1$. *Consequently,* $\gcd(t_k, s_k) = 1$.
(2) $t_k/s_k - t_{k+1}/s_{k+1} = 1/s_k s_{k+1}$. *Consequently, the sequence t_k/s_k is strictly decreasing.*

Unless the ring R_B is of finite representation type, we know that the number of indecomposable finitely presented preinjective left R_B-modules is infinite, and hence the sequences t_k, s_k, d_k are infinite. In particular, the sequence t_k/s_k is infinite and monotone, by Lemma 3.3. Moreover, $dt_k > s_k$ as the linear map $B \otimes_F V \to W$ is surjective for any non-projective indecomposable module, and thus $1/d$ is a lower bound for the terms of the sequence. By these conditions, the sequence t_k/s_k has a limit a, so that $1/d \leq a$.

We study the d-vectors of non-preinjective modules in two steps. When W is a rigid tilting module and $S = \mathrm{End}_{R_B}(W)$, a non-preinjective module X may be such that $H(X)$ is preinjective over the ring S, with $H = \mathrm{Hom}_{R_B}(W, -)$. If the bimodule B_W has the left finite-dimension property, then we can get information about the d-vector of $H(X)$ from the first results in this section applied to the ring S. In order to obtain information about the d-vector of X we need to investigate the relationship between the d-vectors of X and of $H(X)$, and this is our aim in the next results.

Proposition 3.4. *Let $W = M \oplus N$ be a rigid tilting left module of the left artinian ring $R_B = \left[\begin{smallmatrix} F & 0 \\ B & G \end{smallmatrix}\right]$, and consider the splitting torsion theory $(\mathcal{T}, \mathcal{F})$ determined by W. If $X \in \mathcal{T}$ is an indecomposable finitely presented left R_B-module and $X \not\cong N, M$, then there is a short exact sequence*

$$0 \to N^j \to M^r \to X \to 0.$$

Proof. We denote by $H = \mathrm{Hom}_{R_B}(W, -)$ and $H' = \mathrm{Ext}^1_{R_B}(W, -)$ the equivalence functors defined by the tilting module W from $\mathcal{T}$ and $\mathcal{F}$ to the subcategories $\mathcal{Y}$ and $\mathcal{X}$ of S-Mod, S being the endomorphism ring of W. We know that, if P'_0, P'_1 are, respectively, the simple projective and the non-simple projective indecomposable left S-modules, then $H(M) \cong P'_1$ and $H(N) \cong P'_0$. Therefore, there is an epimorphism of left S-modules $H(g) : H(M)^r \to H(X)$. We note first that $g : M^r \to X$ is an epimorphism of left R_B-modules. Indeed, if we had an epimorphism $h : X \to Y$ in R_B-mod such that $h \circ g = 0$, then $Y \in \mathcal{T}$ and $H(h \circ g) = H(h) \circ H(g) = 0$, from which it follows that $H(h) = 0$ and hence $h = 0$.

Observe further that $\mathrm{Hom}_{R_B}(M, X) \cong \mathrm{Hom}_S(P'_1, H(X))$, which is clearly left finitely generated. This entails that we may assume that the epimorphism $g : M^r \to X$ is such that every homomorphism $M \to X$ factors through g.

Let $K_0 = \mathrm{Ker}(g)$. Exactness of the sequence $0 \to K_0 \to M^r \to X \to 0$ implies exactness of the sequence

$$\mathrm{Hom}_{R_B}(M, M^r) \to \mathrm{Hom}_{R_B}(M, X) \to \mathrm{Ext}^1_{R_B}(M, K_0) \to \mathrm{Ext}^1_{R_B}(M, M^r) = 0$$

in S-Mod and the first of these homomorphisms is an epimorphism, hence $\mathrm{Ext}^1_{R_B}(M, K_0) = 0$.

Let Z be an indecomposable direct summand of K_0 so that $\mathrm{Ext}^1_{R_B}(M, Z) = 0$. We may assume $Z \not\cong M$ (this can be achieved by selecting a minimal generating set of $\mathrm{Hom}_{R_B}(M, X)$ to construct the epimorphism g) and, as a consequence, $\mathrm{Hom}_{R_B}(M, Z) = 0$ (because a non-zero homomorphism $M \to Z$ composed with $Z \to M^r$ would give a non-zero automorphism of M factoring through Z, which would entail $Z \cong M$). Since $(\mathcal{T}, \mathcal{F})$ is splitting, Z is either torsion or torsionfree. If it is torsionfree, by Lemma 2.1 we have $0 = \mathrm{Ext}^1_{R_B}(M, Z) \cong \mathrm{Hom}_S(H(M), H'(Z)) = \mathrm{Hom}_S(P'_1, H'(Z))$, which entails that $H'(Z) = P'_0 = H(N)$, a contradiction. Therefore, Z must be torsion. Since we have seen at the beginning of this proof that M generates every indecomposable torsion module not isomorphic to N, we conclude that $Z \cong N$. Thus every indecomposable direct summand of K_0 is isomorphic to N, as was to be seen. ∎

Proposition 3.5. *Let M, N, R_B, $\mathcal{T}$, $\mathcal{F}$ be as in Proposition* 3.4 *and assume that M is not projective. Let $X \in \mathcal{F}$ be an indecomposable finitely presented module. Then there is a short exact sequence*

$$0 \to X \to N^k \to M^r \to 0.$$

Proof. Let $k_0 = \mathrm{r.dim}(\mathrm{Hom}_{R_B}(X, N))$, which is finite by Proposition 2.6. Suppose that $Z \subseteq X$ is the kernel of the homomorphism $X \to N^{k_0}$ obtained from a basis of $\mathrm{Hom}_{R_B}(X, N)$. Then any indecomposable direct summand of X/Z which is not isomorphic to N has to belong to $\mathcal{F}$, since every finitely presented torsion module is N-generated and it cannot have a monomorphism to N^{k_0}. This clearly implies that $\mathrm{Ext}^1_{R_B}(X/Z, N) = 0$, and thus the induced homomorphism $\mathrm{Hom}_{R_B}(X, N) \to \mathrm{Hom}_{R_B}(Z, N)$ is an epimorphism. Consequently, $\mathrm{Hom}_{R_B}(Z, N) = 0$.

By Proposition 2.5, every non-zero homomorphism $Z \to M$ has to be a split epimorphism, which is impossible since Z is torsionfree. Therefore, $\mathrm{Hom}_{R_B}(Z, W) = 0$. But we also clearly have $\mathrm{Ext}^1_{R_B}(Z, W) = 0$, as Z is torsionfree. Then, by [9, Proposition 2.1], $Z = 0$, and we deduce that $X \to N^{k_0}$ is a monomorphism.

Consider now the short exact sequence

$$0 \to X \to N^{k_0} \to C \to 0$$

where C is a torsion module. The induced homomorphism $\mathrm{Hom}_{R_B}(N^{k_0}, N) \to \mathrm{Hom}_{R_B}(X, N)$ is an epimorphism, by construction. Since we have exactness of the sequence

$$\mathrm{Hom}_{R_B}(N^{k_0}, N) \to \mathrm{Hom}_{R_B}(X, N) \to \mathrm{Ext}^1_{R_B}(C, N) \to \mathrm{Ext}^1_{R_B}(N^{k_0}, N) = 0$$

this implies that $\mathrm{Ext}^1_{R_B}(C, N) = 0$. By Proposition 3.4, each indecomposable direct summand of C has to be isomorphic to M or N. Bearing in mind that the endomorphism ring of N is a division ring, we may delete, if necessary, the direct summands of C that are isomorphic to N, and thus we obtain the short exact sequence as stated. ∎

We may now relate the d-vectors of indecomposable finitely presented modules X and of the corresponding modules $H(X)$ or $H'(X)$.

