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A CHARACTERIZATION OF PARTITION POLYNOMIALS AND GOOD BERNOULLI TRIAL MEASURES IN MANY SYMBOLS

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Abstract. Consider an experiment with $d+1$ possible outcomes, d of which occur with probabilities $x_1, \dots, x_d$. If we consider a large number of independent occurrences of this experiment, the probability of any event in the resulting space is a polynomial in $x_1, \dots, x_d$. We characterize those polynomials which arise as the probability of such an event. We use this to characterize those $\vec{x}$ for which the measure resulting from an infinite sequence of such trials is good in the sense of Akin.

1. Introduction. Let m be a probability measure on the finite space $\{0, 1, \dots, d\}$ with $d \geq 1$, and let $x_i = m(\{i\})$ for $i = 1, \dots, d$ (and so $m(\{0\}) = 1 - x_1 - \dots - x_d$). Then the product measure m^n is the measure suggested by n independent trials of m, and is defined on the space $\{0, 1, \dots, d\}^n$. There are $(d+1)^n$ points in this space, and the m-measure of each is given by

$$x_1^{i_1} \cdot \ldots \cdot x_d^{i_d} (1 - x_1 - \dots - x_d)^{n - i_1 - \dots - i_d},$$

where i_t is the number of occurrences of the symbol t in the coordinates of the point. The m^n measure of a subset of $\{0, 1, \dots, d\}^n$ is a sum of such expressions, and so will be a polynomial $p(x_1, \dots, x_d)$. When there exists an integer n and a subset of $\{0, 1, \dots, d\}^n$ such that the probability of this subset is $p(x_1, \dots, x_d)$, we say p is a *partition polynomial.*

The term "partition polynomial" was coined by Austin [4] for $d = 1$ while discussing the homeomorphic measures problem among Bernoulli trial measures. In this case of $d = 1$, it was shown by Dougherty, Mauldin and Yingst [5] that $p(x_1)$ is a partition polynomial if and only if it is either zero or one, or else is a polynomial $p(x_1)$ with integer coefficients having $0 < p(x_1) < 1$ on the interval $0 < x_1 < 1$.

The relevance of partition polynomials can be seen by considering the following elementary problem: Suppose we have a three-sided die which shows each of its three faces with probabilities $\left(\frac{1}{9}, \frac{4}{9}, \frac{4}{9}\right)$. Can we use this die to create an experiment whose probability is $\frac{1}{3}$? If we are able to roll

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the die an unlimited number of times, the answer is certainly yes since we can find a sequence of mutually exclusive events whose probabilities sum to $\frac{1}{3}$; but if we restrict our attention to experiments that will terminate in bounded time, the answer is not immediately obvious. In the above terminology, we are asking whether there exists a partition polynomial $p(x, y)$ with $p\left(\frac{4}{9}, \frac{4}{9}\right) = \frac{1}{3}$. (We will return to this question at the end of the paper.)

As a more abstract way of expressing the same example, consider the Bernoulli measure on $\{0, 1, 2\}^{\mathbb{N}}$ suggested by rolling the above $\left(\frac{1}{9}, \frac{4}{9}, \frac{4}{9}\right)$ die a sequence of times. In this space, the measure of a clopen set is given by evaluating $p\left(\frac{4}{9}, \frac{4}{9}\right)$ for a partition polynomial p. The question of whether there exists a clopen set with a given measure $\left(\text{say, } \frac{1}{3}\right)$ can arise when considering what measure-preserving homeomorphisms exist for this space.

The majority of this paper is spent proving Theorem 3.2, which characterizes for general d when a polynomial $p(x_1, \ldots, x_d)$ is a partition polynomial. In Sections 2 and 3 we introduce some terminology, in Section 4 we prove the theorem in the case of $d = 2$ as a warm-up, while in Section 5 we prove the general case. Finally, in Section 6 we use the theorem to give a characterization of when a Bernoulli trial measure is good in the sense of Akin, and we give some examples of this.

We remark that Section 4 is intended to ease the reader into the ideas for Section 5, and is not strictly necessary.

2. Partition polynomials. A *partition polynomial* in $d + 1$ symbols (or in d variables) is a polynomial $p(x_1, \ldots, x_d)$ for which there exists some (sufficiently large) $n \geq 0$ so that p can be expressed in the form

$$p(\vec{x}) = \sum_{(i_1, \ldots, i_d) \in \mathbb{Z}^d} c^n_{\vec{i}} (1 - x_1 - \cdots - x_d)^{n - i_1 - \cdots - i_d} x_1^{i_1} \cdot \ldots \cdot x_d^{i_d},$$

where each $c^n_{\vec{i}}$ is an integer with $0 \leq c^n_{\vec{i}} \leq \binom{n}{\vec{i}}$. (We will soon see that this corresponds with the earlier definition.)

As a convenience, we write $x_0 = 1 - x_1 - \cdots - x_d$, and when n is understood, we write $i_0 = n - i_1 - \cdots - i_d$. This lets us abbreviate the above as

$$p(\vec{x}) = \sum_{\vec{i}} c^n_{\vec{i}} x_0^{i_0} x_1^{i_1} \cdot \ldots \cdot x_d^{i_d}.$$

Throughout the paper, $\binom{n}{\vec{i}}$ denotes $n!/(i_0! i_1! \cdot \ldots \cdot i_d!)$ if each i_k is non-negative, or denotes 0 if one of them (including i_0) is negative. This necessitates that each $c^n_{\vec{i}}$ in the sum above is zero unless each i_k is non-negative, and so we may interpret any such sum over $\vec{i}$ as a sum of only finitely many terms: those where each i_k including i_0 is non-negative. It is a little counter-intuitive to allow $c^n_{\vec{i}}$ and $\binom{n}{\vec{i}}$ to be defined for negative i values, but it lets us avoid manipulating indices of summation as in the sum above. We also get easy base cases for induction. The product $x_0^{i_0}x_1^{i_1}\cdot\ldots\cdot x_d^{i_d}$ occurs frequently enough that it is worthwhile to occasionally abbreviate it as $\vec{x}^{\vec{i}}$, so a compact expression of the above is $p(\vec{x})=\sum_{\vec{i}}c^n_{\vec{i}}\vec{x}^{\vec{i}}$.

The definition of partition polynomial given here can be easily seen to match the description given in Section 1: For a given $\vec{i}$, the number of points in $\{0,\ldots,d\}^n$ whose m-measure is $\vec{x}^{\vec{i}}$ is the number of points for which the symbol t occurs in exactly i_t coordinates, for each $t=1,\ldots,d$. The number of such points is exactly $\binom{n}{\vec{i}}$, and so the measure of a subset of $\{0,\ldots,d\}^n$ will include a number of these terms between 0 and $\binom{n}{\vec{i}}$.

For a given n, let $\mathcal{B}_n=\{x_0^{i_0}x_1{}^{i_1}\cdot\ldots\cdot x_d^{i_d}:i_0,i_1,\ldots i_d\geq 0\}$. Note that a linear combination of these is clearly a polynomial of degree at most n. Also, if $q(x_1,\ldots,x_d)$ is a polynomial of degree at most n, then we may multiply any term of degree u by an expansion of $1=(x_0+x_1+\cdots+x_d)^{n-u}$ showing that q is in the span of $\mathcal{B}_n$. Hence the span of $\mathcal{B}_n$ is the space of polynomials of degree at most n. Also, there is a clear one-to-one correspondence between elements of $\mathcal{B}_n$ and the usual basis of the space of polynomials of degree at most n. (The usual basis is precisely the elements of $\mathcal{B}_n$ with all factors of x_0 removed.) Therefore $\mathcal{B}_n$ is a basis of the space of polynomials of degree at most n, and so if p is fixed, and $n\geq\deg p$, then there do exist coefficients $c^n_{\vec{i}}$ as in the definition of a partition polynomial, and they are uniquely defined. We will describe these $c^n_{\vec{i}}$ as the *partition coefficients* of p, and $p(\vec{x})=\sum_{\vec{i}}c^n_{\vec{i}}\vec{x}^{\vec{i}}$ as the *partition form* of p. The question of whether p is a partition polynomial is thus not a question of whether $c^n_{\vec{i}}$ exist, but whether they will all lie in the correct range for some (sufficiently large) n.

The well-known relation $\binom{n}{k}=\binom{n-1}{k}+\binom{n-1}{k-1}$ has a generalization to multinomial coefficients:

$$\binom{n}{i_1,\ldots,i_d}=\binom{n-1}{i_1,\ldots,i_d}+\binom{n-1}{i_1-1,\ldots,i_d}+\cdots+\binom{n-1}{i_1,\ldots,i_d-1}.$$

Letting $\vec{e}_t$ denote the standard basis vector, and letting $\vec{e}_0$ denote zero, we may write the above as $\binom{n}{\vec{i}}=\sum_{t=0}^{n}\binom{n-1}{\vec{i}-\vec{e}_t}$. (The use of $\vec{e}_0$ can remind us that n has decreased, but our shorthand $i_0=n-i_1-\cdots-i_d$ did not change with it. The denominator of the binomial coefficient for the $t=0$ term has a factor of $(n-1-i_1-\cdots-i_d)!=(i_0-1)!$, but the vector $\vec{i}=(i_1,\ldots,i_d)$ is not affected.)

We bring up this relation because it mirrors the recursion satisfied by the partition coefficients of a polynomial. If $c^n_{\vec{i}}$ are the partition coefficients of p, then multiplying $p(\vec{x})=\sum_{\vec{i}}c^n_{\vec{i}}\vec{x}^{\vec{i}}$ by $1=x_0+\cdots+x_d$ and collecting like terms gives the relation

$$c^{n+1}_{\vec{j}} = c^n_{\vec{j}-\vec{e}_0} + c^n_{\vec{j}-\vec{e}_1} + \cdots + c^n_{\vec{j}-\vec{e}_d}.$$

With this relation, we see that if there is some n for which the $c^n_{\vec{i}}$ are all non-negative, this holds for all larger n as well, and likewise for the condition $c^n_{\vec{i}} \le \binom{n}{\vec{i}}$.

3. Supporting polynomials. When working in d variables, we say the *closed simplex* is the region

$$x_1, \ldots, x_d \ge 0, \qquad x_1 + \cdots + x_d \le 1.$$

We say the *open simplex* is the same with strict inequality. (Alternatively, we could let x_0 be an independent variable and look at the behavior of polynomials only on the more standard simplex, the portion of the hyperplane $x_0 + \cdots + x_d = 1$ with $x_t \ge 0$ for all t. In that context however, polynomials are only considered unique modulo $(1 - x_0 - \cdots - x_d)$. Thus, we are choosing to work on the less standard simplex but to work with a more standard notion of equality of polynomials.)

Before moving on, we briefly discuss why the case of $d > 1$ is significantly more difficult than $d = 1$. In the $d = 1$ case, the polynomial $p(x)$ has non-negative partition coefficients for some large n if and only if $p > 0$ on $(0, 1)$. (The condition $c^n_i \le \binom{n}{i}$ for some n is equivalent to $p < 1$ on $(0, 1)$.) The hope would be that in two variables, a polynomial $p(x, y)$ which is positive on the open simplex might similarly have non-negative partition coefficients if n is large. This fails for two essentially different reasons.

First consider the polynomial $q(x, y) = x^2 - xy + y^2$. It is already in partition form for $n = 2$, and its next two partition forms are

$$\begin{aligned} q(x, y) &= x^2(1 - x - y) - xy(1 - x - y) + y^2(1 - x - y) + x^3 + y^3, \\ q(x, y) &= x^2(1 - x - y)^2 - xy(1 - x - y)^2 + y^2(1 - x - y)^2 \\ &\qquad\qquad\qquad\qquad\qquad\qquad + \text{additional terms}. \end{aligned}$$

We can verify that this polynomial, despite being positive on the open simplex, is not a partition polynomial, since the negative coefficient on the term $-xy(1 - x - y)^{n-2}$ persists as n increases.

Next, consider the polynomial $r(x, y) = (2x-1)^2 + y$. Again, it is positive on the open simplex, but it cannot be a partition polynomial: as is easily verified, if $r(x, y)$ is a partition polynomial in two variables, then $r(x, 0)$ is a partition polynomial in one variable, and this is not the case since $r(1/2, 0) = 0$, but $r(x, 0)$ is not identically zero. A tighter restriction might be to ask that if the polynomial is positive anywhere on an edge of the simplex then it must be positive along the entire edge. The polynomial $\hat{r}(x, y) = (2x - 1)^2 y + y^2$ similarly foils this plan, hiding the isolated zero at $(0, 1/2)$ by zeroing out the entire edge. The correct requirement is to first divide off all factors of y, and then substitute $y = 0$. The resulting polynomial in x must be positive on $0 < x < 1$.

