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GENERALIZED QUIVERS ASSOCIATED TO REDUCTIVE GROUPS

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Abstract. We generalize the definition of quiver representation to arbitrary reductive groups. The classical definition corresponds to the general linear group. We also show that for classical groups our definition gives symplectic and orthogonal representations of quivers with involution inverting the direction of arrows.

0. Introduction. The representation theory of quivers has played an important role in the representation theory of Artin algebras for more than twenty years. It can be viewed as a formalization of a natural class of linear algebra problems. However if viewed in such a way, this theory has the drawback that it deals only with representations of general linear groups.

In this paper we make an attempt to generalize quivers to arbitrary reductive groups. In this setup a quiver representation is just a representation whose irreducible summands have a particularly simple form—they are the representations occurring in the restriction of the adjoint representation of a bigger reductive group.

This definition has a particularly nice meaning for classical groups. It turns out that in that case the generalized quiver S can be viewed as a quiver S° equipped with a contravariant involution. We call such an object a symmetric quiver.

The symplectic and orthogonal representations of a symmetric quiver S are just the selfdual representations of the quiver S°. The key result about the symmetric quivers is Theorem 2.6, which shows that the symplectic and orthogonal representations of a symmetric quiver S are in a natural bijection with a subset of representations of the associated quiver S°.

This principle allows us to classify the symmetric quivers of finite and tame type, thus extending the results of Gabriel ([G]) and Donovan–Freislich and Nazarova ([D-F], [N]). The main result says that a symmetric quiver S

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is of finite type (resp. tame type) if and only if the quiver S° is of Dynkin (resp. extended Dynkin) type.

The results of this paper are related to the results of the Ukrainian school in the theory of quivers (Roĭter, Sergeĭchuk, Kruglyak). It seems that our approach gives a more natural parametrization of the resulting matrix problems, bringing out the role of the orthogonal and symplectic group. This is important in applications and in relating this kind of problems to other natural classification problems (for example to multiple flag varieties, cf. [M-W-Z]). We explain the relation between the two approaches in Section 5.

In a subsequent paper we plan to investigate the structure of orbits and of rings of semiinvariants for symmetric quivers.

The paper is organized as follows. In Section 1 we recall basic notions related to quivers and define generalized quivers.

In Section 2 we prove that in the case of classical groups the notions of generalized quivers and symmetric quivers overlap. We show that the orthogonal and symplectic representations of symmetric quivers correspond to the selfdual objects with respect to the involution of S°.

In Section 3 we classify the symmetric quivers of finite type. We describe the indecomposable symplectic and orthogonal representations of symmetric quivers of finite type. In Section 4 we classify the symmetric quivers of tame type. We also describe the structure of one-dimensional families of indecomposable symplectic and orthogonal representations.

In Section 5 we explain the relation of our results to the results of Roĭter, Sergeĭchuk and Kruglyak and to the classification of multiple flag varieties of finite type for the symplectic group given in [M-W-Z].

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1. Quivers. Throughout this and the next section we will use some facts concerning algebraic groups, especially the description of their adjoint representations. For general facts concerning reductive groups we refer to Humphreys's book [H]. The description of the adjoint representations for the symplectic and orthogonal groups can be found in [Bou, Ch. VII].

We will work over an algebraically closed base field K of characteristic $\neq 2$. We start with the definitions of basic notions related to representations of quivers.

Definition. A *quiver* is a quadruple $S = (S_0, S_1, i, f)$, where S_0, S_1 are finite sets and $i, f : S_1 \to S_0$ are maps.

We think of a quiver S as of the collection S_0 of vertices, and the collection S_1 of arrows. To the element $g \in S_1$ we associate an arrow from $i(g)$ to $f(g)$.

Definition. A *representation* of a quiver S is a pair (V, ϕ) where V is a collection of finite-dimensional vector spaces V_p, $p \in S_0$, and ϕ is a collection of maps ϕ_g, $g \in S_1$, such that $\phi_g : V_{i(g)} \to V_{f(g)}$ is a linear map.

