C O L L O Q U I U M M A T H E M A T I C U M

VOL. 135 2014 NO. 2

ERRATUM TO "ALGEBRAIC AND TOPOLOGICAL PROPERTIES OF SOME SETS IN ℓ_1"

(COLLOQ. MATH. 129 (2012), 75–85)

BY

TARAS BANAKH (Lviv and Kielce), ARTUR BARTOSZEWICZ (Łódź), SZYMON GŁĄB (Łódź) and EMILIA SZYMONIK (Łódź)

In this note we present a correct proof of Theorem 4.2 from [BBGS], recalled below. Unfortunately, the proof in [BBGS] contains a gap, which was pointed out to the third author by Prof. W. Marciszewski.

THEOREM. *Each infinite-dimensional closed subspace Y of the Banach space ℓ_1 contains an element $y = (y(n))_{n=1}^{\infty}$ whose set of partial sums*

$$E(y) = \Big\{\sum_{n=1}^{\infty} \varepsilon_n y(n) : (\varepsilon_n)_{n=1}^{\infty} \in \{0,1\}^{\mathbb{N}}\Big\}$$

has non-empty interior in the real line.

Proof. Fix a positive real number $\lambda < 1$ such that $\lambda + \lambda^2 > 10/7$ and for every $k \in \mathbb{N}$ put $\varepsilon_k = \lambda^{k+1}/(8k)$. By induction we shall construct a strictly increasing sequence $(n_k)_{k=0}^{\infty}$ of positive integers and a sequence $(x_k)_{k=1}^{\infty}$ of elements of Y such that $n_0 = 1$ and for every $k \in \mathbb{N}$ the following conditions are satisfied:

(1) $x_k(n) = 0$ for all $n < n_{k-1}$;
(2) $\sum_{n=n_{k-1}}^{\infty} |x_k(n)| = 1 + \varepsilon_k$;
(3) $\sum_{n=n_k}^{\infty} |x_m(n)| < \varepsilon_k$ for all $m \le k$.

Assume that for some $k \in \mathbb{N}$ the numbers $n_0 < n_1 < \cdots < n_{k-1}$ and points $x_1, \ldots, x_{k-1} \in Y$ have been constructed. Consider the closed subspace $Y_k = \{y \in Y : y(n) = 0 \text{ for all } n < n_{k-1}\}$ of finite codimension in Y and choose any $x_k \in Y_k$ with $\|x_k\| = 1 + \varepsilon_k$. For every $m \le k$ the series

2010 *Mathematics Subject Classification*: Primary 40A05; Secondary 15A03.
Key words and phrases: subsums of series, achievement set of sequence, algebrability, strong algebrability, lineability.

DOI: 10.4064/cm135-2-8

$\sum_{n=1}^\infty |x_m(n)|$ is convergent, so we can choose n_k satisfying (3). It is clear that n_k and x_k satisfy (1)–(3).

We claim that the set $E(y)$ for $y = \sum_{k=1}^\infty \lambda^k x_k \in Y$ contains an interval.

Consider the non-negative sequences

$$y^+ = \max\{y, 0\} \quad \text{and} \quad y^- = \max\{-y, 0\}$$

in ℓ_1 and observe that $y = y^+ - y^-$ and $|y| = \max\{y^+, y^-\}$. It follows that for every $n \in \mathbb{N}$ we get

$$\begin{aligned} y^-(n) + \{0,1\} \cdot y(n) &= y^-(n) + \{0,1\} \cdot (y^+(n) - y^-(n)) = \{y^-(n), y^+(n)\} \\ &= \{0, |y(n)|\} = \{0,1\} \cdot |y(n)|, \end{aligned}$$

which implies that

$$\begin{aligned} E(y) + \|y^-\| &= \Big\{ \sum_{n=1}^\infty (\varepsilon_n y(n) + y^-(n)) : (\varepsilon_n)_{n=1}^\infty \in \{0,1\}^{\mathbb{N}} \Big\} \\ &= \Big\{ \sum_{n=1}^\infty \varepsilon_n \, |y(n)| : (\varepsilon_n)_{n=1}^\infty \in \{0,1\}^{\mathbb{N}} \Big\} = E(|y|), \end{aligned}$$

and hence $E(y)$ contains an interval if and only if $E(|y|)$ does. Consider the sequence $z = (z(k))_{k=1}^\infty \in \ell_1$ defined by

$$z(k) = \sum_{n=n_{k-1}}^{n_k - 1} |y(n)| = \sum_{n=n_{k-1}}^{n_k-1} \Big| \sum_{m=1}^\infty \lambda^m x_m(n) \Big| = \sum_{n=n_{k-1}}^{n_k-1} \Big| \sum_{m=1}^k \lambda^m x_m(n) \Big|$$

for $k \in \mathbb{N}$.