Proposition 3.6. *Let W, R_B, $\mathcal{T}$, $\mathcal{F}$ be as in Proposition* 3.4. *Assume that (t_0, s_0) and (t_1, s_1) are the d-vectors of M, N respectively. Let $X \in \mathcal{T}$ be a finitely presented indecomposable module with d-vector (t', s'). Assume that (t, s) is the d-vector of $H(X)$ over the ring $S = \mathrm{End}_{R_B}(W)$. Then*

$$t = \frac{t's_1 - t_1s'}{s_1t_0 - s_0t_1}, \qquad s = \frac{t(dt_1 - t_0) + t'}{t_1}$$

where d is the left dimension of $\mathrm{Hom}_{R_B}(N, M)$.

Proof. If X is either M or N, then $H(N) = P_0'$ is the simple projective left S-module, and $H(M) = P_1'$ is the non-simple projective module. Their d-vectors are, respectively, $(0,1)$ and $(1,d)$, so that the equations are obviously satisfied. Otherwise, we know from Proposition 3.4 that there is a short exact sequence

$$0 \to N^k \to M^r \to X \to 0.$$

This gives the equations $kt_1 + t' = rt_0$, $ks_1 + s' = rs_0$, and we may apply the functor H giving the equivalence associated to the tilting module W. This gives again a short exact sequence of left S-modules

$$0 \to H(N)^k \to H(M)^r \to H(X) \to 0.$$

But $H(M)$, $H(N)$ are the projective indecomposable modules over S. Therefore, by considering their d-vectors over S, we have the equations

$$t = r, \qquad k = rd - s = dt - s.$$

By inserting these results into the previous equations, we get $t' + (dt - s)t_1 = t_0 t$ and $s' + (dt - s)s_1 = s_0 t$. We may compute s from any of these equations. For instance,

$$st_1 = t(dt_1 - t_0) + t', \qquad s = \frac{t(dt_1 - t_0) + t'}{t_1}.$$

Now, if we consider the equation $td - s = (t_0 t - t')/t_1$, and substitute in the second equation above, we obtain the value of t as stated in the proposition, completing the proof. ∎

PROPOSITION 3.7. *Let M, N, R_B, $\mathcal{T}$, $\mathcal{F}$ be as in Proposition* 3.4 *and assume that M is not projective. Let (t_0, s_0) and (t_1, s_1) be the d-vectors of M and N respectively. Let $X \in \mathcal{F}$ be a finitely presented indecomposable left R_B-module with d-vector (t', s'). Assume that (t, s) is the d-vector of $H'(X)$ over the ring $S = \operatorname{End}_{R_B}(W)$. Then*

$$t = \frac{s't_1 - t's_1}{s_1 t_0 - s_0 t_1}, \qquad s = \frac{t(dt_1 - t_0) - t'}{t_1}$$

where d is the left dimension of $\operatorname{Hom}_{R_B}(N, M)$.

Proof. This is analogous to the proof of Proposition 3.6, now using Proposition 3.5. ∎

We show next that these results may be improved by extending to all the indecomposable modules the equation we found for preinjective modules in Lemma 3.3.

LEMMA 3.8. *Let $W = M \oplus N$ be a rigid tilting module over the left artinian ring $R_B = \left[\begin{smallmatrix} F & 0 \\ B & G \end{smallmatrix}\right]$. Let (t_0, s_0) and (t_1, s_1) be the d-vectors of M and N respectively. Then $t_0 s_1 - s_0 t_1 = 1$.*

Proof. If M is projective, the result is obvious, and if only N is projective, this is a direct consequence of Proposition 2.5. For the general situation, we start by computing the d-vectors for the modules $H'(P_0)$ and $H'(P_1)$ over the endomorphism ring S of W.

Let us write (t,s) and $(\bar{t},\bar{s})$ for the d-vectors of $H'(P_0)$, $H'(P_1)$ respectively, and let $d = \mathrm{l.dim}(\mathrm{Hom}_{R_B}(N,M))$. We know that $\mathrm{Hom}_{R_B}(P_0,N)$ and $\mathrm{Hom}_{R_B}(P_1,N)$ are right finitely generated, because N is endofinite, by Proposition 2.6. By Proposition 3.7, we have

$$t = \frac{t_1}{s_1t_0 - s_0t_1}, \qquad s = \frac{dt_1 - t_0}{s_1t_0 - s_0t_1}.$$

Thus $s_1t_0 - s_0t_1$ divides both t_1 and $dt_1 - t_0$, hence it divides both t_0, t_1.

Concerning P_1, its d-vector is $(1,d')$ (if we write d' for the left dimension of B) and thus we have

$$\bar{t} = \frac{d't_1 - s_1}{s_1t_0 - s_0t_1},$$

so that $s_1t_0 - s_0t_1$ is a divisor of s_1. Finally,

$$\bar{s} = \frac{(dt_1 - t_0)(d't_1 - s_1)}{t_1(s_1t_0 - s_0t_1)} - \frac{1}{t_1} = \frac{dd't_1 - ds_1 - d't_0 + s_0}{s_1t_0 - s_0t_1},$$

and thus $s_1t_0 - s_0t_1$ is also a divisor of s_0.

Let us write $u = s_1t_0 - s_0t_1$. We have seen that u divides the four values t_0, s_0, t_1, s_1. Therefore $u^2 \mid t_0s_1$ and $u^2 \mid s_0t_1$, so $u^2 \mid u$. Furthermore we infer, for instance from the above equation for t, that $u > 0$, and hence $u = 1$, as was to be seen. ∎

We may now adapt the foregoing results to this new piece of information.

Proposition 3.9. *Let $W = M \oplus N$ be a rigid tilting module over the ring $R_B = \left[\begin{smallmatrix} F & 0 \\ B & G \end{smallmatrix}\right]$ and assume that M is not projective. Letting $K = D(\mathrm{Tr}(M))$, assume further that $\mathrm{Hom}_{R_B}(K,N)$ is left finite-dimensional. Let (t_0,s_0) and (t_1,s_1) be the d-vectors of N and K respectively. Let X be a finitely presented indecomposable left R_B-module with d-vector (t',s'), such that X is torsion (resp., torsionfree). Assume that (t,s) is the d-vector of $H(X)$ (resp., $H'(X)$). Then*

$$t = t's_0 - t_0s', \qquad s = t's_1 - t_1s'$$

(resp., $t = s't_0 - t's_0$, $s = s't_1 - t's_1$).

Proof. The first equation for t follows directly from Proposition 3.6, on taking into account Lemma 3.8. For the s-equation, we have, from Proposition 3.6 and Lemma 3.1,

$$s = \frac{t_1t + t'}{t_0}.$$

By using now the value of t we have just found,

$$s = \frac{t_1(t's_0 - s't_0) + t'}{t_0} = \frac{t'(t_1 s_0 + 1)}{t_0} - t_1 s',$$

and thus all that is left is to show that $t_1 s_0 + 1 = t_0 s_1$. But this is obvious by Lemma 3.8, since $N \oplus K$ is a rigid tilting module by Proposition 2.8.

The proof of the second part is analogous. ∎

When the ring $R_B = \left[\begin{smallmatrix} F & 0 \\ B & G \end{smallmatrix}\right]$ is left pure semisimple there are enough rigid tilting modules, and we may apply the foregoing results in order to obtain a description of the d-vectors of indecomposable modules. To this end, we need some lemmas.