We will soon define the notion of a supporting polynomial, a single idea that generalizes the -1 in the example $q(x, y)$ and the $(2x - 1)^2$ in $r(x, y)$ and $\hat{r}(x, y)$. Above, we considered the behavior of r and $\hat{r}$ on the edge $y = 0$. Clearly, the other two edges of the simplex in $\mathbb{R}^2$ ($x = 0$ and $1 - x - y = 0$) should also be considered in the same way. Our terminology will be easier to work with if instead of defining a supporting polynomial along a general edge, we let $y = 0$ be the standard edge to read from, and we permute the variables of the polynomial when we need to read from the other edges. (Similarly, $(0, 0)$ will be the standard vertex to read coefficients from.)

To this end, if $p(x_1, \dots, x_d)$ is a polynomial in d variables, we say that a polynomial $q(x_1, \dots, x_d)$ is a *simplex permutation* of p if there is a permutation π of $\{0, \dots, d\}$ such that

$$q(x_1, \dots, x_d) = p(x_{\pi(1)}, \dots, x_{\pi(d)}).$$

In this case, we write $q = p \circ \pi$. Note that $x_0 = 1 - x_1 - \cdots - x_d$ is allowed to appear. For example, there are six simplex permutations of $p(x, y) = x^2 y$, two of them being $f(x, y) = p(y, x) = xy^2$ and $g(x, y) = p(1 - x - y, x) = (1 - x - y)^2 x$. Among these six polynomials, the behaviors of p along the various three edges and three vertices are each moved into all six configurations.

Definition 3.1. If $p(\vec{x})$ is a polynomial in $x_1, \dots, x_d$ and $0 \le v \le d$, we say that $s(x_1, \dots, x_v)$ is a *standard supporting polynomial* of p *in* v *variables* if we can write

$$\begin{aligned} (3.1) \qquad p(x_1, \dots, x_d) = {} & x_{v+1}^{a_{v+1}} \cdot \ldots \cdot x_d^{a_d} s(x_1, \dots, x_v) \\ & + x_{v+1}^{1+a_{v+1}} g_{v+1}(x_1, \dots, x_d) + \cdots + x_d^{1+a_d} g_d(x_1, \dots, x_d) \end{aligned}$$

for some integers $a_{v+1}, \dots, a_d$ and some polynomials $g_{v+1}, \dots, g_d$ in d variables. We say $s(x_1, \dots, x_v)$ is a *supporting polynomial* of p if it is a supporting polynomial of some simplex permutation of p.

The g's are not unique, and we call them *garbage polynomials* into which we can throw certain leftover terms.

We think of a standard supporting polynomial s as being some terms of p, all of which have the same powers a_{v+1} through a_d on the variables x_{v+1} through x_d, so that every other term of p has a larger power of one of these and hence can be lumped in with one of the garbage polynomials in the above expression. The vector $(a_{v+1}, \dots, a_d)$ of these powers is called the *exponent vector* associated with this supporting polynomial, and it is minimal in a certain sense: for $k \ge v+1$, no term of p may have its exponent on x_k smaller than a_k while still having its powers on the remaining x's as small as the corresponding a's. That is, if s is non-zero, then under the order $(\mathbb{Z}^{d-v}, \leq)$ the vector $(a_{v+1}, \ldots, a_d)$ must be minimal among exponent vectors appearing at powers of $x_{v+1}, \ldots, x_d$ in terms of p. With this observation, we can see that for a given p, if the $a_{v+1}, \ldots, a_d$ have been given (and are indeed minimal in the above sense) then the corresponding standard supporting polynomial s is uniquely determined by these.

Note that $s = 0$ often counts as a standard supporting polynomial: If p vanishes on $x_{v+1} = \cdots = x_d = 0$, then we can take $(a_{v+1}, \ldots, a_d) = (0, \ldots, 0)$, $s = 0$, and every term of p can be lumped in with one of the garbage polynomials. Even more trivially, we have allowed the a's to be negative in our definition: when $v < d$ we can use $s = 0$, $a_{v+1} = -1$ and $g_{v+1} = p$, with all other a's and g's being zero. (Again, allowing the a's to be negative is non-intuitive, but this adds only more occasions of $s = 0$ as a supporting polynomial, and it is useful to eliminate the need for a base case in some inductions.) For these reasons, we will usually be concerned only with the non-zero standard supporting polynomials.

We allow the special case where $v = 0$, in which case s is a constant, and we call s a *standard supporting coefficient* of p. That is, a standard supporting coefficient of p is a number c such that there is a term of p of the form $cx_1^{a_1} \cdot \ldots \cdot x_d^{a_d}$, where every other term of p has an exponent of some variable greater than this one. We say a *supporting coefficient* of p is a standard supporting coefficient of some simplex permutation of p.

We also allow the special case of $v = d$, in which case the only standard supporting polynomial is p itself. (No garbage polynomials are allowed, so the only expression of this type is $p(\vec{x}) = 1p(\vec{x}) + 0$.) This is the only value of v when zero does not count as a standard supporting polynomial (unless p is zero itself). The case of $v = d$ is often handled separately. Those (standard) supporting polynomials with $v < d$ are called the *proper (standard) supporting polynomials*.

The term "supporting polynomial" is named after "supporting hyperplane". A $d-1$-dimensional hyperplane *supports* a compact convex subset C of $\mathbb{R}^d$ when it intersects C but C lies only on one side of the hyperplane. A *face* of a polytope is the intersection of the polytope with a supporting hyperplane. This is a generalization of edge, face, or vertex of a polyhedron. A line can intersect each edge of a triangle in only one way, and similarly a polynomial in two variables has only three supporting polynomials in one variable (actually six, but occurring in three pairs which are simplex permutations of each other and so are usually equivalent for our purposes). A line can intersect the corner of a triangle in a one-parameter family of ways. Likewise, there is a one-parameter family of standard supporting coefficients for each simplex permutation of $p(x, y)$. (Again those six permutations can be grouped into pairs for which the same vertex is in standard position. These pairs will find the same standard supporting coefficients but in reversed order.) The definition of a face of a polytope mentioned above often comes with an exception, allowing the polytope to count as a face of itself. Similar reasoning motivates us to consider a polynomial (and its simplex permutations) as a supporting polynomial of itself.

We are now able to state the main result of our paper, a complete characterization of partition polynomials. We will prove this theorem in Section 5:

THEOREM 3.2. *If $p(\vec{x}) = p(x_1, \ldots, x_d)$ is a polynomial such that any non-zero supporting polynomial of p is positive on the appropriate open simplex for its domain (or is a positive number in the case of a non-zero supporting coefficient), then there is some n such that p is expressible in the form*

$$p(\vec{x}) = \sum_{i_1,\ldots,i_d} c^n_{\vec{i}} x_1^{i_1} \cdot \ldots \cdot x_d^{i_d} (1 - x_1 - \cdots - x_d)^{n-i_1-\cdots-i_d},$$

where each $c^n_{\vec{i}}$ is non-negative.

Furthermore, p is a partition polynomial iff p and $1 - p$ each have the above property, and p has integer coefficients.

Note that if s is a non-zero standard supporting polynomial of p, then no term of s can be canceled by any term of a garbage polynomial in equation (3.1), and so we must have $a_{v+1} + \cdots + a_d \leq \deg p$. Because of this, we know that the number of supporting polynomials of p is finite, and our ability to algorithmically determine whether a specific polynomial is a partition polynomial is limited only by our ability to determine whether a polynomial is positive on the simplex.

It is easy to verify that the partition coefficients of p are integers if and only if p has integer coefficients in the usual sense, since the change-of-basis processes described earlier preserve integral coefficients.

Also, we can write $1 = \sum_{\vec{i}} \binom{n}{\vec{i}} \vec{x}^{\vec{i}}$, so that if $(c^n_{\vec{i}})_{\vec{i}}$ are the partition coefficients of p, then $\left(\binom{n}{\vec{i}} - c^n_{\vec{i}}\right)_{\vec{i}}$ are the partition coefficients of $1 - p$. From this we can see that p is a partition polynomial if and only if $1 - p$ is, and in the definition of a partition polynomial, the upper bound on the partition coefficients of p is equivalent to the lower bound on the partition coefficients of $1 - p$. With these observations, the second statement in Theorem 3.2 follows from the first.

We briefly consider the simplest non-trivial case, when $d = 1$. A polynomial $p(x)$ can have only two non-zero standard supporting polynomials: the polynomial itself, and one standard supporting coefficient. Suppose $p(x)$ is positive on its open simplex (the open unit interval). If C is the supporting coefficient of p, we have an expression $p(x) = Cx^a + x^{a+1}g(x)$. The term Cx^a is the largest term of p near 0, and since p is positive on the interval $(0, \delta)$, we deduce that C must be positive.

From this we see that if $0 < p(x) < 1$ on the open interval, then every simplex permutation of p or $1 - p$ is positive on its simplex, and so every supporting coefficient of p or $1 - p$ is positive as well. Thus, when $d = 1$, Theorem 3.2 reduces to the characterization given by Dougherty, Mauldin and Yingst [5]: $p(x)$ with integer coefficients is a partition polynomial if and only if p is zero or one or has $0 < p(x) < 1$ on $0 < x < 1$.

Before moving on, we quickly note the following observation about supporting polynomials which will help generate partition polynomials in Section 6.

PROPOSITION 3.3. *If every* (*standard*) *supporting polynomial of* p_1 *and* p_2 *in* v *variables is positive on the appropriate open simplex for its domain* (*or is a positive number if* $v = 0$), *then every* (*standard*) *supporting polynomial of* $p_1 p_2$ *in* v *variables is also positive on the simplex.*

Proof. The property is clearly preserved under simplex permutations and so we prove the theorem for standard supporting polynomials. For each of p_1 and p_2, we may collect the terms according to the powers of $x_{v+1}, \dots, x_d$ and thus express them in the form

$$p_1(\vec{x}) = \sum_{b_{v+1},\dots,b_d} x_{v+1}^{b_{v+1}} \cdot \dots \cdot x_d^{b_d} p_{b_{v+1},\dots,b_d}(x_1, \dots, x_v),$$

$$p_2(\vec{x}) = \sum_{c_{v+1},\dots,c_d} x_{v+1}^{c_{v+1}} \cdot \dots \cdot x_d^{c_d} q_{c_{v+1},\dots,c_d}(x_1, \dots, x_v).$$

Thus we may write

$$p_1 p_2(\vec{x}) = \sum_{\vec{a}=(a_{v+1},\dots,a_d)} x_{v+1}^{a_{v+1}} \cdot \dots \cdot x_d^{a_d} \Big(\sum_{\vec{b},\vec{c}: \vec{b}+\vec{c}=\vec{a}} p_{\vec{b}} q_{\vec{c}} \Big).$$

Let B denote the set of all exponent vectors $(b_{v+1}, \dots, b_d)$ which contribute a non-zero term to the above sum for p_1. (That is, $B = \{\vec{b} : p_{\vec{b}} \neq 0\}$.) Also, let C denote the same for p_2. Note that from our understanding of supporting polynomials, the exponent vectors associated with standard supporting polynomials of p_1 are precisely the minimal elements of B, where "minimal" is with respect to the order $\leq$ on $\mathbb{Z}^{d-v}$.

Note that in the above sum, if the term associated with $\vec{a} = (a_{v+1}, \dots, a_d)$ is non-zero, then there are some $\vec{b}$ and $\vec{c}$ associated with non-zero polynomials in the first two sums respectively with $\vec{b} + \vec{c} = \vec{a}$. So $\vec{a} \in B + C$. (We take the algebraic sum, $B + C = \{\vec{b} + \vec{c} : \vec{b} \in B, \vec{c} \in C\}$.) So we must have $\vec{a} \geq \vec{a}_0$ for some $\vec{a}_0$ which is minimal in $B + C$. We see that $a_0 = \vec{b} + \vec{c}$ for some $\vec{b} \in B$ and $\vec{c} \in C$. There may be many such expressions of a_0, but for each such expression, $\vec{b}$ is minimal in B and $\vec{c}$ is minimal in C. This implies that in the above expression for p_1p_2, the term associated with $\vec{a}_0$ consists of a sum of products of supporting polynomials of p_1 and p_2, and is therefore positive. Thus, if $a \neq a_0$, then a is not the exponent vector of a supporting polynomial, while if $a = a_0$, it is the exponent vector of a supporting polynomial, and that polynomial is positive. ∎

4. Three symbols. Before attempting the difficult proof of Theorem 3.2, we will warm up by doing the special case when $d+1=3$. As above, the problem reduces to showing that

THEOREM 4.1. *If $p(x,y)$ is a polynomial with real coefficients such that any non-zero supporting polynomial of p is positive on its open simplex (or is a positive number in the case of a supporting coefficient), then there is n such that we may write p as a non-negative linear combination of $\{(1-x-y)^{n-i-j}x^iy^j\}_{i\geq 0,\, j\geq 0,\, i+j\leq n}$.*

Proof. Fix p as in the statement. When $n \geq \deg p$, we let $c^n_{i,j}$ denote the (unique) values such that

$$p(x,y) = \sum_{i,j} c^n_{i,j}(1-x-y)^{n-i-j}x^iy^j.$$

We want to show that for some sufficiently large n, we have $c^n_{i,j} \geq 0$ for all i and j.