Definition. A *morphism* between two quiver representations $\psi : (V, \phi) \to (V', \phi')$ is a collection of linear maps $\psi_p : V_p \to V'_p$, $p \in S_0$, such that $\phi'_g \circ \psi_{i(g)} = \psi_{f(g)} \circ \phi_g$ for all $g \in S_1$. The identity morphism $\mathrm{id} : (V, \phi) \to (V, \phi)$ is the collection id_p, $p \in S_0$, where id_p is the identity map $V_p \to V_p$ for all $p \in S_0$. If $\psi : (V, \phi) \to (V', \phi')$ and $\tau : (V', \phi') \to (V'', \phi'')$ are two morphisms, then the composition $\tau \circ \psi$ is given by $(\tau \circ \psi)_p = \tau_p \circ \psi_p$, $p \in S_0$. Two quiver representations (V, ϕ) and (V', ϕ') are called *isomorphic* if there exist morphisms $\psi : (V, \phi) \to (V', \phi')$ and $\tau : (V', \phi') \to (V, \phi)$ such that $\psi \circ \tau = \mathrm{id}$ and $\tau \circ \psi = \mathrm{id}$.

The representations of a quiver S form a category $\mathrm{Rep}(S)$, whose objects and morphisms are defined above. The category $\mathrm{Rep}(S)$ is an abelian category, which is equivalent to the category of modules over the so-called path algebra of S (cf. [D-R]).

Let us recall the definition of the direct sum of objects (V, ϕ) and (V', ϕ').

Definition. If (V, ϕ) and (V', ϕ') are representations of a quiver S, then the direct sum is the pair (W, ψ) where $W_p = V_p \oplus V'_p$ and where $\psi_g : V_{i(g)} \oplus V'_{i(g)} \to V_{f(g)} \oplus V'_{f(g)}$ is given by $(v, w) \mapsto (\phi_g(v), \phi'_g(w))$ for all $g \in S_1$. A representation (V, ϕ) is called *indecomposable* if it is nontrivial and it is not isomorphic to the direct sum of two nontrivial representations.

Let us reformulate the definition of a quiver representation. Let S be a quiver and (V, ϕ) a representation. Suppose that $S_0 = \{1, \ldots, k\}$ and $S_1 = \{1, \ldots, l\}$ and we have vector spaces $V_1, \ldots, V_k$ where V_i is isomorphic to K^{d_i}. We call $\underline{d} := (d_1, \ldots, d_k)$ the *dimension vector* of the representation (V, ϕ). For each element $g \in S_1$ we have a map $\phi_g \in \mathrm{Hom}(V_{i(g)}, V_{f(g)})$. So the representation (V, ϕ) gives an element of

$$\mathrm{Hom}(V_{i(1)}, V_{f(1)}) \oplus \mathrm{Hom}(V_{i(2)}, V_{f(2)}) \oplus \ldots \oplus \mathrm{Hom}(V_{i(l)}, V_{f(l)}).$$

Two representations with dimension vector $\underline{d}$ are isomorphic if and only if they are in the same $R := \mathrm{GL}(V_1) \times \ldots \times \mathrm{GL}(V_k)$-orbit.

Let $V = V_1 \oplus \ldots \oplus V_k$. Let $H = {K^*}^k$ be a torus consisting of matrices which are multiples of the identity matrix in each block V_i. The centralizer of H is equal to $R = \mathrm{GL}(V_1) \times \ldots \times \mathrm{GL}(V_k)$. Let $\mathfrak{gl}(V)$ be the Lie algebra of $\mathrm{GL}(V)$. We have an isomorphism of R-modules $\mathfrak{gl}(V) = \bigoplus_{i,j} \mathrm{Hom}(V_i, V_j)$.

This means that a variety of representations of a fixed dimension vector of the quiver S is a representation of R whose irreducible summands occur in $\mathfrak{gl}(V)$ considered as an R-module.