LEMMA 3.10. *Let $R_B = \left[\begin{smallmatrix} F & 0 \\ B & G \end{smallmatrix}\right]$ be left artinian, and $W = M \oplus N$ a rigid tilting module such that M is not projective. Consider the splitting torsion theory determined by W. Suppose that X, Y are finitely presented indecomposable left R_B-modules which are either both torsion or both torsionfree. Let (t, s), (t', s') be the d-vectors of X, Y respectively and assume $t/s < t'/s'$. For S the endomorphism ring of W, let (u, v), (u', v') be the d-vectors of $H(X)$, $H(Y)$ (or of $H'(X)$, $H'(Y)$) over S. Then $uv' < vu'$.*

Proof. This is a straightforward computation from Propositions 3.6 and 3.7 and Lemma 3.8. ∎

PROPOSITION 3.11. *Let $R_B = \left[\begin{smallmatrix} F & 0 \\ B & G \end{smallmatrix}\right]$ be left artinian and not of finite representation type, and assume that B has the left finite-dimension property. Let $X_0, X_1, \ldots$ be the sequence of preinjective modules, and write the d-vector of X_k as (t_k, s_k). Let a be the limit of the ratios t_k/s_k and let X be an indecomposable finitely presented module with d-vector (t, s) such that $t/s > a$. Then X is preinjective.*

Proof. Suppose, to the contrary, that there exists an indecomposable finitely presented module X with d-vector (t, s) which is not preinjective and such that $t/s > a$. Since we must have $t/s < d^*$ (because every non-injective indecomposable finitely presented module satisfies this inequality) and $t_1/s_1 = d^*$, we see that there is a smallest $k \geq 0$ such that $t/s \geq t_{k+2}/s_{k+2}$. If we choose the rigid tilting module $M_k \oplus M_{k+1}$ with endomorphism ring S, which determines the splitting torsion theory $(\mathcal{T}, \mathcal{F})$, then $M_{k+2} \in \mathcal{F}$ and $X \in \mathcal{F}$ too, as torsion modules are all preinjective.

Over the ring S, the module M_{k+2} gives the simple injective $H'(M_{k+2})$ as M_{k+2} has no non-zero homomorphism to any other torsionfree module, by the construction in Proposition 2.5. Thus it has d-vector $(1, 0)$. By applying Lemma 3.10, we find that if (t', s') is the d-vector of $H'(X)$ then $s' < 0$, a contradiction that proves the result. ∎

LEMMA 3.12. *Let $R_B = \left[\begin{smallmatrix} F & 0 \\ B & G \end{smallmatrix}\right]$ be left artinian and not of finite representation type, and assume that B has the left finite-dimension property. Consider the chain of preinjective modules X_k $(k = 0, 1, \dots)$ with d-vectors (t_k, s_k). Then there is an infinite sequence of positive integers $i_1 < i_2 < \cdots$ such that $s_{i_1} < s_{i_2} < \cdots$ and for any $k = 0, 1, \dots$ and $j > i_k$, we have $s_{i_k} < s_j$.*

Proof. As induction hypothesis, we assume that there is a sequence $i_1 < \cdots < i_r$ satisfying the conditions of the statement. We will show how to choose i_{r+1}.

Consider the set of all indices $j > i_r$ (so that $s_j > s_{i_r}$). Then the set of all integers s_j for those j has a minimum value s and we have $s > s_{i_r}$. Consider now all the preinjective modules X_m with $m > i_r$ and such that $s_m = s$. We note that there are only finitely many such modules. This is because if we have $m_1 > m_2$ and $s_{m_1} = s = s_{m_2}$, then $t_{m_1}/s < t_{m_2}/s$ and hence $t_{m_1} < t_{m_2}$ and the set of possible t-values is finite. So, we may choose the largest possible index m with the property that the d-vector of X_m has $s_m = s$. Then we set $i_{r+1} = m$. This entails that $s_{i_1} < \cdots < s_{i_{r+1}}$; and if $j > i_{r+1}$, then necessarily $s_j > s_{i_{r+1}}$, showing that the conditions hold for the extended sequence. ∎

We now arrive at the last of our preliminary results.

LEMMA 3.13. *Let $R_B = \left[\begin{smallmatrix} F & 0 \\ B & G \end{smallmatrix}\right]$ be left artinian and not of finite representation type, and assume that B has the left finite-dimension property. Consider the chain of preinjective modules X_k $(k = 0, 1, \dots)$ with d-vectors (t_k, s_k). Let a be the limit of the sequence t_k/s_k. Then either there exists a non-preinjective indecomposable finitely presented module with d-vector (t, s) such that $t/s = a$, or else there is an infinite chain of non-preinjective indecomposable finitely presented modules $Y_1, Y_2, \dots$ with d-vectors (u_i, v_i) such that the sequence u_i/v_i is strictly increasing and bounded above by a.*

Proof. Suppose that no finitely presented indecomposable module has d-vector (t, s) with $t/s = a$. Choose any non-preinjective indecomposable finitely presented module Y_1 with d-vector (t, s) so that $t/s \neq a$. It follows from Proposition 3.11 that $t/s < a$. Let us set $\epsilon = a - t/s$, and let X_k be any preinjective module. Since $t_k/s_k > a$, we have $t_k/s_k - t/s > \epsilon$. Therefore $t_k s - s_k t > s_k s \epsilon$.

By Lemma 3.12, there exists k such that $s_k \epsilon > 1$, and moreover, if $j > k$, then $s_j > s_k$. For this k, one has $t_k s - s_k t > s$. On the other hand, consider a maximal submodule L of X_k. Since the quotient must be simple, it is isomorphic to the simple injective whose d-vector is $(1, 0)$. Accordingly, the d-vector of L is $(t_k - 1, s_k)$. We observe that each indecomposable direct summand of L is non-preinjective. This is because such a direct summand cannot be of the form X_i for $i < k$, because $\mathrm{Hom}_{R_B}(X_i, X_k) = 0$. But if we take $j > k$, then $s_j > s_k$ and hence there is no monomorphism $X_j \to X_k$. Consequently, each direct summand of L has a d-vector (u, v) such that $u/v < a$, in view of Proposition 3.11 and our assumption. If we had $u/v \leq t/s$ for all those direct summands, then we would also have $(t_k - 1)/s_k \leq t/s$ and hence $t_k s - s_k t \leq s$, which contradicts our choice of k. Therefore, there is some indecomposable non-preinjective finitely presented module with d-vector (u, v) such that $t/s < u/v < a$. By repeating the argument, we see that we may construct the announced infinite sequence of indecomposable modules. ■

When the ring R_B is left pure semisimple, we know from Proposition 2.3 that the indecomposable left R_B-modules form a unique chain indexed by ordinals, $X_0, X_1, \ldots$, such that $\alpha < \beta$ implies $\mathrm{Hom}_{R_B}(X_\alpha, X_\beta) = 0$. By applying to this case the results in the current section, we obtain the following consequences.

THEOREM 3.14. *Let* $R_B = \left[\begin{smallmatrix} F & 0 \\ B & G \end{smallmatrix}\right]$ *be a left pure semisimple ring which is not of finite representation type, and consider the unique chain of indecomposable left* R_B*-modules* $X_0, X_1, \ldots, X_\rho$ *satisfying* $\mathrm{Hom}_{R_B}(X_\alpha, X_\beta) = 0$ *if* $\alpha < \beta \leq \rho$. *Denote the d-vector of* X_α *as* (t_α, s_α). *Then:*

(i) *For any* $\alpha < \rho$, *we have* $t_\alpha s_{\alpha+1} - s_\alpha t_{\alpha+1} = 1$.
(ii) $\alpha < \beta \leq \rho$ *implies* $t_\alpha / s_\alpha > t_\beta / s_\beta$.
(iii) *If* $\mu < \rho$ *is a limit ordinal and* $\lambda = \mu + \omega$, *then*

$$\frac{t_\lambda}{s_\lambda} = \lim_{k \to \infty} \frac{t_{\mu+k}}{s_{\mu+k}}.$$

Proof. (i) follows by a direct application of Lemma 3.8, since $X_\alpha \oplus X_{\alpha+1}$ is a rigid tilting module.