Recall that for ease of indexing we take the convention that $c^n_{i,j} = 0$ whenever $i<0$ or $j<0$ or $i+j>n$. With this, whenever $n \geq \deg p$ we have

$$c^{n+1}_{i,j} = c^n_{i,j} + c^n_{i-1,j} + c^n_{i,j-1}.$$

Let A denote the largest integer such that for all j and n, $i<A$ implies $c^n_{i,j} = 0$. Similarly, let B denote the largest integer such that for all i and n, $j<B$ implies $c^n_{i,j}=0$. We may note that $A, B \geq 0$.

CLAIM 4.2. *There are $u_1 \geq A+B$ and $N \geq \deg p$ such that*

- *$c^n_{i,j} \geq 0$ whenever $n \geq N$, $i+j \leq u_1$, and*
- *$c^n_{i,j} > 0$ whenever $n \geq N$, $i+j = u_1$, $i \geq A$, $j \geq B$.*

(Note: It is possible that $c^n_{A,B}$ happens to be zero for $n = \deg p$, in which case it will remain zero for all n. If it is zero, it is further possible that $c^n_{A+1,B}$ or $c^n_{A,B+1}$ is zero for $n=\deg p$, in which case they will also remain zero. The above claim says that there is some u_1 beyond all these possibilities, where A and B capture the only i,j for which $c^n_{i,j}$ is always zero, and all others will be strictly positive for large n. The issue of A and B could be avoided by removing a factor of $x^A y^B$ from p, but we would like to illustrate a difficult aspect of the proof for $d > 3$.)

To show Claim 4.2, we prove (by induction on $i + j$) that for each non-negative i and j, the function $f(t) = c^t_{i,j}$ is a polynomial with non-negative leading coefficient, which is constant only when it is a standard supporting coefficient for the exponent vector (i, j). A base for our induction can be any case when $i + j < 0$, as these are trivially supporting coefficients and $f(t)$ is identically zero here.

This induction step is straightforward, and comes from noting that $f(t)$ satisfies the recurrence relation $f(t + 1) = f(t) + c^t_{i-1,j} + c^t_{i,j-1}$. By our induction hypothesis, the last two terms are polynomials with non-negative leading coefficient, and so the solution is also such a polynomial. Our solution will be constant only if the sum of the last two terms is zero. Since they have non-negative leading coefficient, this implies that each is zero, so each is a supporting coefficient. In this case, all terms of p have a power of x greater than i (since $c^t_{i,j-1}$ is a supporting coefficient) or a power of y greater than j (since $c^t_{i-1,j}$ is a supporting coefficient). So $c^t_{i,j}$ is a supporting coefficient.

Now let u_1 be so large that the exponent vector (i, j) of any standard supporting coefficient has $i+j \le u_1$. Claim 4.2 becomes a requirement that finitely many polynomials with non-negative leading coefficient are eventually all positive.

Claim 4.3. *There are $\delta_1 > 0$ and $N_1 \ge \deg p$ such that for all $n > N_1$, the assumption $i + j < \delta_1 n$ implies that $c^n_{i,j} \ge 0$.*

With N as in Claim 4.2, let $\epsilon = \min\{c^N_{i,j} : i \ge A,\ j \ge B,\ i+j = u_1\}$. Let $-M = \min\{c^N_{i,j}\}_{i,j}$. We assume $M \ge 0$, since if not, all $c^N_{i,j}$ are non-negative, and we are done. Now, our relation $c^{n+1}_{i,j} = c^n_{i,j} + c^n_{i,j-1} + c^n_{i-1,j}$ inducts the following:

$$c^{N+n}_{i,j} = \sum_{\alpha=0}^{n} \sum_{\beta=0}^{n-\alpha} \binom{n}{\alpha, \beta, n-\alpha-\beta} c^N_{i-\alpha,j-\beta}. \tag{4.1}$$

For the terms of the above with $i - \alpha + j - \beta < u_1$, or $i - \alpha < A$, or $j - \beta < B$ we may use the approximation $c^N_{i-\alpha,j-\beta} \ge 0$. For those with $i - \alpha + j - \beta = u_1$, $i - \alpha \ge A$, and $j - \beta \ge B$, we have $c^N_{i-\alpha,j-\beta} \ge \epsilon$. For the remaining terms, we use $c^N_{i-\alpha,j-\beta} \ge -M$. These yield

$$\begin{aligned} c^{N+n}_{i,j} \ge \epsilon & \sum_{i-\alpha\ge A,\, j-\beta\ge B,\, \alpha+\beta=i+j-u_1} \binom{n}{\alpha, \beta, n-\alpha-\beta} \\ & - M \sum_{i-\alpha\ge A,\, j-\beta\ge B,\, \alpha+\beta<i+j-u_1} \binom{n}{\alpha, \beta, n-\alpha-\beta}. \end{aligned}$$

Recall that $\binom{n}{\alpha,\beta,n-\alpha-\beta}$ gives the number of ways to color a set of n elements so that α of the elements are red, β of the elements are blue, and $n-\alpha-\beta$ of the elements are green. Let $C(x)$ denote the number of red-green-blue colorings of a set with n elements such that the number of reds does not exceed $i-A$, the number of blues does not exceed $j-B$, and the number of greens is exactly $n-i-j+u_1+x$. The above becomes

$$c_{i,j}^{N+n} \geq \epsilon C(0) - M \sum_{x=1}^{i+j-u} C(x).$$

Let $\rho = (i+j)/n$. We need to show that for sufficiently large n, there is $\delta_1 > 0$ such that if $\rho < \delta_1$, we have $\sum_{x=1}^{i+j-u} C(x)/C(0) < \epsilon/M$.

We now compare $C(x)$ with $C(x+1)$. For the moment, fix some $0 \leq x < i+j-u$, and let D denote the number of colorings of a set of n elements such that the number of reds does not exceed $i-A$, the number of blues does not exceed $j-B$, the number of greens is exactly $n-i-j+u+x$, and one element is colored white.

It is easy to compare D with $C(x+1)$: Painting the white element green matches each coloring in $C(x+1)$ with exactly $n-i-j+u+x+1$ elements of D. (There are that many greens which could have been the white.) So $D = (n-i-j+u+x+1)C(x+1)$.

Comparing D with $C(x)$ is trickier. For each coloring of $C(x)$, the number of reds and blues is exactly $i+j-u-x$. If we choose a coloring, and choose a red or blue to paint white, there are exactly $(i+j-u-x)C(x)$ possible choices. Doing this will create every coloring in D, but not necessarily uniquely, since we may or may not be able to tell whether the new white element came from a red or a blue. However, there are at most two possibilities, and we know $D \leq (i+j-u-x)C(x) \leq 2D$.

Combining these, we get

$$C(x+1)/C(x) \leq \frac{i+j-u-x}{n-i-j+u+x+1} \leq \frac{i+j}{n-i-j} = \frac{\rho}{1-\rho}.$$

So $C(x)/C(0) \leq \left(\frac{\rho}{1-\rho}\right)^x$, and we have

$$\begin{aligned}\sum_{x=1}^{i+j-u} C(x)/C(0) &\leq \sum_{x=1}^{i+j-u} \left(\frac{\rho}{1-\rho}\right)^x \\ &= \frac{\rho}{1-\rho} \frac{1-\left(\frac{\rho}{1-\rho}\right)^{i+j-u}}{1-\frac{\rho}{1-\rho}} \leq \frac{\rho}{1-2\rho}.\end{aligned}$$

Clearly, this will be $< \epsilon/M$ for sufficiently small ρ. This ends the proof of Claim 4.3.

Claim 4.4. *There are* $\hat{\delta}_1 > 0$ *and* $\hat{N}_1$ *such that for* $n > \hat{N}_1$ *we have* $c^n_{i,j} \geq 0$ *whenever* $i + j < \hat{\delta}_1 n$ *or* $n - i < \hat{\delta}_1 n$ *or* $n - j < \hat{\delta}_1 n$.

This comes immediately by applying Claim 4.3 to simplex permutations of p.

We have now essentially shown that all partition coefficients *near the corners* are non-negative, in two senses, first for those within u_1 of a corner, and then with that done we had sure footing to prove the statement in the stronger way, for those coefficients within $\delta_1 n$ of a corner. Our next step is to repeat, moving up a dimension, showing that all partition coefficients near an *edge* are non-negative.

Recall that we let B denote the largest integer such that for all i and n, $j < B$ implies $c^n_{i,j} = 0$. Note that in the following claim, the label u_2 is not strictly necessary, as in the end we will be taking $u_2 = B$. The name u_2 is used only to help the reader keep track of the parallel with Claim 4.2 of this proof and Claim 5.5 in the future.

Claim 4.5. *There are* $u_2 \geq B$ *and* $N_2 \geq \deg p$ *such that*

- $c^n_{i,j} \geq 0$ *whenever* $n \geq N_2$, $j \leq u_2$, *and*
- $c^n_{i,j} > 0$ *whenever* $j = u_2$, $i \geq \hat{\delta}_1 n$, $i \leq (1 - \hat{\delta}_1)n$, $n \geq N_2$.

In the expression $p(x, y) = \sum_{i,j} c^n_{i,j}(1 - x - y)^{n-i-j} x^i y^j$, every non-zero term has $j \geq B$, while some non-zero term has $j = B$. Collecting those terms where $j = B$, we can write this as

$$p(x, y) = y^B \Big[\sum_i c^n_{i,B}(1 - x - y)^{n-i-B} x^i \Big] + y^{B+1} g(x, y),$$

where g is some polynomial. Furthermore, if we selectively expand our power of $(1 - x) - y$, we can throw all terms having a y in with $g(x, y)$. This lets us write

$$p(x, y) = y^B \Big[\sum_i c^n_{i,B}(1 - x)^{n-i-B} x^i \Big] + y^{B+1} \hat{g}(x, y).$$

The sum here is written in partition form for a polynomial in one variable. In particular, if $s(x)$ is the sum above, and d^n_i are the partition coefficients of s, then $d^{n-B}_i = c^n_{i,B}$. By choice of B, these d's are not all zero, and so $s(x)$ is a non-zero supporting polynomial of p. It is therefore positive on $(0, 1)$, and so from the one-variable case, it has non-negative partition coefficients for sufficiently large n. Taking $u_2 = B$, we deduce for large n that $c^n_{i,j} \geq 0$ whenever $j \leq u_2$.

We must still show that all coefficients not too close to a corner are strictly positive. Fix some M for which $j \leq u_2$ implies $c^M_{i,j} \geq 0$, and fix some I with $c^M_{I,u_2} > 0$. (Such an I exists by the definition of $B = u_2$.)

The expansion of c_{i,u_2}^{M+n} in (4.1) becomes

$$c_{i,u_2}^{M+n} = \sum_{\alpha=0}^{n} \sum_{\beta=0}^{n-\alpha} \binom{n}{\alpha, \beta, n-\alpha-\beta} c_{i-\alpha, u_2-\beta}^{M}.$$

This sum contains only non-negative terms, and will be positive if it contains a term of c_{I,u_2}^{n}. This happens when the value $\alpha = i - I$ lies in the range $0 \leq \alpha \leq n$, and so we only need $I \leq i \leq n + I$. We are now done, since whenever n is sufficiently large, we will have $I \leq \hat{\delta}_1(M+n) \leq i \leq (1-\hat{\delta}_1)(M+n) < n + I$.