This fact inspires us to generalize the notion of quiver:

DEFINITION. A *generalized quiver with dimension vector* is a triple (G, R, V) where G is a reductive group, R is a centralizer of a Zariski closed abelian reductive subgroup H of G (R is also a reductive group) and V is a representation of R which decomposes into irreducible representations which also appear in $\mathfrak{g}$, the Lie algebra of G, seen as an R-module. We assume that the trivial representation does not occur as a summand of V.

DEFINITION. A *generalized quiver representation* is a quadruple (G, R, V, Rv) where (G, R, V) is a generalized quiver with dimension vector, $v \in V$ and Rv is an R-orbit.

REMARK 1.1. The subgroup H is not unique. For example instead of the torus K^* we might choose its cyclic subgroup of large enough order (so that it has the same centralizer).

REMARK 1.2. Suppose that (G, R, V) is a generalized quiver with dimension vector. If H' is an abelian closed reductive subgroup of R, then $(Z_G(H'), Z_R(H'), V^{H'})$ (where $V^{H'}$ denotes the set of H'-invariants) is again a generalized quiver with dimension vector, unless some trivial direct summands appear. Indeed, $H \subset Z_G(H')$, and $Z_R(H') = Z_G(H') \cap R$ is the centralizer of H in $Z_G(H')$. The irreducible representations appearing in $V^{H'}$ all appear in $\mathfrak{g}^{H'}$, which is the Lie algebra of $Z_G(H')$.

REMARK 1.3. Our assumption that if (G, R, V) is a quiver with dimension vector then the representation V does not contain a trivial direct summand means that when the trivial representation appears as a direct summand in $\mathfrak{gl}(V)$ considered as an R-module, we disregard it.

A lot of interesting representations appear as a generalized quiver with dimension vector.

In fact several results from [K1] can be restated in that language.

Suppose that (G, R, V) is a generalized quiver with dimension vector. The subgroup R is the centralizer of some abelian subgroup $H \subseteq G$. We have a decomposition

$$\mathfrak{g} = \bigoplus_{\chi} \mathfrak{g}_{\chi}$$

where χ runs through all characters of H. Suppose that $V \subset \mathfrak{g}_{\chi}$ for some χ. The following facts were proven by Kac in [K1].

PROPOSITION 1.4 (Kac, [K1]). (a) *If χ is not of finite order, then G has only finitely many orbits in V.*

(b) *If χ is of finite order, then the quotient map $V \to V//G$ is equidimensional and each fiber contains finitely many orbits.*

Kac also shows that most irreducible R-representations V with finitely many orbits appear as a generalized quiver (G, R, V) with dimension vector.

Also, most of the irreducible cofree representations (classified in [L]) appear in this way.

EXAMPLE 1.5. Let G be an algebraic group of type E_6. The Dynkin diagram of E_6 is:

$$\begin{array}{ccccccccc} & & & & 2 & & & & \\ & & & & \circ & & & & \\ & & & & | & & & & \\ \circ & — & \circ & — & \circ & — & \circ & — & \circ \\ 1 & & 3 & & 4 & & 5 & & 6 \end{array}$$

The fundamental weight λ_4 corresponds to a subgroup H of E_6 which is isomorphic to K^*. The centralizer R of H is isomorphic to $\mathrm{SL}_2 \times \mathrm{SL}_3 \times \mathrm{SL}_3$. Write $\mathrm{SL}_2 \times \mathrm{SL}_3 \times \mathrm{SL}_3 = \mathrm{SL}(U) \times \mathrm{SL}(V) \times \mathrm{SL}(W)$ where U is a vector space of dimension 2, and V, W are vector spaces of dimension 3.

Then we have an isomorphism of $\mathrm{SL}(U) \times \mathrm{SL}(V) \times \mathrm{SL}(W)$-modules

$$\begin{aligned} \mathfrak{g}|_R = {} & S_{1,-1}U \oplus S_{1,0,-1}V \oplus S_{1,0,-1}W \oplus (U \otimes V \otimes W) \\ & \oplus (U^* \otimes V^* \otimes W^*) \oplus (V \otimes W) \oplus (V^* \otimes W^*) \oplus U \oplus U^* \oplus K. \end{aligned}$$