(ii) follows by induction on β. The induction step is clear by (i) when β is not a limit ordinal, so suppose that $t_\alpha / s_\alpha \leq t_\beta / s_\beta$ for some limit ordinal β and $\alpha < \beta$. Take $W = X_\alpha \oplus X_{\alpha+1}$ and $S = \mathrm{End}_{R_B}(W)$. If H and H' denote the equivalence functors associated to the rigid tilting module W, then the ratio t/s for the preinjective S-module $H'(X_{\alpha+2})$ is greater than the ratio for the non-preinjective S-module $H'(X_\beta)$ by Proposition 3.11. But this contradicts our supposition about the ratio of X_β by Lemma 3.10.

(iii) First, let $a = \lim_{k \to \infty} t_k / s_k$. By (ii), an infinite chain of indecomposable modules with an increasing sequence of ratios t/s cannot exist, hence by Lemma 3.13 there exists a non-preinjective indecomposable module with d-vector (t, s) such that $t/s = a$. By applying (ii) again, we see that this module has to be X_ω.

Let now μ be any non-zero limit ordinal and $\lambda = \mu + \omega$. Take $W = X_\mu \oplus X_{\mu+1}$ and $S = \mathrm{End}_{R_B}(W)$. Then $H'(X_\lambda)$ is the first non-preinjective

S-module in the ordering of the indecomposable modules, and hence its d-vector has a ratio that is the limit of the ratios for the preinjective S-modules $H'(X_{\mu+2+k})$, by the first part of this proof. The result follows from the relationship between the ratios of X_α and $H'(X_\alpha)$ obtained in Proposition 3.9. ∎

4. The weak pure semisimplicity conjecture. Suppose that the ring $R_B = \left[\begin{smallmatrix} F & 0 \\ B & G \end{smallmatrix}\right]$ is left pure semisimple. Then we will consider the chain $\{X_\alpha \mid 0 \le \alpha \le \delta+1\}$ of indecomposable left R_B-modules of Proposition 2.3. Let us write d_α (for $\alpha = 0, 1, \dots, \delta$) to denote the left dimension of $\mathrm{Hom}_{R_B}(X_{\alpha+1}, X_\alpha)$, and also $d^* = d_{\delta+1}$ for the left dimension of B^*. We introduce the following concept.

Definition 4.1. Let $R_B = \left[\begin{smallmatrix} F & 0 \\ B & G \end{smallmatrix}\right]$ be left pure semisimple, and consider the dimensions d_α for $\alpha = 0, 1, \dots, \delta+1$. Then R_B will be called a *sporadic ring* if $d_\alpha > 1$ for any $0 \le \alpha \le \delta+1$.

Any left pure semisimple sporadic ring is a counterexample to the pssC (see [12]). Suppose that a ring R_B of the form (1) is a non-sporadic counterexample to the pssC, with $d_\alpha = 1$ for some $\alpha \le \delta$ (using the notation at the beginning of this section). Thus $W = X_\alpha \oplus X_{\alpha+1}$ is a rigid tilting module with endomorphism ring $S = \left[\begin{smallmatrix} D_1 & 0 \\ B' & D_2 \end{smallmatrix}\right]$ which is still a counterexample to the pssC, and there are corresponding equivalence functors H, H'. Here, D_1, D_2 are division rings, and if P_0, P_1 are the indecomposable projective left S-modules, then $P_0 = H(X_{\alpha+1})$, $P_1 = H(X_\alpha)$, and $B' \cong \mathrm{Hom}_S(P_0, P_1)$ has left dimension 1 so that $B' \cong D_2$. This suggests the following weak version of the pssC, which we shall call the *weak pure semisimplicity conjecture* (briefly, wpssC).

(wpssC) *If the ring*

$$R_G = \left[\begin{smallmatrix} F & 0 \\ G & G \end{smallmatrix}\right] \tag{2}$$

is left pure semisimple, then it is of finite representation type.

Proposition 4.2. *The pure semisimplicity conjecture holds if and only if the weak pure semisimplicity conjecture holds and there do not exist left pure semisimple sporadic rings.*

Proof. One way is obvious. For the converse, if the pssC does not hold but there do not exist left pure semisimple sporadic rings, then there exists a left pure semisimple ring R_B of the form (1) such that $d_\alpha = 1$ for some $\alpha \le \delta+1$ (following the notation at the beginning of this section). If $\alpha = d_{\delta+1}$, then the left dimension of B^* is 1, i.e., the left dimension of $\mathrm{Ext}^1_{R_B}(E_0, P_0)$ is 1, where P_0, E_0 are the simple projective and simple injective modules. If W is any non-projective rigid tilting left R_B-module with endomorphism ring S and equivalence functors H, H', then $H(E_0)$, $H'(P_0)$

are consecutive modules in the ordering of the modules over the left pure semisimple ring S, and the left dimension of $\mathrm{Hom}_S(H(E_0), H'(P_0))$ is 1 by Lemma 2.1. Thus we may assume that $\alpha \leq \delta$ and we have just seen above that this will give a left pure semisimple ring of the form (2) which is not of finite representation type, hence the wpssC does not hold. ∎

In this section we study the wpssC and show that it is equivalent to a property of embeddings of division rings that is a purely linear algebra property. Let us define, for any division ring embedding $F \subseteq G$, a couple of notions.

Lemma 4.3. *Let $F \subseteq G$ be a division ring embedding. Let $m, n \geq 1$ be integers, and A any G-matrix of size $m \times n$. Consider the rings of square matrices $\mathbb{M}_m(F)$ and $\mathbb{M}_n(G)$. Then the sets*

$$\mathbb{M}_n^A(G) = \{M \in \mathbb{M}_n(G) \mid A \cdot M = X \cdot A \text{ for some } X \in \mathbb{M}_m(F)\}$$

and

$$\mathbb{M}_m^A(F) = \{N \in \mathbb{M}_m(F) \mid N \cdot A = A \cdot X \text{ for some } X \in \mathbb{M}_n(G)\}$$

are subrings of $\mathbb{M}_n(G)$ and $\mathbb{M}_m(F)$, respectively.

Proof. This is straightforward. ∎

Lemma 4.4. *Let F, G, A, m, n be as in Lemma 4.3. Suppose that the right column rank of the matrix A is n (i.e., the columns of A are right linearly independent vectors in G^m) and that the rows of A are left F-linearly independent (i.e., they are vectors of G^n that are independent when G^n is viewed as a left F-vector space). Then the map $\mathbb{M}_m^A(F) \to \mathbb{M}_n^A(G)$ which assigns to each matrix $N \in \mathbb{M}_m^A(F)$ the unique G-matrix X such that $NA = AX$, is a ring isomorphism.*

Proof. By hypothesis, the columns of A are right linearly independent, and thus it is clear that X is unique. On the other hand, if $M \in \mathbb{M}_n^A(G)$, then $NA = AM$ for some matrix $N \in \mathbb{M}_m^A(F)$, from which the surjectivity of the map follows. Similarly, this F-matrix N is also unique by the hypothesis on the rows, and thus the map is injective. It is trivially a ring homomorphism. ∎

We are interested in certain bimodules defined for these rings. Let A_1 and A_2 be G-matrices with respective sizes $m_i \times n_i$, i.e., they belong to the $\mathbb{M}_{m_i}(G)$-$\mathbb{M}_{n_i}(G)$-bimodule of matrices $\mathbb{M}_{m_i,n_i}(G)$. Then we define

$$\mathbb{M}_{m_1,m_2}^{A_1,A_2}(F) = \{N \in \mathbb{M}_{m_1,m_2}(F) \mid N \cdot A_2 = A_1 \cdot X \text{ for some } X \in \mathbb{M}_{n_1,n_2}(G)\},$$
$$\mathbb{M}_{n_1,n_2}^{A_1,A_2}(G) = \{M \in \mathbb{M}_{n_1,n_2}(G) \mid A_1 \cdot M = X \cdot A_2 \text{ for some } X \in \mathbb{M}_{m_1,m_2}(F).$$