Claim 4.6. *There are $\delta_2 > 0$ and $N_2 \geq \deg p$ such that for all $n \geq N_2$ the assumption $j < \delta_2 n$ implies $c_{i,j}^{n} \geq 0$.*

We repeat the method of proving Claim 4.3. Let $\epsilon(N) = \min\{c_{i,u}^{N} : \hat{\delta}_1 N \leq i \leq (1-\hat{\delta}_1)N\}$. By Claim 4.5, if N is sufficiently large, we have $\epsilon(N) > 0$. Fix $N > \hat{N}_1$ sufficiently large that this $\epsilon(N)$ (now just called ϵ) is positive. Let $-M$ be the least value of $c_{i,j}^{N}$. Again, we assume $M \geq 0$, as otherwise every $c_{i,j}^{N}$ is non-negative and we are done. We have

$$c_{i,j}^{N+n} = \sum_{\alpha=0}^{n} \sum_{\beta=0}^{n-\alpha} \binom{n}{\alpha, \beta, n-\alpha-\beta} c_{i-\alpha, j-\beta}^{N}.$$

First note that if $(i-\alpha) > (1-\hat{\delta}_1/2)N$, then $N - (i-\alpha) < \hat{\delta}_1 N$, and by Claim 4.4, we have $c_{i-\alpha,j-\beta}^{N} \geq 0$. Secondly, we may assume that $\delta_2 < \hat{\delta}_1/2$ so we need only consider the case of $j < \hat{\delta}_1 n/2$. When $i - \alpha < \hat{\delta}_1 N/2$, these combine to give $(i-\alpha) + (j-\beta) < \hat{\delta}_1 n$, and we again have $c_{i-\alpha,j-\beta}^{N} \geq 0$. Removing these non-negative c's, applying our ϵ bound when it applies, and our $-M$ bound to all remaining cases, we get

$$c_{i,j}^{N+n} \geq \epsilon C(0) - M \sum_{x=1}^{\infty} C(x),$$

where $C(x) = \sum \binom{n}{\alpha,\beta,n-\alpha-\beta}$ with the sum being over all (α, β) such that $\hat{\delta}_1 N/2 \leq i - \alpha \leq (1-\hat{\delta}_1/2)N$ and $j - \beta = u + x$.

Again, we view $C(x)$ as a counting of colorings: $C(x)$ gives the number of red-blue-green colorings of n objects such that the number of reds, α, the number of blues, β, and the number of greens, $n-\alpha-\beta$, satisfy $\beta = j-u-x$ and $i - (1-\hat{\delta}_1/2)N \leq \alpha \leq i - \hat{\delta}_1 N/2$.

Fixing x for the moment, let D denote the number of red-green-blue-white colorings of n objects such that exactly one object is white, and the numbers $(\alpha, \beta, n-\alpha-\beta-1)$ of red, blue, and green objects, respectively, satisfy $\beta = j-u-x-1$ and $i - (1-\hat{\delta}_1/2)N \leq \alpha \leq i - \hat{\delta}_1 N/2$.

Again, we use D as a stepping stone to compare $C(x)$ with $C(x+1)$. First $C(x)$: To convert a D coloring to a $C(x+1)$ coloring, we must paint the white marble either red or green. Because our restrictions to become a $C(x+1)$ coloring depend only on α and β, green is certainly a legal choice, but perhaps red is also. Given a $C(x+1)$ coloring, the number of ways it can be so produced is at most the total number of reds and greens. We have

$$2D \geq (n-j+u+x+1)C(x+1).$$

To convert a $C(x)$ coloring into a D coloring, we must choose one of the $j-u-x$ blue objects to paint white. Each D coloring will be chosen exactly once this way, and we get

$$D = (j-u-x)C(x).$$

Combining these and relaxing, we get

$$(n-j)C(x+1) \leq 2jC(x).$$

If we write $j = \rho n$, then we have $C(x+1) \leq \frac{2\rho}{1-\rho}C(x)$, and hence

$$c_{i,j}^{N+n} \geq \left(\epsilon - \sum_{x=1}^{\infty}\left(\frac{2\rho}{1-\rho}\right)^x\right)C(0).$$

Again, the right-hand side is a rational function of ρ which is positive at $\rho = 0$, and hence is positive for all sufficiently small ρ, as desired, and Claim 4.6 is proved.

Again, we may apply Claim 4.6 to simplex permutations of p and get the following:

Claim 4.7. *There are $\hat{\delta}_2 > 0$ and $\hat{N}_2 \geq \deg p$ such that for all $n \geq \hat{N}_2$ the assumption $j < \hat{\delta}_2 n$ or $i < \hat{\delta}_2 n$ or $n-i-j < \hat{\delta}_2$ implies $c_{i,j}^n \geq 0$.*

To wrap up our proof, we will need the following theorem.

Lemma 4.8. *Let $p(x,y)$ be a polynomial, and let $\{c_{i,j}^n\}_{i,j,n-i-j\geq 0}$ be the unique coefficients (for $n \geq \deg p$) such that*

$$p(x,y) = \sum_{i=0}^{n}\sum_{j=0}^{n-i} c_{i,j}^n x^i y^j (1-x-y)^{n-i-j}.$$

Then

$$\lim_{n\to\infty} \sup\left\{\left|p\left(\frac{i}{n},\frac{j}{n}\right) - \frac{c_{i,j}^n}{\binom{n}{i,j}}\right| : 0 \leq i,j,n-i-j\right\} = 0.$$

We will give a detailed proof of this later for more general d in Lemma 5.1. A brief sketch is: The class of all polynomials for which this theorem is true is linear, contains 1, and is closed under multiplication by x or y. It therefore must contain all polynomials.

We finally conclude our proof of Theorem 4.1. Since p is a supporting polynomial of itself, we know that either it is zero (in which case we are done) or it is positive on the open simplex $x, y, 1-x-y > 0$. But then it is positive on the compact region $x, y, 1-x-y \geq \hat{\delta}_2$, and so greater than some $\epsilon > 0$ on that region. The above lemma tells us that for large n, we have $\left|p\left(\frac{i}{n}, \frac{j}{n}\right) - c^n_{i,j}/\binom{n}{i,j}\right| < \epsilon/2$. So those $c^n_{i,j}$ with $i, j, n-i-j > \hat{\delta}_2$ are all positive for large n, while Claim 4.7 handles all other $c^n_{i,j}$. ■

We will soon begin proving Theorem 3.2, but first we will take stock of some lessons learned from the case of $d = 2$. Within this proof, we needed to refer to the $d = 1$ theorem, which indicates we will be arguing by induction on d. Lemma 4.8 and its generalization, Lemma 5.1, will show that all coefficients of a distance more than δn from the edge are positive, so our difficulty is in dealing with these coefficients near the edges. As indicated by the similarity of Claims 4.2–4.4 and Claims 4.5–4.7, we will do this by induction on the dimension of the edge. We first show that those coefficients within u of the corner are positive, and then those within δn for some δ. We repeat this process for coefficients near an edge, then a 2-dimensional face, eventually working our way up to the $d-1$-dimensional faces, after which Lemma 5.1 applies and we are done.

5. Characterizing partition polynomials. We begin with the following generalization of Lemma 4.8.

LEMMA 5.1. *If $p(\vec{x}) = p(x_1, \ldots, x_d)$ is a polynomial, and if $c^n_{\vec{i}}$ are the unique coefficients for each $n \geq \deg p$ such that*

$$p(\vec{x}) = \sum_{i_0, i_1, \ldots, i_d \geq 0} c^n_{\vec{i}} x_0^{i_0} x_1^{i_1} \cdot \ldots \cdot x_d^{i_d},$$

then

$$\lim_{n \to \infty} \sup \left\{ \left| p\left(\frac{\vec{i}}{n}\right) - \frac{c^n_{\vec{i}}}{\binom{n}{\vec{i}}} \right| : i_0, i_1, \ldots, i_d \geq 0 \right\} = 0.$$

REMARK. Readers familiar with Bernstein polynomials may note we are showing a reverse of the usual. When g is a function defined on the simplex, we can define a higher-dimensional version of the nth Bernstein polynomial of g by

$$B_n(g)(\vec{x}) = \sum_{\vec{i}} g\left(\frac{\vec{i}}{n}\right) \binom{n}{\vec{i}} x_0^{i_0} \cdot \ldots \cdot x_d^{i_d}.$$

A common proof of Weierstrass' theorem is to show that for continuous g, $B_n(g)$ converges uniformly to g on the simplex. In Lemma 5.1, we are fixing the Bernstein polynomial p, and showing that is is well approximated by the function $g_n : (\frac{\vec{i}}{n}) \mapsto c^n_{\vec{i}}/\binom{n}{\vec{i}}$. That is, if g_n is the function for which p is the

nth Bernstein polynomial, we are showing that g_n converges uniformly to p. This g_n is defined only on a discrete set which changes with n, and so we get the awkward version of uniform convergence described in the theorem.

Proof. We first show that if the assertion holds for some $p(\vec{x})$, then it holds for $q(\vec{x}) = x_1 p(\vec{x})$ as well. Write $p(\vec{x}) = \sum_{\vec{i}} c^n_{\vec{i}} \vec{x}^{\vec{i}}$ and $q(\vec{x}) = \sum_{\vec{i}} d^n_{\vec{i}} \vec{x}^{\vec{i}}$. If we use these expansions of $q(\vec{x}) = x_1 p(\vec{x})$, apply a change of variable and equate like terms, we find that $d^{n+1}_{\vec{i}} = c^n_{\vec{i}-\vec{e}_1}$ if $i_1 > 0$, while $d^{n+1}_{\vec{i}} = 0$ if $i_1 = 0$. We are looking at the limit of

$$\sup_{i_1,\ldots,i_d,n+1-i_1-\cdots-i_d\geq 0} \left| q\left(\frac{\vec{i}}{n+1}\right) - \frac{d^{n+1}_{\vec{i}}}{\binom{n+1}{\vec{i}}} \right|.$$

Those elements where $i_1 = 0$ are all zero, so we may remove them from consideration. Doing this, and replacing i_1 with $i_1 + 1$, gives

$$\begin{aligned}
&\sup_{i_1+1,i_2,i_3,\ldots,i_d,\, n-i_1-\cdots-i_d\geq 0} \left| \frac{i_1+1}{n+1} p\left(\frac{\vec{i}+e_1}{n+1}\right) - \frac{c^n_{\vec{i}}}{\binom{n+1}{\vec{i}+e_1}} \right| \\
&\quad \leq \sup \left| \frac{i_1+1}{n+1} p\left(\frac{\vec{i}+e_1}{n+1}\right) - \frac{i_1}{n} p\left(\frac{\vec{i}}{n}\right) \right| + \sup \left| \frac{i_1}{n} p\left(\frac{\vec{i}}{n}\right) - \frac{i_1+1}{n+1} p\left(\frac{\vec{i}}{n}\right) \right| \\
&\qquad + \sup \left| \frac{i_1+1}{n+1} p\left(\frac{\vec{i}}{n}\right) - \frac{i_1+1}{n+1} \frac{c^n_{\vec{i}}}{\binom{n}{\vec{i}}} \right| + \sup \left| \frac{i_1+1}{n+1} \frac{c^n_{\vec{i}}}{\binom{n}{\vec{i}}} - \frac{c^n_{\vec{i}}}{\binom{n+1}{\vec{i}+e_1}} \right|.
\end{aligned}$$

The first of these is the distance between the value of q at two nearby points; this goes to zero by the uniform continuity of q on the compact simplex. The second of these goes to zero trivially: p is bounded on the simplex. The third (after removing the bounded common factor) goes to zero by the assumption that the theorem holds for p, and the fourth is identically zero by a property of multinomial coefficients.

We have now shown that the set of p for which the theorem holds is closed under multiplication by x_1. When $p = 1$, we have $c^n_{\vec{i}} = \binom{n}{\vec{i}}$, so $p = 1$ is in this set also, as the sequence whose limit we are taking is identically zero. Further, it is trivial to verify that the set of all p for which the claim holds is linear, and is closed under simplex permutations. Finally, any set which has these four properties contains all polynomials, and so the theorem holds always. ∎

Proof of Theorem 3.2. The second statement clearly follows from the first; we show the first statement. Our proof has nested inductions that go down to four layers deep. For clarity, we state each of the outer three inductions as Proposition 5.2, Proposition 5.3, and Claim 5.4. The fourth induction occurs in the proof of Claim 5.4 and is brief enough not to need a name.

Our first induction is on d. We will need something slightly stronger than the first statement of Theorem 3.2 as an induction hypothesis: we prove the following (by induction on d):

Proposition 5.2. *Let $p(x_1, \ldots, x_d)$ be a polynomial in d variables with partition coefficients $(c^n_{\vec{i}})_{\vec{i}}$. If the non-zero supporting polynomials of p are positive on the simplices of their respective domains* (*or are positive numbers for supporting coefficients*), *then for sufficiently large n we have $c^n_{\vec{i}} \geq 0$ for all $\vec{i}$. Furthermore, if p is non-zero and $0 < \delta < 1/d$ then*

$$\liminf_{n\to\infty} \min\left\{ c^n_{\vec{i}} \Big/ \binom{n}{\vec{i}} : i_0, i_1, \ldots, i_d > \delta n \right\} > 0.$$

(Note: The condition that $\delta < 1/d$ is present only to ensure the set minimized above is non-empty.)

Fix some $d \geq 1$, and if $d > 1$ assume that this statement holds for lower values of d. Let p be as in this statement. (We begin with our application of Lemma 5.1, which was the last step when we proved 4.1.)