EXAMPLE 1.6. Let G be an algebraic group of type G_2. The Dynkin diagram of G_2 is:

$$\begin{array}{ccc} 1 & & 2 \\ \circ & \equiv & \circ \end{array}$$

Let α_1, α_2 be the simple roots, and let λ_1, λ_2 be the fundamental weights. If T is a maximal torus of G, then there is an element $a \in T$ of order 2 which acts trivially on $\mathfrak{g}_{\alpha_1}$ and by multiplication by -1 on $\mathfrak{g}_{\alpha_2}$. Let H be the cyclic group generated by a. Then the centralizer R of H is isomorphic to $\mathrm{SL}_2 \times \mathrm{SL}_2$. If V, W are the standard representations of the two copies of $\mathrm{SL}_2 \times \mathrm{SL}_2$, then $\mathfrak{g}|_R = S^2V \oplus S^2W \oplus S^3V \otimes W$.

2. Symmetric quivers. In this section we describe the generalized quivers in the case when G is a symplectic (resp. orthogonal) group. It turns out that the representations of generalized quivers in this case are the same as symplectic (resp. orthogonal) representations of symmetric quivers, i.e. the quivers with contravariant involution (see the definition and Proposition 2.3 below).

DEFINITION. A *symmetric quiver* is a pentuple $S := (S_0, S_1, i, f, \sigma)$ where $S^\circ := (S_0, S_1, i, f)$ is a quiver (called the *underlying quiver*), and σ is a bijective map from the disjoint union $S_0 \amalg S_1$ to itself such that $\sigma(S_0) = S_0$ and $\sigma(S_1) = S_1$, $\sigma^2 = \mathrm{id}$, $i(\sigma(g)) = \sigma(f(g))$ and $f(\sigma(g)) = \sigma(i(g))$ for all $g \in S_1$, and $\sigma(g) = g$ whenever $g \in S_1$ and $\sigma(i(g)) = f(g)$. The assumptions just mean that σ is an involution of the vertices of the quiver S and of its arrows, reversing the orientation of arrows.

If S is a quiver and (V, ϕ) is a representation of S, then we set $V_\Sigma = \bigoplus_{p \in S_0} V_p$. If (V', ϕ') is another representation and $\psi : (V, \phi) \to (V', \phi')$ is a morphism, then $\psi_\Sigma : V_\Sigma \to V'_\Sigma$ is the linear map which maps v to $\psi_p(v)$ for all $p \in S_0$ and all $v \in V_p$.

If S is a symmetric quiver, and (V, ϕ) is a representation of the underlying quiver S°, then the dual $(V, \phi)^*$ is defined as (V^*, ϕ^*), where $(V^*)_p := (V_{\sigma(p)})^*$ for all $p \in S_0$ and $(\phi^*)_g = -(\phi_{\sigma(g)})^*$ for all $g \in S_1$. If (V', ϕ') is another representation of S° and $\psi : (V, \phi) \to (V', \phi')$ is a morphism, then $\psi^* : (V', \phi')^* \to (V, \phi)^*$ is defined by $(\psi^*)_p = (\psi_{\sigma(p)})^* : ((V')^*)_p \to (V^*)_p$.

Throughout this section we follow the convention that if $(V, \langle\cdot,\cdot\rangle)$ is a vector space with a nondegenerate symmetric (skew-symmetric) scalar product on V, then V can be canonically identified with V^* via the map $x \mapsto \langle x, \cdot\rangle$.

Definition. An *orthogonal* (resp. *symplectic*) *representation* of a symmetric quiver $S = (S_0, S_1, i, f, \sigma)$ is a triple $(V, \phi, \langle\cdot,\cdot\rangle)$, where (V, ϕ) is a representation of the underlying quiver S°. The scalar product $\langle\cdot,\cdot\rangle$ is a nondegenerate symmetric (resp. skew-symmetric) scalar product on V_Σ such that the restriction of $\langle\cdot,\cdot\rangle$ to $V_p \times V_q$ is 0 if $q \neq \sigma(p)$, and $\langle \phi_g(v), w\rangle + \langle v, \phi_{\sigma(g)}(w)\rangle = 0$ for all $v \in V_{i(g)}$ and all $w \in V_{\sigma(f(g))}$.