It easily turns out that $\mathbb{M}_{m_1,m_2}^{A_1,A_2}(F)$ is an $\mathbb{M}_{m_1}^{A_1}(F)$-$M_{m_2}^{A_2}(F)$-bimodule, and $\mathbb{M}_{n_1,n_2}^{A_1,A_2}(G)$ is an $\mathbb{M}_{n_1}^{A_1}(G)$-$\mathbb{M}_{n_2}^{A_2}(G)$-bimodule. Moreover, we have:

LEMMA 4.5. *Let F, G, A_1, A_2 be as above. Suppose that the matrices A_1, A_2 have right column rank n_1, n_2, respectively; and that the rows of each matrix are left F-linearly independent. Then the map $\mathbb{M}^{A_1,A_2}_{m_1,m_2}(F) \to \mathbb{M}^{A_1,A_2}_{n_1,n_2}(G)$ which carries a matrix N in the first bimodule to the unique G-matrix X such that $NA_2 = A_1X$, is a semilinear bimodule isomorphism.*

Proof. This is a routine check. ∎

We now develop a connection between these rings and bimodules of matrices and the modules over the ring $R_G = \left[\begin{smallmatrix} F & 0 \\ G & G \end{smallmatrix}\right]$ of the form (2). We know that each non-zero finitely generated left R_G-module without simple projective direct summands is given through a G-linear surjective map $G \otimes_F V \to W$, where V and W are respectively left F- and G-vector spaces of finite dimension. Therefore, if m, n are the dimensions of V, W respectively, then $n \le m$. Moreover, $n > 0$ if the module is not semisimple injective.

Let us define the category $\mathcal{C}_{G,F}$ whose objects are all the G-matrices of size $m \times n$ such that $0 < n \le m$, whose right column rank is n and such that the rows are left F-linearly independent. As the set of morphisms from one such matrix A to another one B, we take $\mathbb{M}^{A,B}_{m_1,m_2}(F)$. Like this, the endomorphism ring of each object A is $\mathbb{M}^{A}_{m}(F)$. Composition of morphisms is given by matrix multiplication. It is easy to see that this is indeed a category, whose sets of morphisms have compatible abelian group structures.

PROPOSITION 4.6. *Let $F \subseteq G$ be an embedding of division rings such that the left F-dimension of G is finite, and let $R_G = \left[\begin{smallmatrix} F & 0 \\ G & G \end{smallmatrix}\right]$. There is an equivalence between the full subcategory of R_G-*mod *consisting of non-zero modules which have no simple direct summands, and the category $\mathcal{C}_{G,F}$.*

Proof. We define a functor giving the equivalence from a skeleton of the above subcategory of R_G-mod to $\mathcal{C}_{G,F}$. Thus, for each isomorphism class of finitely generated left R_G-modules we choose a module M represented by a surjective G-linear map $h_M : G \otimes F^m \to G^n$, where $0 < n \le m$. We then associate to M the matrix A_M of the linear map h_M relative to the canonical bases. Since h_M is surjective, the matrix A_M has right column rank n. Moreover, if we had some left F-linear dependence relation between the rows of A_M, then the module would have a direct summand isomorphic to the simple injective module E_0, and hence A_M is indeed an object of the category $\mathcal{C}_{G,F}$.

Now, for modules M and N we have $\mathrm{Hom}_{R_B}(M,N)$ identified to the set of pairs (f,g) of linear maps $f : F^{m_1} \to F^{m_2}$ and $g : G^{n_1} \to G^{n_2}$ with the property $h_M \cdot g = (1 \otimes f) \cdot h_N$. This means that $A_M \cdot X = C \cdot A_N$, where C, X are, respectively, the matrices of the maps f, g. Then we associate the matrix C to the homomorphism (f,g). Since the group of morphisms in $\mathcal{C}_{G,F}$

from A_M to A_N is $\mathbb{M}_{m_1,m_2}^{A_M;A_N}(F)$, in this way we clearly have an isomorphism $\operatorname{Hom}_{R_G}(M,N) \cong \operatorname{Hom}_{\mathcal{C}_{G,F}}(A_M, A_N)$. This shows that our functor is full and faithful. On the other hand, any object of $\mathcal{C}_{G,F}$ is a matrix A which can always be interpreted as A_M for a non-zero module M which has no simple direct summands. This shows that the functor is an equivalence. ■

Next, we give a characterization of rings of the form (1) which are left pure semisimple, and we then use the above equivalence to translate the characterization into a linear algebra condition.

THEOREM 4.7. *Let $R_B = \left[\begin{smallmatrix} F & 0 \\ B & G \end{smallmatrix}\right]$ be left artinian. The ring R_B is left pure semisimple if and only if the following conditions hold:*

(i) *$B^* = \operatorname{Hom}_G(B,G)$ is a finite-dimensional left F-module* (*equivalently, the non-simple injective left R_B-module E_1 is finitely presented*).

(ii) *Given any non-empty family $\{M_i \mid i \in I\}$ of finitely generated left R_B-modules such that each M_i is non-zero and without simple direct summands, there exists an index $j \in I$ such that for every $i \in I$ and homomorphism $g : M_i \to E_1$, there exist homomorphisms $h_1, \ldots, h_n : M_i \to M_j$ and $f_1, \ldots, f_n : M_j \to E_1$ such that $g = \sum_{k=1}^n h_k f_k$.*

Furthermore, if R_B is left pure semisimple, then it is a ring of finite representation type if and only if the following condition holds (*here P_1 will denote the non-simple projective indecomposable left R_B-module*).

(iii) *Given any non-empty family $\{M_i \mid i \in I\}$ of finitely generated left R_B-modules such that each M_i is non-zero and without simple direct summands, there exists $j \in I$ such that for every $i \in I$ and homomorphism $g : P_1 \to M_i$, there exist homomorphisms $h_1, \ldots, h_n : M_j \to M_i$ and $f_1, \ldots, f_n : P_1 \to M_j$ such that $g = \sum_{k=1}^n f_k h_k$.*

Proof. We prove the first part of the theorem. Suppose that R_B is left pure semisimple. Since it has a left Morita duality by [31, Proposition 2.4], we know that E_1, being the injective hull of the simple projective P_0, is finitely generated, so condition (i) holds. Suppose now we are given a family $\{M_i \mid i \in I\}$ of finitely generated left R_B-modules which are non-zero and without simple direct summands. Consider the set $\mathcal{C}$ of all indecomposable direct summands of the modules M_i, so that each module M_i belongs to $\operatorname{add}(\mathcal{C})$. If E_1 belongs to $\mathcal{C}$, then the property is obviously true with $M_j = E_1 \oplus N$ and taking $f_1 : M_j \to E_1$ as the canonical projection. If $E_1 \notin \mathcal{C}$, then we know from Proposition 2.3 that there is a smallest ordinal $\beta > 0$ such that $X_\beta \in \mathcal{C}$, where the X_α give the well-ordered set of indecomposable modules. Also, X_β is the only splitting injective module of $\mathcal{C}$ and thus X_β cogenerates every module M_i [8, Theorem 2.3]. We then choose M_j such that $M_j = X_\beta \oplus N$. Now, we take any homomorphism $g : M_i \to E_1$

and a monomorphism $h : M_i \to M_j^n$. We may factor g through h, getting $f : M_j \to E_1$ such that $g = hf$. This proves (ii).

Conversely, suppose that the conditions hold. Then we show that R_B-ind has a strong preinjective partition, so that R_B is left pure semisimple by Proposition 2.3. Consider first the set $\mathcal{C}_0$ of all non-injective indecomposable finitely presented left R_B-modules. By (ii), there is a module $M_1 \in \mathcal{C}_0$ satisfying the condition. Since any indecomposable finitely presented left R_B-module without simple injective direct summands is cogenerated by E_1, we may find a monomorphism $M_i \to E_1^r$ for any module $M_i \in \mathcal{C}_0$. By (ii), this monomorphism can be factored through some direct sum of copies of M_1, and hence M_i is cogenerated by M_1. Applying again [8, Theorem 2.3], we see that M_1 is the only splitting injective module in $\mathrm{add}(\mathcal{C}_0)$. This gives the first steps of the preinjective partition of R_B-ind, that is, $\mathcal{I}_0 = \{E_0, E_1\}$ and $\mathcal{I}_1 = \{M_1\}$.