Since p is a supporting polynomial of itself, we know that p is zero or is positive on $x_0, x_1, \ldots, x_d > 0$. If p is zero we are done. For small $\delta > 0$ consider the region $K = \{\vec{x} : x_0, x_1, \ldots, x_d \geq \delta\}$. Since p is strictly positive on this compact region, there is ϵ such that $0 < \epsilon < p(\vec{x})$ on K. But by Lemma 5.1, for large n we have $\left|p\left(\frac{\vec{i}}{n}\right) - c^n_{\vec{i}}/\binom{n}{\vec{i}}\right| < \epsilon/2$. Hence the lim inf above is at least $\epsilon/2$, and $c^n_{\vec{i}} > 0$ when $\vec{i}/n \in K$.

All that remains is to show that $c^n_{\vec{i}} \geq 0$ for those $\vec{i}$ where $\vec{i}/n \notin K$. That is, we must show that $c^n_{\vec{i}} \geq 0$ when $i_t/n < \delta$ for some $t = 0, \ldots, d$. Since our requirement is symmetric under simplex permutations, it is sufficient to show that there is $\delta > 0$ such that $i_d < \delta n$ implies $c^n_{\vec{i}} \geq 0$.

Note that the condition $i_d < \delta n$ is equivalent to $i_0 + \cdots + i_{d-1} \geq (1-\delta)n$. We are done then if we can show the following, which we do by induction on k.

Proposition 5.3. *If $0 \leq k \leq d-1$, there are N_k and $\delta_k > 0$ such that $i_0 + \cdots + i_k \geq (1 - \delta_k)n$ and $n > N_k$ implies $c^n_{\vec{i}} \geq 0$.*

Fix $0 \leq k \leq d-1$, and if $k > 0$, assume that we have found δ_{k-1} and N_{k-1}. By applying this induction hypothesis to simplex permutations of p, we may replace the sum $i_0 + \cdots + i_{k-1}$ with a sum of any k of the indices. That is, if $k > 0$ we assume that we have found $\hat{\delta} < 1/d$ and $\hat{N}$ with the property that whenever $S \subseteq \{0, \ldots, d\}$ with $|S| = k$, then $n \geq \hat{N}$ and $\sum_{s\in S} i_s \geq (1 - \hat{\delta})n$ implies $c^n_{\vec{i}} \geq 0$.

(In the base case we have no induction hypothesis, but the conditions required of $\hat{\delta}$ and $\hat{N}$ are vacuous: for $k = 0$, we can let $\hat{\delta} = 1/(d + 1)$ and

$\hat{N} = \deg p$ and the above requirement holds. In either case, we treat $\hat{\delta}$ and $\hat{N}$ as fixed until we finish the proof of Proposition 5.3.)

We will find δ_k and N_k as in Proposition 5.3.

To get our c's near the edge to be non-negative, we will need to know that the c's closest to the edge are strictly positive, except for those which are always zero. This set of c's which are always zero must be treated carefully, so we introduce the following notation:

$$Z = \{(i_{k+1}, \ldots, i_d) : \text{for all } j_1, \ldots, j_k,$$
$$(j_{k+1}, \ldots, j_d) \le (i_{k+1}, \ldots, i_d) \Rightarrow c_{\vec{j}}^{\deg p} = 0\}.$$

(Here and elsewhere, we write $\vec{u} \le \vec{v}$ to mean that each coordinate of $\vec{u}$ is less than or equal to the corresponding coordinate of $\vec{v}$.)

We will now show

Claim 5.4. *For each $(i_{k+1}, \ldots, i_d)$ we have:*

- *If $(i_{k+1}, \ldots, i_d) \in Z$, then $c_{\vec{i}}^n = 0$ (for all $i_1, \ldots, i_k$ and all $n \ge \deg p$).*
- *If $(i_{k+1}, \ldots, i_d) \notin Z$, then*

$$\liminf_{n\to\infty} \min\left\{ c_{\vec{i}}^n \Big/ \binom{i_0 + i_1 + \cdots + i_k}{i_0, i_1, \ldots, i_k} : i_0, i_1, \ldots, i_k > \frac{\hat{\delta}}{2} n \right\} > 0.$$

We proceed by induction on $i_{k+1} + \cdots + i_d$. As a base for our induction, we can take those cases where $i_{k+1} + \cdots + i_d < 0$; in such cases, $(i_{k+1}, \ldots, i_d) \in Z$, and $c_{\vec{i}}^n = 0$.

Let $v \ge 0$. Assume the above holds whenever $i_{k+1} + \cdots + i_d < v$, and fix some $(i_{k+1}, \ldots, i_d)$ with $i_{k+1} + \cdots + i_d = v$. We will focus on the identity

$$c_{\vec{i}}^n = (c_{\vec{i}-\vec{e}_0}^{n-1} + \cdots + c_{\vec{i}-\vec{e}_k}^{n-1}) + (c_{\vec{i}-\vec{e}_{k+1}}^{n-1} + \cdots + c_{\vec{i}-\vec{e}_d}^{n-1}) \tag{5.1}$$

We need to consider three cases.

Case 1: $(i_{k+1}, \ldots, i_d) \in Z$. We must show $c_{\vec{i}}^n = 0$. In this simplest case, we need a brief induction on n. When $n = \deg p$, $c_{\vec{i}}^n = 0$ by the definition of Z. For larger n, we examine equation (5.1). Every term on the right-hand side is zero: the first batch are zero by our current induction hypothesis on n. Since we have $(i_{k+1}, \ldots, i_d) - \vec{e}_t \in Z$ for all t, each term of the second batch is zero by our induction hypothesis from Claim 5.4.

Note that the vector $(i_{k+1}, \ldots, i_d)$ has $d - k$ coordinates, so when we subtract $\vec{e}_t$, t is ranging from 1 to $d - k$, above and in the following.

Case 2: $(i_{k+1}, \ldots, i_d) \notin Z$, and $(i_{k+1}, \ldots, i_d) - \vec{e}_t \notin Z$ for some $t = 1, \ldots, d-k$. We may apply our Claim 5.4 induction hypothesis to the second collection of terms. We see that those whose subscript is in Z are identically zero, while at least one has its subscript not in Z. For some $\epsilon > 0$, the second collection of terms is greater than $\binom{i_0+i_1+\cdots+i_k}{i_0,i_1,\ldots,i_k}\epsilon$ for sufficiently large n.

Let $f(n)$ denote the minimum whose lower limit we must show to be positive, and let $c^n_{\vec{i}}$ be an element appearing in this minimum. (So we are assuming that $i_0, \ldots, i_k > \hat{\delta} n/2$.) We examine the right-hand side of (5.1). We divide terms of the first batch of terms into two types. For some $t = 0, \ldots, k$, we may have $i_t - 1 > \hat{\delta}(n-1)/2$. When this is the case, Claim 5.4 applies to the corresponding term, immediately giving us

$$c^{n-1}_{\vec{i}-e_t} \geq f(n-1)\binom{i_0 + i_1 + \cdots + i_k - 1}{(i_0, i_1, \ldots, i_k) - \vec{e}_t},$$

so we have a lower bound for some terms of the first batch in (5.1).

For those terms with $i_t - 1 \leq \hat{\delta}(n-1)/2$, we argue that if n is large, then $c^{n-1}_{\vec{i}-e_t} \geq 0$. This occurs since if n is so large that $\hat{\delta}(n-1)/2 > v$, we will have

$$i_t - 1 + i_{k+1} + \cdots + i_d \leq \frac{\hat{\delta}(n-1)}{2} + v \leq \hat{\delta}(n-1).$$

Let $S = \{0, \ldots, k\} \setminus \{t\}$. Subtracting both sides from $n-1$ gives $\sum_{s \in S} i_s \geq (1-\hat{\delta})(n-1)$. By choice of $\hat{\delta}$, this implies $c^{n-1}_{\vec{i}-e_t} \geq 0$ for large n.

We now combine these, noting that in the above three paragraphs, our notion of *sufficiently large* n did not depend on $\vec{i}$ itself, but only on v. Assume n is sufficiently large in this sense. We will use (5.1) to compare $f(n)$ with $f(n-1)$. We will first deal with the case that $f(n-1)$ is negative. We know that the second group of terms is greater than $\epsilon\binom{i_0+i_1+\cdots+i_k}{i_0,i_1,\ldots,i_k}$. Each term of the first group is greater than either 0 or $f(n-1)\binom{i_0+i_1+\cdots+i_k-1}{(i_0,i_1,\ldots,i_k)-\vec{e}_t}$. When $f(n-1)$ is negative, the second possibility is the lesser, and we get

$$\begin{aligned} c^n_{\vec{i}} &\geq \sum_{t=0}^{k} f(n-1)\binom{i_0 + i_1 + \cdots + i_k - 1}{(i_0, i_1, \ldots, i_k) - \vec{e}_t} + \epsilon\binom{i_0 + i_1 + \cdots + i_k}{i_0, i_1, \ldots, i_k} \\ &= (f(n-1) + \epsilon)\binom{i_0 + i_1 + \cdots + i_k}{i_0, i_1, \ldots, i_k}. \end{aligned}$$

Dividing and taking a minimum gives $f(n) \geq f(n-1) + \epsilon$. This means that for large n, $f(n)$ will be positive. Once $f(n)$ is positive, we can use the same argument, but replace the first collection of terms in (5.1) with zero, yielding $f(n) \geq \epsilon$. So $\liminf f(n) > 0$.

Case 3: $(i_{k+1}, \ldots, i_d) \notin Z$, but $(i_{k+1}, \ldots, i_d) - \vec{e}_t \in Z$ for all $t = 1, \ldots, d-k$. Using the definition of Z, we see that if $(j_{k+1}, \ldots, j_d) \leq (i_{k+1}, \ldots, i_d)$ and $(j_{k+1}, \ldots, j_d) \neq (i_{k+1}, \ldots, i_d)$, then $(j_{k+1}, \ldots, j_d) \in Z$, and so $c^n_{\vec{j}} = 0$. Hence, if $c^n_{\vec{j}}$ is non-zero, we must have either $(j_{k+1}, \ldots, j_d) = (i_{k+1}, \ldots, i_d)$, or else $j_t > i_t$ for some $t \in \{k+1, \ldots, d\}$.

Thus, we can write

$$\begin{aligned} p(\vec{x}) &= \sum_{\vec{j}} c^n_{\vec{j}} x_0{}^{j_0} x_1^{j_1} \cdot \ldots \cdot x_d^{j_d} \\ &= x_{k+1}^{i_{k+1}} \cdot \ldots \cdot x_d^{i_d} \sum_{\vec{j} \in I} c^n_{\vec{j}} x_0^{j_0} x_1^{j_1} \cdot \ldots \cdot x_k^{j_k} \\ &\quad + x_{k+1}^{i_{k+1}+1} g_{k+1}(\vec{x}) + \cdots + x_d^{i_d+1} g_d(\vec{x}), \end{aligned}$$

where $g_{k+1}, \ldots, g_d$ are some polynomials, and I is the set of all $\vec{j}$ such that the last $d-k$ coordinates of $\vec{j}$ are $(j_{k+1}, \ldots, j_d) = (i_{k+1}, \ldots, i_d)$.

We now expand the multinomial $x_0^{j_0}$, writing it as

$$((1 - x_1 - \cdots - x_k) - x_{k+1} - \cdots - x_d)^{j_0}.$$

We will get one term of the form $(1 - x_1 - \cdots - x_k)^{j_0}$, and all other terms will have a factor of x_t for some $k+1 \le t \le d$, and so can be lumped in with the g's. We get

$$\begin{aligned} p(\vec{x}) &= x_{k+1}^{i_{k+1}} \cdot \ldots \cdot x_d^{i_d} \sum_{\vec{j} \in I} c^n_{\vec{j}} (1 - x_1 - \cdots - x_k)^{(n-v)-j_1-\cdots-j_k} x_1^{j_1} \cdot \ldots \cdot x_k^{j_k} \\ &\quad + x_{k+1}^{i_{k+1}+1} \hat{g}_{k+1}(\vec{x}) + \cdots + x_d^{i_d+1} \hat{g}_d(\vec{x}). \end{aligned}$$

Let $s(x_1, \ldots, x_k)$ denote the sum over $\vec{j}$ in the above display. We notice that s is a supporting polynomial of p. Furthermore, for a given choice of $(i_{k+1}, \ldots, i_d)$, this defines s, and so $s(x_1, \ldots, x_k)$ does not depend on n, and the expression of s in partition form above holds for all n. Letting $d^n_{j_1,\ldots,j_k}$ denote the partition coefficient of s, the above display shows that $d^n_{j_1,\ldots,j_k} = c^{n+v}_{j_1,\ldots,j_k,i_{k+1},\ldots,i_d}$.