Since $E(z) \subset E(|y|)$, it is enough to show that $E(z)$ contains an interval. To establish this, we will prove the Kakeya condition [K], which is sufficient for $E(z)$ to be an interval:

$$0 < z(k) \le \sum_{j>k} z(j) \qquad \textit{for all } k \in \mathbb{N}.$$

Observe that for every $k \in \mathbb{N}$,

$$\begin{aligned} z(k) &= \sum_{n=n_{k-1}}^{n_k-1} \Big| \sum_{m=1}^k \lambda^m x_m(n) \Big| \le \sum_{m=1}^k \sum_{n=n_{k-1}}^{n_k-1} \lambda^m |x_m(n)| \\ &= \lambda^k \sum_{n=n_{k-1}}^{n_k-1} |x_k(n)| + \sum_{m=1}^{k-1} \sum_{n=n_{k-1}}^{n_k-1} \lambda^m |x_m(n)| \\ &\le \lambda^k (1 + \varepsilon_k) + (k-1)\varepsilon_{k-1} = \lambda^k \Big(1 + \frac{\lambda^{k+1}}{8k} \Big) + \frac{\lambda^k}{8} < \frac{5}{4} \lambda^k \end{aligned}$$

and

$$\begin{aligned} z(k) &= \sum_{n=n_{k-1}}^{n_k-1} \Big|\sum_{m=1}^{k} \lambda^m x_m(n)\Big| = \sum_{n=n_{k-1}}^{n_k-1} \Big|\lambda^k x_k(n) + \sum_{m=1}^{k-1} \lambda^m x_m(n)\Big| \\ &\geq \sum_{n=n_{k-1}}^{n_k-1} \Big(\lambda^k |x_k(n)| - \sum_{m=1}^{k-1} \lambda^m |x_m(n)|\Big) \\ &= \lambda^k \sum_{n=n_{k-1}}^{n_k-1} |x_k(n)| - \sum_{m=1}^{k-1} \sum_{n=n_{k-1}}^{n_k-1} \lambda^m |x_m(n)| \\ &\geq \lambda^k - (k-1)\varepsilon_{k-1} = \lambda^k - \frac{1}{8}\lambda^k = \frac{7}{8}\lambda^k. \end{aligned}$$

Then

$$\begin{aligned} \sum_{j>k} z(j) &\geq z(k+1) + z(k+2) \geq \frac{7}{8}\lambda^{k+1} + \frac{7}{8}\lambda^{k+2} > \frac{7}{8}(\lambda + \lambda^2)\lambda^k \\ &> \frac{7}{8} \cdot \frac{10}{7}\lambda^k = \frac{5}{4}\lambda^k \geq z(k), \end{aligned}$$

and we are done. ∎

Actually, we have proved something more. By the Kakeya condition, $E(z)$ is the interval $[0, \sum_{k=1}^{\infty} z(k)]$. Moreover,

$$E(z) \subset E(|y|) \subset [0, \|y\|] = \Big[0, \sum_{n=1}^{\infty} |y(n)|\Big] = \Big[0, \sum_{k=1}^{\infty} z(k)\Big].$$

Hence $E(|y|)$ and $E(y)$ are intervals, and we obtain the following corollary:

Corollary 1. *The set $\mathcal{C} \cup \mathcal{MC} \cup c_{00}$ is not spaceable.*

Let us mention that Corollary 1 solves Problem 4.4(3) from [BBGS]; the notation in its statement can be found there.

Acknowledgments. We would like to thank Witold Marciszewski for informing us about the error.

REFERENCES

[BBGS] T. Banakh, A. Bartoszewicz, S. Głąb and E. Szymonik, *Algebraic and topological properties of some sets in ℓ_1*, Colloq. Math. 129 (2012), 75–85.

[K] S. Kakeya, *On the partial sums of an infinite series*, Tôhoku Sci. Rep. 3 (1914), 159–164.

Taras Banakh
Wydział Matematyczno-Przyrodniczy
Uniwersytet Jana Kochanowskiego
Świętokrzyska 15
25-406 Kielce, Poland
and
Department of Mathematics
Ivan Franko National University of Lviv
Universytetska 1
79000 Lviv, Ukraine
E-mail: t.o.banakh@gmail.com

Artur Bartoszewicz, Szymon Głąb,
Emilia Szymonik
Institute of Mathematics
Łódź University of Technology
Wólczańska 215
93-005 Łódź, Poland
E-mail: arturbar@p.lodz.pl
szymon.glab@p.lodz.pl
szymonikemilka@wp.pl

Received 8 May 2014;
revised 9 June 2014 (5746a)