As inductive step, assume that we have constructed the strong preinjective partition for all $\mathcal{I}_\beta$ with $\beta < \alpha$. As above, we take the set $\mathcal{C}_\alpha$ of all finitely presented indecomposable left R_B-modules not belonging to the sets $\mathcal{I}_\beta$. By hypothesis, there is M_α in this set satisfying condition (ii) of our statement. The same argument above shows that any other indecomposable finitely presented module of $\mathcal{C}_\alpha$ is cogenerated by M_α, and thus M_α is the only splitting injective of $\mathcal{C}_\alpha$; this gives $\mathcal{I}_\alpha = \{M_\alpha\}$. Since we may proceed in this way until we exhaust all isomorphism classes of finitely presented indecomposable modules, we see that there is a strong preinjective partition of R_B-ind, and we are done.

The second part of the theorem can be proved by using dual arguments, since P_1 generates all non-simple modules. ■

Then, we may translate this into a linear algebra condition.

THEOREM 4.8. *Let $F \subseteq G$ be a division ring embedding such that G is left F-finite-dimensional, and let $R_G = \left[\begin{smallmatrix} F & 0 \\ G & G \end{smallmatrix}\right]$. Then R_G is left pure semisimple if and only if the following holds: Given any family $\{A_i \mid i \in I\}$ of G-matrices with respective sizes $m_i \times n_i$ where $1 \le n_i \le m_i$ and such that each A_i has right column rank n_i and left F-linearly independent rows, there exists $j \in I$ such that for each $i \in I$, the canonical map*

$$\mathbb{M}_{n_i,n_j}^{A_i,A_j}(G) \otimes_{\mathbb{M}_{n_j}^{A_j}(G)} G^{n_j} \to G^{n_i}$$

is a surjection.

Proof. Let A be any G-matrix of size $m \times n$ with $1 \le n \le m$ and with right column rank equal to n and with left F-linearly independent rows. Let M be the finitely generated left R_G-module that corresponds to A in the equivalence of Proposition 4.6. Since we have assumed that G is left

F-finite-dimensional, we know that E_1 is finitely generated. It is easy to see that $\mathrm{Hom}_{R_G}(M, E_1) \cong G^n$, with the left structure of G^n obtained through the isomorphism $\mathbb{M}_m^A(F) \cong \mathbb{M}_n^A(G)$ of Lemma 4.4. Also, $\mathrm{Hom}_{R_G}(M_i, M_j) \cong \mathbb{M}_{n_i,n_j}^{A_i,A_j}(G)$ for matrices A_i and A_j as above, in view of Lemma 4.5.

Suppose that R_G is left pure semisimple and we want to check the condition above. Take any family $\{A_i \mid i \in I\}$ of G-matrices as in the statement. The matrices A_i correspond by the equivalence of Proposition 4.6 to a family of finitely generated left R_G-modules M_i without simple direct summands. Theorem 4.7 implies that there exists $j \in I$ with property (ii) of that theorem. Given any $i \in I$ and $H \in G^{n_i}$, we may consider the corresponding $h : M_i \to E_1$ as shown above, and find homomorphisms $g_1, \dots, g_r : M_i \to M_j$ and $f_1, \dots, f_r : M_j \to E_1$ with $h = \sum_{k=1}^r g_k f_k$. Now, each $f_k : M_j \to E_1$ gives naturally a column G-matrix $H_k \in G^{n_j}$. Similarly, each homomorphism g_k corresponds through the equivalence of Proposition 4.6 to a matrix L_k in $\mathbb{M}_{n_i,n_j}^{A_i,A_j}(G)$, and thus we get $\sum_{k=1}^r L_k H_k = H$. This justifies the stated condition.

For the converse, suppose now that we are given a set of finitely generated modules M_i without simple direct summands. Then we may obtain the corresponding set of matrices A_i by the equivalence of Proposition 4.6. It is straightforward to see that our condition now implies (ii) of Theorem 4.7 by the equivalence of Proposition 4.6, and thus the proof is complete. ∎

To achieve our goal, we need a characterization, in these linear algebra terms, of the rings of finite representation type inside the class of left pure semisimple rings of the form (2).

Theorem 4.9. *Let $F \subseteq G$ be an embedding of division rings, and let $R_G = \left[\begin{smallmatrix} F & 0 \\ G & G \end{smallmatrix}\right]$ be left pure semisimple. Then R_G is of finite representation type if and only if any of the following equivalent conditions holds:*

(a) *For any G-matrix A of size $m \times n$ with $1 \le n \le m$ with right column rank n and left F-linearly independent rows, G^n is finitely generated as a right module over the ring $\mathbb{M}_n^A(G)$.*

(b) *Given any family $\{A_i \mid i \in I\}$ of G-matrices with respective sizes $m_i \times n_i$ where $0 < n_i \le m_i$ and A_i has right column rank n_i and left F-linearly independent rows, there exists $j \in I$ such that for each $i \in I$, the canonical map*
$$F^{m_j} \otimes_{\mathbb{M}_{m_j}^{A_j}(F)} \mathbb{M}_{m_j,m_i}^{A_j,A_i}(F) \to F^{m_i}$$
is a surjection.

Proof. Assume that R_G is left pure semisimple. The fact that R_G is of finite representation type if and only if (b) holds follows from the second part of Theorem 4.7 in a way similar to the proof of Theorem 4.8. So, we set to show that R_G is of finite representation type if and only if (a) holds.

Let A be any G-matrix as in the statement. By Proposition 4.6, there is a finitely generated left R_G-module M without simple direct summands such that the map $G \otimes_F V \to W$ defining M has matrix A (with respect to the canonical bases). By this same equivalence of categories, we know that the endomorphism ring of M is isomorphic to $\mathbb{M}^A_m(F)$. By Lemma 4.4, $\mathbb{M}^A_m(F) \cong \mathbb{M}^A_n(G)$ and so the right structure of $\operatorname{Hom}_{R_G}(P_0, M) \cong G^n$ is the natural structure of G^n as an $\mathbb{M}^A_n(G)$-module. Thus, G^n is right finitely generated over $\mathbb{M}^A_n(G)$ if and only if $\operatorname{Hom}_{R_G}(P_0, M)$ is right finitely generated.

If R_G is of finite representation type, then this property holds because every left module is endofinite (see, e.g., [11] or [26]).

Conversely, suppose that the property holds. Therefore $\operatorname{Hom}_{R_G}(P_0, M)$ is right finitely generated for every indecomposable left R_G-module M which is not simple. Since the simple indecomposable modules P_0, E_0 are clearly endofinite, it only remains to show that $\operatorname{Hom}_{R_G}(P_1, M)$ is right finitely generated for every non-simple indecomposable left R_G-module M, and the result holds by [14, Theorem 4.1].

Take any such module M. We have a short exact sequence $0 \to P_0 \to P_1 \to E_0 \to 0$, and $\operatorname{Hom}_{R_G}(E_0, M) = 0$. Consequently, we have a short exact sequence $0 \to \operatorname{Hom}_{R_G}(P_1, M) \to \operatorname{Hom}_{R_G}(P_0, M) \to \operatorname{Ext}^1_{R_G}(E_0, M) \to 0$. Since this is a sequence of right $\operatorname{End}_{R_G}(M)$-modules and $\operatorname{Hom}_{R_G}(P_0, M)$ is right finite-dimensional, we find immediately that $\operatorname{Hom}_{R_G}(P_1, M)$ is also right finite-dimensional. ∎

We may finally state the wpssC in terms of linear algebra.