The relation "is a supporting polynomial of" is transitive, and so s satisfies the conditions of the induction hypothesis of Proposition 5.2. For any $0 < \rho < 1/d$, if n is sufficiently large, we must have $d^n_{j_1,\ldots,j_k} \ge 0$, and also

$$\liminf_{n\to\infty} \min\left\{ d^n_{j_1,\ldots,j_k} \Big/ \binom{n}{j_1, \ldots, j_k} : (n - j_1, \ldots - j_k), j_1, \ldots, j_k > \rho n \right\} > 0.$$

Replacing n with $n - v$ in the above, and choosing $\rho < \hat{\delta}/2$, will give the desired statement, concluding our proof of Claim 5.4 in Case 3.

Before moving on, we make an observation about when Case 3 can occur. Note that s cannot be zero, as this would imply that each $c^n_{\vec{j}}$ in the sum above is zero, meaning that $(i_{k+1}, \ldots, i_d) \in Z$, which we have assumed it is not. Simplifying the above, s yields a non-zero term of p not canceled elsewhere, so we must have $\deg p \ge v = i_{k+1} + \cdots + i_d$.

That is: If $(u_{k+1}, \ldots, u_d) \notin Z$, and $u_{k+1} + \cdots + u_d > \deg p$, then there is some $1 \le t \le d-k$ such that $(u_{k+1}, \ldots, u_d) - e_t \notin Z$.

We have now finished showing Claim 5.4, whose awkward statement was necessary for the induction to work. What we actually need is the following simpler statement, which comes from applying Claim 5.4 to a finite number of choices of $(i_{k+1}, \ldots, i_d)$. In the following, $\hat{\delta}$ and $\hat{N}$ are still as defined shortly after the statement of Proposition 5.3.

CLAIM 5.5. *Let $u = \deg p$. There are $\epsilon > 0$ and $\tilde{N} \geq \hat{N}$ such that $c_{\vec{i}}^{\tilde{N}} \geq 0$ whenever $i_{k+1} + \cdots + i_d \leq u$, and $c_{\vec{i}}^{\tilde{N}} > \epsilon$ whenever $i_{k+1} + \cdots + i_d = u$, $(i_{k+1}, \ldots, i_d) \notin Z$, and $i_t > \hat{\delta}\tilde{N}/2$ for each $t = 0, \ldots, k$.*

All that remains is to show Proposition 5.3. We want to prove that there are some $\delta_k > 0$ and N_k such that $i_{k+1} + \cdots + i_d < \delta_k n$, $n \geq N_k$, implies $c_{\vec{i}}^n \geq 0$.

We may assume that $\delta_k < \hat{\delta}/2$. This assumption means we are concerned only with the case that $i_{k+1} + \cdots + i_d < \hat{\delta} n/2$.

Let $-M = \min\{c_{\vec{i}}^{\tilde{N}}\}$. We can assume that $M > 0$, since if not, then all $c_{\vec{i}}^{\tilde{N}}$ are positive, and hence all $c_{\vec{i}}^n$ are positive for $n > \tilde{N}$, and we are done.

The recursion relation $c_{\vec{i}}^{n+1} = \sum_{t=0}^{d} c_{\vec{i}-e_t}^n$ easily generalizes to

$$c_{\vec{i}}^{N+z} = \sum_{\vec{\alpha} \in \mathbb{Z}^n} \binom{z}{\vec{\alpha}} c_{\vec{i}-\vec{\alpha}}^N.$$

We use this to write a general $c_{\vec{i}}^{\tilde{N}+n}$ in terms of the level $\tilde{N}$ values, for which we have inequalities. Fix $\vec{i}$, and assume $i_{k+1} + \cdots + i_d < \hat{\delta} n/2$. We may write

$$c_{\vec{i}}^{\tilde{N}+n} = \sum_{\vec{\alpha}} \binom{n}{\vec{\alpha}} c_{\vec{i}-\vec{\alpha}}^{\tilde{N}} = \sum_{\vec{\alpha} \in I_1} \binom{n}{\vec{\alpha}} c_{\vec{i}-\vec{\alpha}}^{\tilde{N}} + \sum_{\vec{\alpha} \in I_2} \binom{n}{\vec{\alpha}} c_{\vec{i}-\vec{\alpha}}^{\tilde{N}} + \sum_{\vec{\alpha} \in I_3} \binom{n}{\vec{\alpha}} c_{\vec{i}-\vec{\alpha}}^{\tilde{N}},$$

where

$$\begin{aligned}
I_1 = \{\vec{\alpha} : {} & (i_{k+1} - \alpha_{k+1}) + \cdots + (i_d - \alpha_d) < u \text{ or} \\
& (i_{k+1} - \alpha_{k+1}, \ldots, i_d - \alpha_d) \in Z \text{ or } i_t - \alpha_t \leq \hat{\delta}\tilde{N}/2 \text{ for some } t = 0, \ldots, k\}, \\
I_2 = \{\vec{\alpha} : {} & (i_{k+1} - \alpha_{k+1}) + \cdots + (i_d - \alpha_d) = u \\
& \text{and } (i_{k+1} - \alpha_{k+1}, \ldots, i_d - \alpha_d) \notin Z \text{ and } i_0 - \alpha_0, \ldots, i_k - \alpha_k \geq \hat{\delta}\tilde{N}/2\}, \\
I_3 = \{\vec{\alpha} : {} & (i_{k+1} - \alpha_{k+1}) + \cdots + (i_d - \alpha_d) > u \\
& \text{and } (i_{k+1} - \alpha_{k+1}, \ldots, i_d - \alpha_d) \notin Z \text{ and } i_0 - \alpha_0, \ldots, i_k - \alpha_k \geq \hat{\delta}\tilde{N}/2\}.
\end{aligned}$$

On each of these sets we use a lower bound on $c_{\vec{i}-\vec{\alpha}}^{\tilde{N}}$. On I_2, we have $c_{\vec{i}-\vec{\alpha}}^{\tilde{N}} \geq \epsilon$, by choice of ϵ. On I_3, we have no good bound, so we use $c_{\vec{i}-\vec{\alpha}}^{\tilde{N}} \geq -M$.

On I_1, we have $c_{\vec{i}-\vec{\alpha}}^{\tilde{N}} \geq 0$: This is clear for either of the first two conditions. If the third holds, we may combine it with our assumption that

$i_{k+1}+\cdots+i_d < \hat{\delta}\tilde{N}/2$ to find $(i_t-\alpha_t)+(i_{k+1}-\alpha_{k+1})+\cdots+(i_d-\alpha_d) < \hat{\delta}\tilde{N}$. Subtracting both sides from $\tilde{N}$ and letting $S = \{0,\ldots,k\} \setminus \{t\}$, we have $\sum_{s\in S}(i_s-\alpha_s) > (1-\hat{\delta})\tilde{N}$. This gives $c^{\tilde{N}}_{\vec{i}-\vec{\alpha}} \geq 0$ by the induction hypothesis of Proposition 5.3.

Thus, we have

$$c^{\tilde{N}+n}_{\vec{i}} \geq 0 \sum_{\vec{\alpha}\in I_1} \binom{n}{\vec{\alpha}} + \epsilon \sum_{\vec{\alpha}\in I_2} \binom{n}{\vec{\alpha}} - M \sum_{\vec{\alpha}\in I_3} \binom{n}{\vec{\alpha}}.$$

We wish to group terms together according to what value (greater than or equal to u) they have for $(i_{k+1}-\alpha_{k+1})+\cdots+(i_d-\alpha_d)$. We write $C(x) = \sum \binom{n}{\vec{\alpha}}$, where the sum is over all $\vec{\alpha}$ with $(i_{k+1}-\alpha_{k+1})+\cdots+(i_d-\alpha_d) = u+x$, with $(i_{k+1}-\alpha_{k+1},\ldots,i_d-\alpha_d) \notin Z$, and with $i_0-\alpha_0,\ldots,i_k-\alpha_k \geq \hat{\delta}n/2$.

Under this notation, we have shown that

$$c^{\tilde{N}+n}_{\vec{i}} \geq \epsilon C(0) - M\sum_{t=1}^{\infty} C(t).$$

For our $\vec{i}$, let $\rho = (i_{k+1}+\cdots+i_d)/n$. We are then trying to find δ_k such that $c^{N_1+n}_{\vec{i}}$ is non-negative as long as $\rho < \delta_k$. We will do this by showing $C(x+1) \leq \frac{(k+1)\rho}{1-\rho}C(x)$ for all $x \geq 0$. From this, we will know that

$$c^{\tilde{N}+n}_{\vec{i}} \geq \left(\epsilon - M\sum_{t=1}^{\infty}\left(\frac{(k+1)\rho}{1-\rho}\right)^t\right)C(0).$$

We have $C(0) \geq 1$ for any legal choice of $\vec{i}$. (This is true but tricky to verify and unnecessary for Proposition 5.3; if $C(0)$ were zero, the above would hold and we would have $c^{\tilde{N}+n}_{\vec{i}} \geq 0$ for all ρ. Assume therefore that $C(0) \geq 1$.) Also this geometric series converges if ρ is sufficiently small. The right-hand side is therefore a rational function which depends only on ρ. This rational function equals $\epsilon C(0)$ at zero, and so the existence of δ_k follows from the continuity of this rational function at zero.

All that remains is to verify $C(x+1) \leq \frac{(k+1)\rho}{1-\rho}C(x)$. We do this by a counting argument. (We eschew the coloring metaphor from $d=2$, as it is less helpful in the case of d colors.) Recall that $\binom{n}{\vec{\alpha}}$ counts the number of functions $f : \{1,\ldots,n\} \to \{0,1,\ldots,d\}$ such that the number of elements in $f^{-1}(j)$ is α_j for all $0 \leq j \leq d$. (We have defined the multinomial coefficient to be zero if any α_j is negative, so this fact remains true even for nonsensical $\vec{\alpha}$.)

We may thus interpret $C(x)$ as the number of functions f which map $\{1,\ldots,n\}$ to $\{0,1,\ldots,d\}$ and which satisfy:

- $\#f^{-1}(\{k+1,\ldots,d\}) = i_{k+1} + \cdots + i_d - u - x$
  (or equivalently, $\#f^{-1}(\{0,1,\ldots,k\}) = n + u + x - (i_{k+1} + \cdots + i_d)$),
- $(i_{k+1} - \#f^{-1}(k+1), \ldots, i_d - \#f^{-1}(d)) \notin Z$,
- $i_0 - \#f^{-1}(0), \ldots, i_k - f^{-1}(k) > \hat{\delta}n/2$.

Fixing x, we define D to be the number of functions f from $\{1,\ldots,n\}$ to $\{-1,0,1,\ldots,d\}$ that satisfy:

- $\#f^{-1}(\{k+1,\ldots,d\}) = i_{k+1} + \cdots + i_d - u - x - 1$,
- $\#f^{-1}(-1) = 1$,
- $(i_{k+1} - \#f^{-1}(k+1), \ldots, i_d - \#f^{-1}(d)) \notin Z$,
- $i_0 - \#f^{-1}(0), \ldots, i_k - f^{-1}(k) > \hat{\delta}n/2$.

(If the second condition holds, the first says $\#f^{-1}(\{0,\ldots,k\}) = n+u+x-i_{k+1} - \cdots - i_d$.)

Choose an f of the type counted by $C(x+1)$. If y is an element of $f^{-1}(\{0,\ldots,k\})$, we may change the value of $f(y)$ to be -1 and we will get a function of the type counted by D. Any function from D can be so constructed at most $k+1$ times (once for each possible previous value of $f(y)$). Counting the number of ways to choose such a y, we find

$$(n + u + x + 1 - (i_{k+1} + \cdots + i_d))C(x+1) \leq (k+1)D.$$

Now let f be a function of the type counted by D. By the observation preceding Claim 5.5, we know that there is $v \in \{1,\ldots,d-k\}$ such that $(i_{k+1} - \#f^{-1}(k+1), \ldots, i_d - \#f^{-1}(d)) - e_v \notin Z$. If x is the unique value with $f(x) = -1$, we may adjust $f(x)$ to be v, and the resulting function will be of the type counted by $C(x)$. Any $C(x)$-type function constructed in this way will be so constructed at most once for each element of $f^{-1}(\{k+1,\ldots,d\})$. (Only such an element could have been the x in the construction.) Thus

$$D \leq (i_{k+1} + \cdots + i_d - u - x)C(x).$$

Combining these two inequalities yields

$$(n+u+x+1-(i_{k+1}+\cdots+i_d))C(x+1) \leq (k+1)(i_{k+1}+\cdots+i_d-u-x)C(x).$$

Decreasing the lower bound and increasing the upper shows that

$$(1-\rho)nC(x+1) \leq (k+1)\rho nC(x).$$

Dividing both sides by $(1-\rho)n$, we are done and Theorem 3.2 is proved. ■

Finally, we conclude by stating a corollary which allows one to build partition polynomials without discussing supporting polynomials.

COROLLARY 5.6. *If $p(x_1,\ldots,x_d)$ is a non-constant partition polynomial, and $k = \deg p$, then for any integer polynomial $g(\vec{x})$, the polynomial*

$$q(\vec{x}) = p(\vec{x}) + (1 - x_1 - \cdots - x_d)^{k+1} x_1^{k+1} \cdot \ldots \cdot x_d^{k+1} g(\vec{x})$$

is a partition polynomial if and only if it satisfies $0 < q(\vec{x}) < 1$ *on the region of* $x_1, \dots, x_d$ *with* $1 - x_1 - \cdots - x_d > 0$.

Proof. Note that in the contribution made to p to produce q, every term has a factor of x_d^{k+1} which is greater than occurs on any term of p, and such terms do exist since $p \neq 0$. Thus, an exponent vector $(a_{v+1}, \dots, a_d)$ of q which uses $a_d \geq k+1$ cannot be minimal (so any proper standard supporting polynomial of q is also one for p). Similarly, for any minimal exponent vector of p, we have $a_d < k+1$, meaning that all extra terms for q can be lumped in with the $x_d^{1+a_d}$ garbage polynomial (so any proper standard supporting polynomial of p is one for q).

The contribution to p has similar factors for all x_i, so the same argument applies to any simplex permutation of p and q, hence p and q have exactly the same proper supporting polynomials, and thus every supporting polynomial of q is positive.

Finally, for the non-proper case, we clearly see that the simplex permutations of q and $1-q$ are positive on the simplex, so q is a partition polynomial by Theorem 3.2. ∎

6. Good measures. A *Cantor space* is any compact perfect totally disconnected metrizable space. (Any such space is homeomorphic to $\{0,1\}^{\mathbb{N}}$.) Let μ be a probability measure on (the Borel subsets of) a Cantor space X. We assume μ is *full* (has positive measure on open sets) and *non-atomic* (points have measure zero).

We say such a measure is *good in the sense of Akin* or simply *good* if whenever U and V are clopen sets in X with $\mu(U) \leq \mu(V)$, there is a clopen set $\hat{U} \subseteq V$ with $\mu(\hat{U}) = \mu(U)$. That is, if α is the measure of some clopen set, then there are many clopen sets of measure α, and they can be found wherever there is room for one.

Good measures are interesting for a number of reasons, mainly to do with constructing measure-preserving homeomorphisms. If μ and ν are probability measures on a Cantor space X, one might expect it to be trivial to verify that there is a homeomorphism of Cantor space which sends μ to ν. This is not the case however: The *clopen values set* of μ, defined to be $\{\mu(E) : E \text{ is clopen}\}$, is a countable, dense subset of $[0,1]$. Two measures will typically have different clopen values sets, and thus not be homeomorphic. This invariant is not sufficient in general to determine if two measures are homeomorphic, but among good measures, it is (Akin [2]).

Another notable property of good measures is that they are precisely the invariant measures of uniquely ergodic homeomorphisms of Cantor space. That is, if T is a uniquely ergodic homeomorphism of a Cantor space, then its unique invariant measure is good (Glasner and Weiss [6]), and if a measure

μ is good, then there is a uniquely ergodic transformation for which it is the unique invariant measure (Akin [2]).

Given a probability vector $\vec{p} = (p_0, p_1, \ldots, p_d)$, we define $\beta(\vec{p})$ to be the Bernoulli trial measure on (the Borel subsets of) $X = \{0, 1, \ldots, d\}^{\mathbb{N}}$. That is, it is the countable product of independent copies of $\vec{p}$. Surprisingly, there is a large class of good measures found among the Bernoulli trial measures.

Dougherty, Mauldin and Yingst [5] showed that the measure $\beta(1-r, r)$ is good when there is an integer polynomial $R(x)$ with $R(0) = \pm 1$, $R(1) = \pm 1$, and r is the unique root of R in $[0, 1]$. (Equivalently, $\beta(1 - r, r)$ is good precisely when r is an algebraic number in $(0, 1)$ with no conjugates in $(0, 1)$ and such that $1/r$ and $1/(1 - r)$ are algebraic integers.)

Akin, Dougherty, Mauldin and Yingst [3] looked at the case of more than two symbols, finding many partial results in the general case, and completely solving it for a rational measure, in a result we will state shortly.

The notion of an algebraic conjugate is tricky in multiple symbols. (The ring $\mathbb{Z}[x, y]$ is not a principal ideal domain and so a point in $\mathbb{R}^2$ does not have a single minimal polynomial.) Given a point $(p_1, \ldots, p_d)$ in $\mathbb{R}^d$, we let $Z(p_1, \ldots, p_d)$ denote the set of all integer polynomials which equal zero at $(p_1, \ldots, p_d)$. The notion of the set of algebraic conjugates of $(p_1, \ldots, p_d)$ is made precise by considering (as we often shall) the set of points at which every element of $Z(p_1, \ldots, p_d)$ vanishes.

Also, we note the connection between partition polynomials and Bernoulli trial measures: From our understanding of partition polynomials, it is clear that given a subset E of $\{0, \ldots, d\}^{\mathbb{N}}$ which depends only on finitely many coordinates, there is a partition polynomial f such that $\beta(x_0, \ldots, x_d)(E) = f(x_1, \ldots, x_d)$. But in the space $\{0, \ldots, d\}^{\mathbb{N}}$, a set depends on only finitely many coordinates precisely when it is clopen. (Such a set is clearly clopen. A clopen set is open and so is covered by the basic open sets inside it; there is a finite subcover, each element of which depends on finitely many coordinates.) Thus, each clopen set in $\{0, \ldots, d\}^{\mathbb{N}}$ has an *associated* partition polynomial, which gives its Bernoulli trial measure. Conversely, given a partition polynomial f in d variables, there is a clopen set E for which f gives the Bernoulli trial measure of E. This clopen set *associated* with f is usually not unique, while the polynomial associated with a given clopen set clearly is unique.

THEOREM 6.1. *Let $(p_0, \ldots, p_d)$ be a probability vector. Then $\beta(p_0, \ldots, p_d)$ is good if and only if there exists an integer polynomial $f(x_1, \ldots, x_d)$ such that $(p_1, \ldots, p_d)$ is the only point in the simplex at which f equals zero, and f equals one on the boundary of the simplex.*

Equivalently, $\beta(p_0, \ldots, p_d)$ is good if and only if the point $(p_1, \ldots, p_d)$ is the only point in the simplex at which all polynomials in $Z(p_1, \ldots, p_d)$ equal *zero, and also $Z(p_1, \ldots, p_d)$ contains $f_0, f_1, \ldots, f_d$ such that f_t equals one identically on the hyperplane $x_t = 0$ for $t = 0, 1, \ldots, d$.*

Proof. First we show that these two conditions are equivalent. The first clearly implies the second; suppose the second holds. There must be a finite set of polynomials in $Z(p_1, \ldots, p_d)$ among which $(p_1, \ldots, p_d)$ is the only common zero in the simplex. (This follows from Hilbert's basis theorem.) If we square these and add them, we will get a single polynomial ϕ for which $(p_1, \ldots, p_d)$ is the only zero in the simplex. Now consider the polynomial

$$(1 - (1 - f_0)(1 - f_1) \cdot \ldots \cdot (1 - f_d))^2 + x_0 \cdot x_1 \cdot \ldots \cdot x_d \phi.$$

This polynomial is zero at $(p_1, \ldots, p_d)$, is positive elsewhere in the simplex, and equals one when any x_t is zero, and so the first condition holds.

We will prove that a good measure satisfies the second condition. Suppose that $\beta(p_0, \ldots, p_d)$ is good. Let $\vec{b} = (b_1, \ldots, b_d)$ be a point in the simplex other than $\vec{p} = (p_1, \ldots, p_d)$. It is simple to verify that there are partition polynomials q_U and q_V with $q_V(\vec{p}) > q_U(\vec{p})$, with $q_U(\vec{b}) > q_V(\vec{b})$, and with $q_V = 0$ and $q_U = 1$ on the boundary of the simplex.

(One strategy is to let $(m_1/n, \ldots, m_d/n)$ be a rational vector very close to $\vec{p}$. The polynomial $x_1^{m_1} \cdot \ldots \cdot x_d^{m_d} \cdot x_0^{n-m_1-\cdots-m_d}$ is positive on the simplex, with most of its mass centered at its local maximum at $(m_1/n, \ldots, m_d/n)$. For some positive integer A, the polynomial $q_V(\vec{x}) = A x_1^{m_1} \cdot \ldots \cdot x_d^{m_d} \cdot x_0^{n-m_1-\cdots-m_d}$ will be sufficiently large at $\vec{p}$ (say greater than $1/2$) but still less than 1 on the simplex. If n is sufficiently large then we will have $q_V(\vec{b}) < 1/2$. Let $q_U = 1 - q_V$. Any non-zero supporting polynomial of a simplex permutation of q_V is an expression of a similar form to q_V, and so it is positive on its simplex. Since q_U equals one on the boundary, we can verify that the only non-zero proper supporting polynomial of q_U is one. So q_U and q_V are partition polynomials as desired, by Theorem 3.2.)

Let U and V be clopen sets in $\{0, \ldots, d\}^{\mathbb{N}}$ associated with the partition polynomials q_U and q_V, respectively. We have $q_U(\vec{p}) < q_V(\vec{p})$, and so $\beta(p_0, \ldots, p_d)(U) < \beta(p_0, \ldots, p_d)(V)$. Since this measure is good, there must be $\hat{U}$ with $\hat{U} \subseteq V$ and $\beta(p_0, \ldots, p_d)(U) = \beta(p_0, \ldots, p_d)(\hat{U})$. Let $\hat{q}$ be the partition polynomial associated with $\hat{U}$. From $\hat{U} \subseteq V$, it follows that $\beta(x_0, \ldots, x_d)(\hat{U}) \le \beta(x_0, \ldots, x_d)(V)$, and so $\hat{q} \le q_V$ on the simplex, and in particular $\hat{q} = 0$ on the boundary.

Combining the above, we have $\hat{q}(\vec{p}) = q_U(\vec{p})$, but $\hat{q}(\vec{b}) \le q_V(\vec{b}) < q_U(\vec{b})$. Hence, $q_U - \hat{q}$ is a polynomial in $Z(\vec{p})$ which equals $1 - 0$ on the boundary of the simplex. Furthermore, $q_U - \hat{q}$ is not zero at $\vec{b}$, and such a polynomial exists for any $\vec{b}$ in the simplex other than $\vec{p}$.

Next, we show that if f exists as in the first statement of the theorem, then $\beta(\vec{p})$ is good. Note that $f \ge 0$ on the simplex. It is possible that f

gets as large as two on the simplex. If this occurs, we may replace f with $f_1(\vec{x}) = (1 - x_0x_1 \cdot \ldots \cdot x_d)^N f(\vec{x})$ for some large N. (We already have $f < 2$ near the boundary, and as N increases, $f_1 \to 0$ uniformly on compact subsets of the interior of the simplex.) This f_1 still equals one on the boundary and has $\vec{p}$ as its only zero in the simplex.

Now observe that $1 - f_1$ vanishes on the boundary of the simplex, and so has a factor of $x_0x_1 \cdot \ldots \cdot x_d$. If we replace f_1 with $f_2 = 1 - (1 - f_1)^2$, then f_2 still has $\vec{p}$ as its only zero in the simplex (since we have ensured $f_1 \neq 2$), f_2 equals one on the boundary, and $1 - f_2$ has a factor of $(x_0x_1 \cdot \ldots \cdot x_d)^2$. Furthermore, we have $f_2 \leq 1$ on the interior of the simplex.

Next consider replacing f_2 with $f_3(\vec{x}) = (1 - x_0x_1 \cdot \ldots \cdot x_d) f_2(\vec{x})$. We may note that again f_3 also has $\vec{p}$ as its only zero in the simplex, and equals one on the boundary of the simplex. Further, the non-zero supporting polynomials of $1 - f_3$ are all positive on the interiors of their respective simplices. To see this, first note that $f_2 \leq 1$ on the closed simplex implies that $1 - f_3$ itself is positive on the interior of the simplex. For the proper supporting polynomials, recall that $1 - f_2$ is divisible by $(x_0x_1 \cdot \ldots \cdot x_d)^2$, and so we may write $f_2(\vec{x}) = 1 - (x_0x_1 \cdot \ldots \cdot x_d)^2 g_2(\vec{x})$ for some integer polynomial $g_2(\vec{x})$. Hence,

$$1 - f_3(\vec{x}) = x_0x_1 \cdot \ldots \cdot x_d + (x_0x_1 \cdot \ldots \cdot x_d)^2 g_2(\vec{x})(1 - x_0x_1 \cdot \ldots \cdot x_d).$$

Since $x_0x_1 \cdot \ldots \cdot x_d$ divides $1 - f_3$, it follows that if $s(x_1, \ldots, x_v)$ is a non-zero standard supporting polynomial of $1 - f_3$ with exponent vector $(a_{v+1}, \ldots, a_d)$, we must have $(a_{v+1}, \ldots, a_d) \geq (1, \ldots, 1)$. But inspecting the above expression of $1 - f_3$ we see that the exponent vector $(a_{v+1}, \ldots, a_d) = (1, \ldots, 1)$ does yield a non-zero standard supporting polynomial of $x_1x_2 \cdot \ldots \cdot x_v(1 - x_1 - \cdots - x_v)$, and so this can be the only non-zero supporting polynomial for this choice of v, and it is positive on the interior of its simplex. Looking at the form of the above expression for $1 - f_3$, the same argument also applies to simplex permutations of $1 - f_3$.