Proposition 4.10. *Consider the following conditions for an embedding $F \subseteq G$ of division rings:*

(1) *Given any family $\{A_i \mid i \in I\}$ of G-matrices with respective sizes $m_i \times n_i$ where $0 < n_i \leq m_i$ and A_i has right column rank n_i and left F-linearly independent rows, there exists $j \in I$ such that for each $i \in I$, the canonical map*
$$\mathbb{M}^{A_i,A_j}_{n_i,n_j}(G) \otimes_{\mathbb{M}^{A_j}_{n_j}(G)} G^{n_j} \to G^{n_i}$$
is a surjection.

(2) *For any G-matrix A of size $m \times n$ with $0 < n \leq m$, right column rank n and left F-linearly independent rows, G^n is finitely generated as a right module over the ring $\mathbb{M}^A_n(G)$.*

(2′) *Given any family $\{A_i \mid i \in I\}$ of G-matrices with respective sizes $m_i \times n_i$ where $0 < n_i \leq m_i$ and A_i has right column rank n_i and left F-linearly independent rows, there exists $j \in I$ such that for each $i \in I$, the canonical map*
$$F^{m_j} \otimes_{\mathbb{M}^{A_j}_{m_j}(F)} \mathbb{M}^{A_j,A_i}_{m_j,m_i}(F) \to F^{m_i}$$
is a surjection.

The weak pure semisimplicity conjecture is equivalent to the assertion that whenever G is left F-finite-dimensional and (1) *holds, then one of the equivalent conditions* (2) *or* (2′) *holds.*

Proof. Apply Theorems 4.8 and 4.9. ∎

5. Sporadic pure semisimple rings. Let $R_B = \left[\begin{smallmatrix} F & 0 \\ B & G \end{smallmatrix}\right]$ be left pure semisimple. Following the notation at the beginning of Section 4, the indecomposable left R_B-modules form a chain $\{M_\alpha \mid 0 \leq \alpha \leq \delta+1\}$ with $\alpha < \beta$ precisely when $\mathrm{Hom}_{R_B}(M_\alpha, M_\beta) = 0$. We denote by d_α the left dimension of $\mathrm{Hom}_{R_B}(M_{\alpha+1}, M_\alpha)$ when $\alpha \leq \delta$, and $d^* = d_{\delta+1}$ is the left dimension of B^*.

Suppose that $\delta + 1 = \rho + n$ for some limit ordinal ρ and $n > 0$. As shown in [16], if $0 \leq \beta \leq \rho$ is a limit ordinal, then $\mathcal{U}^\beta = \{M_{\beta+k} \mid k < \omega\}$ is one of the Auslander–Reiten components of R_B-ind. Thus $\mathcal{U}^0$ is the set of preinjective modules and $\mathcal{U}^\rho$ is the finite set of preprojective modules.

Recall that we say that R_B is sporadic if $d_\alpha > 1$ for every ordinal $\alpha \leq \delta + 1$. Now, if R_B is such that $d_\alpha > 1$ for all ordinals α that are not infinite limit ordinals, then we shall say that the ring R is *almost sporadic.*

The crucial property of pure semisimple almost sporadic rings is the following:

Theorem 5.1. *Let $R_B = \left[\begin{smallmatrix} F & 0 \\ B & G \end{smallmatrix}\right]$ be a left pure semisimple almost sporadic ring. Suppose that $\mathcal{U}^\lambda$ is any Auslander–Reiten component of R_B-ind which is not the preprojective component. Then there exists $n \geq 0$ such that for all $k \geq n$ we have $d_{\lambda+k} = 2$.*

Proof. Let us denote the d-vector of each module M_α as (t_α, s_α) and let $\mathcal{U}^\beta$ be any Auslander–Reiten component which is not the preprojective component. For each natural number $k \geq 0$, let us consider the values $v_k = dt_{\beta+k} - s_{\beta+k}$. We start by establishing the following claim: Either there exists some $m \geq 0$ such that the sequence v_k is strictly increasing for $k \geq m$, or else there exists $m \geq 0$ such that the sequence v_k is constant for $k \geq m$. Note that $v_k > 0$ because if $M_{\beta+k}$ is indecomposable and not projective, the linear map $B \otimes_F V \to W$ defining $M_{\beta+k}$ is a proper surjection.

To prove the claim, we observe that if $v_k < v_{k+1}$ with $k > 0$, then the sequence $v_k, v_{k+1}, \ldots$ is strictly increasing. To see this, it is enough to apply Lemma 3.1:

$$v_{k+2} = d(d_{\beta+k}t_{\beta+k+1} - t_{\beta+k}) - (d_{\beta+k}s_{\beta+k+1} - s_{\beta+k}) = d_{\beta+k}v_{k+1} - v_k.$$

Therefore

$$v_{k+2} - v_{k+1} = (d_{\beta+k} - 1)v_{k+1} - v_k \geq v_{k+1} - v_k > 0$$

as $d_{\beta+k} > 1$ by our hypothesis that the ring is almost sporadic.

As a consequence, if the sequence is not eventually strictly increasing, then we have $v_{k+1} \leq v_k$ for any $k > 0$. But since all these values are ≥ 1, this implies that the sequence is eventually constant. This proves the claim.

Now, since $\mathcal{U}^\lambda$ is not the preprojective component, we know that there is a next component $\mathcal{U}^\mu$ with $\mu = \lambda + \omega$. The module $M_\mu \oplus M_{\mu+1}$ is a tilting module whose endomorphism ring S is again a left pure semisimple ring of the form (1). Moreover, in view of the equivalences H, H' of the tilting theorem, the sequence of the indecomposable left S-modules consists of the images (in the corresponding order) through the equivalence H' of the indecomposable torsionfree left R_B-modules, followed by the images through H of the indecomposable torsion modules (again, in the same order as in R_B-ind). Moreover, the left dimension of $\operatorname{Hom}_S(X_{\alpha+1}, X_\alpha)$ for two consecutive modules over this ring is the same as the left dimension of the Hom of the corresponding left R_B-modules. Therefore, S is again almost sporadic. Thus, without loss of generality, we may assume that $\mathcal{U}^\mu$ is the preprojective component and consists only of the two projective modules P_1 and P_0.

We know that for each $k \geq 0$, there is an irreducible map $h_k : M_{\lambda+k+1} \to M_{\lambda+k}$. This has to be either a monomorphism or an epimorphism. But if some h_k with $k > 0$ is an epimorphism, then h_{k+1} cannot be a monomorphism, hence it is an epimorphism too. This is because, if h_k were an epimorphism and h_{k+1} were a monomorphism, then $t_{\lambda+k+1} \geq t_{\lambda+k}, t_{\lambda+k+2}$ and similarly for $s_{\lambda+k+1}$; by the equation of Lemma 3.1, we would have

$$d_{\lambda+k} t_{\lambda+k+1} = t_{\lambda+k+2} + t_{\lambda+k}, \qquad d_{\lambda+k} s_{\lambda+k+1} = s_{\lambda+k+2} + s_{\lambda+k}.$$

But this would contradict the hypothesis that $d_{\lambda+k} \geq 2$ because the ring is almost sporadic. Therefore, if h_k is an epimorphism for some positive k, then all the successive maps h_{k+m} are epimorphisms. On the other hand, it cannot be the case that all the mappings are monomorphisms, and hence, from some index onwards, all h_k are epimorphisms. Consequently, the lengths of the modules $M_{\lambda+k}$ are growing from some index onwards and hence the d-vectors $(t_{\lambda+k}, s_{\lambda+k})$ have non-decreasing components (with at least one of them strictly increasing).