Suppose U and V are clopen sets in $\{0, \ldots, d\}^{\mathbb{N}}$ such that the $\beta(p_0, \ldots, p_d)$ measure of U is less than that of V. Let q_U and q_V be their associated partition polynomials, respectively. So $q_U(\vec{p}) < q_V(\vec{p})$. Consider $\hat{q}_V(\vec{x}) = q_V(\vec{x})(1 - f_3(\vec{x}))^N$ for some sufficiently large N. We argue that $\hat{q}_V$ and $q_U - \hat{q}_V$ have supporting polynomials that are positive on their respective simplices. The first of these is easy: by Proposition 3.3, the supporting polynomials of $\hat{q}_V$ are positive since those of q_V and $1 - f_3$ are.

Next we show that the supporting polynomials of $q_U - \hat{q}_V$ are positive on their open simplices. The important properties of $\hat{q}_V$ are preserved under simplex permutations, while q_U is arbitrary. The argument that follows will therefore apply to simplex permutations of $q_U - \hat{q}_V$, and we will concern ourselves only with the standard supporting polynomials: first the proper ones and then $q_U - \hat{q}_V$ itself.

Assume N was chosen so large that it exceeds $\deg q_U$. Then x_d^N is not a factor of q_U but is a factor of $\hat{q}_V$. As in the proof of Corollary 5.6, we have adjusted q_U by adding only terms with a larger power of x_d than any term of q_U, and so the proper standard supporting polynomials of q_U and $q_U - \hat{q}_V$ are the same. The standard supporting polynomials of $q_U - \hat{q}_V$ are therefore positive because q_U is a partition polynomial.

All that remains is to show that $q_U - \hat{q}_V$ itself (and therefore any simplex permutation) is positive on the open simplex. We do this over three regions: At $\vec{p}$, we have $\hat{q}_V = q_V < q_U$, so on some neighborhood of $\vec{p}$ we have $\hat{q}_V \leq q_V < q_U$. Since q_U is a non-zero partition polynomial, we may write it in partition form for some n and find a term $x_0^{i_0} \cdot \ldots \cdot x_d^{i_d}$ which is less than q_U on the simplex. Since $1 - f_3$ has a factor of $x_0 \cdot \ldots \cdot x_d$, there will be some large N for which $\hat{q}_V = q_V(1 - f_3)^N$ has more than i_t factors of x_t, and so $\hat{q}_V(\vec{x}) < x_0^{i_0} \cdot \ldots \cdot x_d^{i_d} < q_U(\vec{x})$ when $\vec{x}$ is inside the simplex and within δ of the boundary. (The same δ will also work for larger N since $\hat{q}_V$ will only decrease.) Away from the boundary and away from $\vec{p}$ we have $q_U > 0$ and $1 - f_3 < 1$. This region is compact, so $q_U > \epsilon$ and $1 - f_3 < 1 - \epsilon$ for some $\epsilon > 0$. When N is so large that $(1 - \epsilon)^N < \epsilon$, we have $q_V(1 - f_3)^N < q_U$ on this region.

Finally, $\hat{q}_V$, q_U, and $q_U - \hat{q}_V$ all have positive supporting polynomials, and so have non-negative partition coefficients for large n. Since each level n partition coefficient of q_U is the sum of the corresponding level n partition coefficients of $\hat{q}_V$ and $q_U - \hat{q}_V$, we see that if n is large then the partition coefficients of $\hat{q}_V$ are positive and are less than those of q_U.

Let n also be sufficiently large that U depends only on the first n coordinates of $\{0, \ldots, d\}^{\mathbb{N}}$. Recall that the partition coefficient $a_{\vec{i}}^n$ of q_U gives the number of words of length n which are permitted to appear as the first n coordinates of a point in U such that the symbol 't' occurs in this word exactly i_t times for each t. By throwing away some of these words leaving a number equal to the corresponding partition coefficient of $\hat{q}_V$, we will find a clopen subset of U whose associated partition polynomial is $\hat{q}_V$. The $\beta(p_0, \ldots, p_d)$ measure of this set is $\hat{q}_V(\vec{p}) = q_V(\vec{p}) = \beta(p_0, \ldots, p_d)(V)$. So $\beta(p_0, \ldots, p_d)$ is good. ∎

The existence of polynomials which equal one on the faces of the boundary can be a tricky question. A direct approach is as follows: Hilbert's basis theorem ensures that the ideal

$$Z(\vec{p}) = \{f(\vec{x}) \in \mathbb{Z}[x_1, \ldots, x_d] : f(\vec{p}) = 0\}$$

is finitely generated. That is, there are integer polynomials $f_1, \ldots, f_k$ such that

$$Z(\vec{p}) = [f_1, \dots, f_d] = \{f_1 g_1 + \dots + f_k g_k : g_1, \dots, g_k \in \mathbb{Z}[x_1, \dots, x_d]\}.$$

(We are being precise since much of the literature is more focused on $\mathbb{Q}[x_1, \dots, x_d]$, but Hilbert's basis theorem does apply in this sense.) Now the question of whether this ideal contains an element which equals one on the face $x_1 = 0$ is equivalent to the question of whether 1 is in the ideal $[\hat{f}_1, \dots, \hat{f}_k] \subseteq \mathbb{Z}[x_2, \dots, x_d]$ where $\hat{f}_t(x_2, \dots, x_d) = f_t(0, x_2, \dots, x_d)$. To determine whether 1 is in this ideal, we need only calculate a strong Gröbner basis (over $\mathbb{Z}$) of this ideal and check whether 1 is in it, and similarly we can handle the other edges. (Again, one has to be careful since it is more common to find discussion of Gröbner bases over a field. One nice treatment of Gröbner bases over a ring is by Adams and Loustaunau [1].)

The computation of a Gröbner basis can give an algorithm for determining whether a given measure is good, but in many simple cases is unnecessary. Note that $p(\vec{x})$ equals one on the hyperplane $x_t = 0$ if and only if x_t is a factor of $p(\vec{x}) - 1$. Thus, such a polynomial is precisely one of the form

$$f(\vec{x}) = 1 + x_t g(\vec{x})$$

for some integer polynomial g. Asking such a polynomial to equal zero at $\vec{p}$ is then equivalent to asking $1/p_t = -g(\vec{p})$ for some integer polynomial g. But the elements that can appear as $-g(\vec{p})$ are exactly those elements of the ring $\mathbb{Z}[p_1, \dots, p_d] = \mathbb{Z}[p_0, p_1, \dots, p_d]$. (It is clear that these are the same ring.) We therefore have the following equivalent formulation of Theorem 6.1:

Corollary 6.2. *Suppose that $(p_0, \dots, p_d)$ is a probability vector. Then $\beta(p_0, \dots, p_d)$ is good if and only if the point $(p_1, \dots, p_d)$ is the only point in the simplex at which all polynomials in $Z(p_1, \dots, p_d)$ equal zero, and also each p_t for $t = 0, \dots, d$ is a unit in the ring $\mathbb{Z}[p_0, \dots, p_d]$.*

A few examples are in order.

Consider $\beta\big(1 - \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{8}}, \frac{1}{\sqrt{8}}\big)$. Is this measure good? The polynomials $8x^2 - 1$ and $8y^2 - 1$ have only one common zero in the simplex, so the first condition holds easily. It is simple to verify that the ring $\mathbb{Z}\big[\frac{1}{\sqrt{8}}, \frac{1}{\sqrt{8}}\big]$ is precisely those numbers of the form $(A + B\sqrt{2})/2^n$ for integers A, B, n with $n \geq 0$. This set includes $\sqrt{8} = 2\sqrt{2}$ and $1/(1 - 1/\sqrt{2}) = 2 + \sqrt{2}$, so each coordinate of (p_0, p_1, p_2) is a unit in this ring, and the measure is good.

Is the measure $\beta\big(1 - \frac{3}{\sqrt{11}}, \frac{1}{\sqrt{11}}, \frac{2}{\sqrt{11}}\big)$ good? Again it is clear that $\big(\frac{1}{\sqrt{11}}, \frac{2}{\sqrt{11}}\big)$ is the unique root in the simplex of the ideal of polynomials equaling zero there. The ring we consider is those numbers expressible in the form $\frac{A + B\sqrt{11}}{11^n}$. Even denominators are not allowed, so $\frac{2}{\sqrt{11}}$ is not a unit in this ring, and the measure is not good.

We conclude with the following theorem, originally proved in [3], which shows when a rational vector is good.

THEOREM 6.3 (Akin, Dougherty, Mauldin, Yingst). *Let* $(p_0, \ldots, p_d)$ *be a rational probability vector. Then* $\beta(p_0, \ldots, p_d)$ *is good iff every prime factor of the numerator of* (*the reduced form of*) *any of the* p_t *appears as a prime factor of the denominator of* (*the reduced form of*) *one of the* p_t.

Proof. It is clear that the linear polynomials which specify each rational coordinate specify $\vec{p} = (p_1, \ldots, p_d)$ as their only common zero in $\mathbb{R}^d$. Let u be the least common denominator of $\{p_0, \ldots, p_d\}$. It is also easy to see that the ring generated by $\{p_0, \ldots, p_d\}$ is precisely the set of fractions of the form A/u^n, where A, n are integers and $n \geq 0$. The coordinate p_t is a unit in this ring if and only if it is expressible in the form u^n/A for some n and A. A number can be written in this form exactly when every prime factor of its numerator is a factor of u. ∎

With this theorem, we can tell at a glance that $\beta\left(\frac{2}{5}, \frac{2}{5}, \frac{1}{5}\right)$ is not good, but $\beta\left(\frac{2}{5}, \frac{2}{5}, \frac{1}{10}, \frac{1}{10}\right)$ is good.

Wrapping up, we return to the problem stated in the introduction.

PROBLEM. *Using a die with* 3 *sides which occur with probability* $\left(\frac{1}{9}, \frac{4}{9}, \frac{4}{9}\right)$, *is there some* n *such that an event depending on* n *rolls of the die has probability* $\frac{1}{3}$? *That is, is there a partition polynomial* $p(x, y)$ *with* $p\left(\frac{4}{9}, \frac{4}{9}\right) = \frac{1}{3}$?

First, we can note that $f(x, y) = 3(1 - x - y)$ is one polynomial with $f\left(\frac{4}{9}, \frac{4}{9}\right) = \frac{1}{3}$. We can compute that the polynomials h with $h\left(\frac{4}{9}, \frac{4}{9}\right) = 0$ are precisely those of the form $(x - y)g_1(x, y) + (9x - 4)g_2(x, y)$ for some integer polynomials g_1 and g_2. (Some experience with Hilbert's basis theorem and Gröbner bases over rings is useful here but not necessary: we can use multiples of $x - y$ to remove all occurrences of y from $h(x, y)$, leaving a polynomial of x which equals zero at $\frac{4}{9}$ and hence is a multiple of $9x - 4$.) Thus, we are looking to determine whether there exist integer polynomials g_1 and g_2 such that

$$f(x, y) = (3 - 3x - 3y) + (x - y)g_1(x, y) + (9x - 4)g_2(x, y)$$

is a partition polynomial.

We can start thinking about choices of g_1 and g_2 that will give positive supporting polynomials, but we will hit a roadblock when it comes to look at the standard supporting coefficients of f and $1 - f$. The exponent vector $(0, 0)$ gives a standard supporting coefficient of f as $3 - 4g_2(0, 0)$, while for $1 - f$ the same is $-2 + 4g_2(0, 0)$. It is easy to see that these cannot both be positive while $g_2(0, 0)$ is an integer, and so we have answered the above question in the negative.

(Since above we were considering the $(0,0)$ standard supporting coefficient, the same result could have been answered by considering $f(0,0) = 3 - 4g_2(0,0)$. If f is a supporting polynomial, it lies between 0 and 1 in the interior of the simplex, and so $f(0,0)$, an integer, must be zero or one, which is impossible for the above function.)

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