Let $m > 0$ be such an index. Then $(t_{\lambda+m}, s_{\lambda+m})$ will be the d-vector of $M_{\lambda+m}$. If L is a maximal submodule of $M_{\lambda+m}$, then each indecomposable direct summand of L has to be projective, since the other modules in $\mathcal{U}^\lambda$ either have length greater than $M_{\lambda+m}$, or else have no non-zero homomorphisms to $M_{\lambda+m}$. Thus, we have a short exact sequence

$$0 \to P_0^l \oplus P_1^h \to M_{\lambda+m} \to E_0 \to 0.$$

This entails the equation

$$l(0,1)+h(1,d)+(1,0)=(1+h,l+dh)=(t_{\lambda+m},s_{\lambda+m})$$

so that $1+h=t_{\lambda+m}$ and $l+dh=s_{\lambda+m}$.

Note that this is also valid for any index $k\geq m$, by the same argument. Of course, the numbers l and h will in general depend on that index. But we must always have $d>l$. Indeed, since $v_k>0$ as we have seen above, it follows that

$$v_m=d(1+h)-(l+dh)=d-l>0,$$

and the same is true for each v_k with $k\geq m$. This shows that $v_k\leq d$ for all those indices, and hence the sequence v_k cannot be strictly increasing.

Thus, as a consequence of our claim, there exists some index $i>0$ such that the sequence v_k for $k\geq i$ is constant. We then choose an index n such that both conditions hold, so that for $k\geq n$ the mapping h_k is an epimorphism and the value $v_k=d-l$ is constant. Accordingly, the value l is constant for those indices.

Consider now three consecutive modules in $\mathcal{U}^\lambda$ whose indices are $\lambda+k$, $\lambda+k+1$ and $\lambda+k+2$, with $k\geq n$. Then their respective d-vectors will have the form

$$(1+h,l+dh),\quad (1+h_1,l+dh_1),\quad (1+h_2,l+dh_2).$$

Now, the value $d_{\lambda+k}=d'\geq 2$ satisfies, by Lemma 3.1, $d'(1+h_1)=(1+h)+(1+h_2)$ and $d'(l+dh_1)=(l+dh)+(l+dh_2)$. So we have

$$(1+h_1)(2l+dh+dh_2)=(l+dh_1)(2+h+h_2),$$

and thus

$$2l+2lh_1+d(h+h_2)+dhh_1+dh_1h_2=2l+hl+h_2l+2dh_1+dhh_1+dh_1h_2.$$

Simplifying gives $2lh_1+d(h+h_2)=(h+h_2)l+2dh_1$, from which it follows that $(h+h_2)(d-l)=2h_1(d-l)$ and $h+h_2=2h_1$.

Now, since we had $d'(1+h_1)=2+h+h_2$, we conclude that

$$d'(1+h_1)=2+2h_1=2(1+h_1),$$

which shows that $d'=2$. Since $d'=d_{\lambda+k}$ for any $k\geq n$, the result is proven. ∎

An easy consequence follows:

Corollary 5.2. *If the pssC is false, then there exists a counterexample which is a ring of the form* (1)*, i.e.* $R_B=\left[\begin{smallmatrix}F & 0\\ B & G\end{smallmatrix}\right]$*, such that* $\mathrm{l.dim}(B)\leq 2$*.*

Proof. If some d_α equals 1, then the endomorphism ring of the tilting module $M_{\alpha+1}\oplus M_\alpha$ is a ring $R_B=\left[\begin{smallmatrix}F & 0\\ B & G\end{smallmatrix}\right]$ such that $B=\mathrm{Hom}_R(M_{\alpha+1},M_\alpha)$. Its left dimension is 1, by our assumption. On the other hand, if $d_\alpha>1$ for each possible ordinal α, then the ring is sporadic and hence $d_\alpha=2$

for some α, by Theorem 5.1. By taking $M_\alpha \oplus M_{\alpha+1}$ as a tilting module, its endomorphism ring would be a counterexample which is a ring of the form (1) such that the left dimension of the defining bimodule is 2. ∎

We observe that left pure semisimple sporadic rings seem to be relatively scarce (if there are any at all). In this connection, we show now that there is no sporadic left pure semisimple ring with only two Auslander–Reiten components. This follows also from Simson's description of potential counterexamples to the pssC with two components (see, e.g., [35]), but we give an independent and simple proof here.

PROPOSITION 5.3. *Let* $R_B = \left[\begin{smallmatrix} F & 0 \\ B & G \end{smallmatrix}\right]$ *be a left pure semisimple almost sporadic ring with only two Auslander–Reiten components. Up to replacement by the endomorphism ring of a basic tilting module, the only preprojective modules are the projective modules* P_0, P_1*; furthermore,* B *has left dimension* 1 *and the sequence of the left dimensions of the dualizations* $B^*, \ldots$ *has all its values equal to* 2*. In particular, there are no sporadic left pure semisimple rings with only two Auslander–Reiten components.*

Proof. Suppose R_B-ind has only the preinjective and the preprojective component. By Theorem 5.1, we may choose $k \geq 0$ such that $d_j = 2$ for all $j \geq k$. By replacing R_B with the endomorphism ring of the tilting module $M_{k+1} \oplus M_{k+2}$, we may assume that our ring is such that $d^* = d_0 = d_1 = \cdots = 2$. Hence, the sequence of the d-vectors of the preinjective left R_B-modules will be

$$(1,0), (2,1), (3,2), \ldots, (h+1,h), \ldots.$$

The limit of the ratios t_h/s_h is 1, and thus by Theorem 3.14, the d-vector of M_ω is $(1,1)$. By Lemma 3.8, the d-vector of $M_{\omega+1}$ is $(t, t+1)$ for some $t \geq 0$. If $t = 0$, then $M_{\omega+1} = P_0$ and $M_\omega = P_1$, so that $d = 1$ and the ring has the form given in the statement of the proposition.

Thus, let $t > 0$. Let us introduce an order for d-vectors with: $(t,s) < (t',s')$ if $t \leq t'$, $s \leq s'$ and either $t < t'$ or $s < s'$. Then, assume we are given three consecutive indecomposable modules, say with indices α, $\alpha+1$, $\alpha+2$ such that $(t_\alpha, s_\alpha) < (t_{\alpha+1}, s_{\alpha+1})$. We are going to show that if $d' = d_\alpha > 1$, then $(t_{\alpha+1}, s_{\alpha+1}) < (t_{\alpha+2}, s_{\alpha+2})$.

This is an easy computation. If we write (t,s), (t',s'), (t'',s'') for the d-vectors in this sequence and $d' = d_\alpha$, then we deduce by Lemma 3.1 and Proposition 2.3(b2) that

$$d't' = t + t'', \qquad d's' = s + s''.$$

Thus $t'' - t' = (d't' - t) - t' = (d'-1)t' - t \geq t' - t \geq 0$, and similarly for s. This shows the claim.

Since $t > 0$, we have $(t_\omega, s_\omega) < (t_{\omega+1}, s_{\omega+1})$. If $d_\omega > 1$, then the sequence of pairs $(t_{\omega+k}, s_{\omega+k})$ would be increasing, which is impossible as it has to end with $(0, 1)$. It follows that $d_\omega = 1$ and hence the ring R cannot be sporadic. According to Lemma 3.1, the d-vector of $M_{\omega+2}$ is $(t-1, t)$.

Since the sequence of pairs $(t_{\omega+k}, s_{\omega+k})$ must be decreasing, we easily see that each $d_{\omega+k}$ with $k \geq 1$ has to be 2.

If we now substitute the endomorphism ring of the tilting module $M_\omega \oplus M_{\omega+1}$ for R_B, we get the ring described in the statement of the proposition. ∎

REMARK 5.4. The (essentially unique) class of potential almost sporadic left pure semisimple rings of Proposition 5.3 was constructed by Simson [34].

Acknowledgments. The author wishes to acknowledge financial support from the Spanish Ministry of Education (MTM2010-20940-C02-02) and from the 'Fundación Séneca' of the Region of Murcia (04555-GERM/06). He also wishes to thank Professors N. V. Dung and D. Simson for helpful remarks and suggestions.

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Received 10 March 2014;
revised 10 April 2014 (6190)