If $(V, \phi, \langle\cdot,\cdot\rangle)$ is an orthogonal or symplectic representation of a symmetric quiver S, then we can identify (V, ϕ) and $(V, \phi)^*$ in a natural way, since V_p and $V_{\sigma(p)}$ are dual to each other, so $(V^*)_p = (V_{\sigma(p)})^* = V_p$. Furthermore, ϕ_p is the dual of $-\phi_{\sigma(p)}$, so $(\phi^*)_p = -(\phi_{\sigma(p)})^* = \phi_p$.

Definition. Suppose that S is a symmetric quiver. Two orthogonal (resp. symplectic) representations $(V, \phi, \langle\cdot,\cdot\rangle)$ and $(V', \phi', \langle\cdot,\cdot\rangle')$ are *isomorphic* if there exists a morphism $\psi : (V, \phi) \to (V', \phi')$ of representations of S° such that $\psi^* \circ \psi = \mathrm{id}$ and $\psi \circ \psi^* = \mathrm{id}$.

Note that $\psi_\Sigma : V_\Sigma \to V'_\Sigma$ preserves the scalar product.

Definition. If $(V, \phi, \langle\cdot,\cdot\rangle)$, $(V', \phi', \langle\cdot,\cdot\rangle')$ are two orthogonal (or *symplectic*) representations of a symmetric quiver $S = (S_0, S_1, i, f, \sigma)$, then their direct sum is given by $(W, \psi, \langle\cdot,\cdot\rangle_W)$ where (W, ψ) is the direct sum of (V, ϕ) and (V', ϕ'), and the scalar product $\langle\cdot,\cdot\rangle_W$ on $W_\Sigma \cong V_\Sigma \oplus V'_\Sigma$ is the sum of $\langle\cdot,\cdot\rangle$ and $\langle\cdot,\cdot\rangle'$.

An orthogonal (resp. symplectic) representation is called *indecomposable* if it is nontrivial and it is not isomorphic to the direct sum of two nontrivial orthogonal (resp. symplectic) representations.

Graphically we represent a symmetric quiver S by drawing the quiver S°, indicating the involution and drawing the nodes fixed under the involution as closed nodes, with other nodes drawn as open nodes. Sometimes we skip the definition of the involution σ when there is only one nontrivial choice of σ.

Example 2.1. Consider the symmetric quiver

$$\begin{array}{ccccc} & & \circ & & \\ & & \downarrow & & \\ S: \circ & \rightarrow & \bullet & \rightarrow & \circ \\ & & \downarrow & & \\ & & \circ & & \end{array}$$

where σ interchanges the antipodal nodes and leaves the closed node fixed. An orthogonal representation of S is an orthogonal space W and two vector spaces V_1 and V_2 together with maps $g_1 : V_1 \to W$ and $g_2 : V_2 \to W$. Of course this also induces the dual map $-g_1^* : W \to V_1^*$ and $-g_2^* : W \to V_2^*$ so we get the following diagram:

$$\begin{array}{ccccc} & & V_1 & & \\ & & \downarrow & & \\ V_2 & \rightarrow & W & \rightarrow & V_2^* \\ & & \downarrow & & \\ & & V_1^* & & \end{array}$$

Orthogonal representations of S are $\mathrm{GL}(V_1) \times \mathrm{GL}(V_2) \times \mathrm{O}(W)$-orbits in $\mathrm{Hom}(V_1, W) \oplus \mathrm{Hom}(V_2, W)$.

Example 2.2. Consider the symmetric quiver

$$S : \circ \to \circ \to \circ \to \circ$$

The involution σ is determined uniquely here, as it has to reverse the orientation of arrows. A symplectic representation of S is a set of two vector spaces V_1 and V_2 together with a map $g : V_1 \to V_2$ and an element $h \in S^2V_2^*$ (a symmetric form on V_2). We also have the dual map $-g^* : V_2 \to V_1$. So we get the following diagram:

$$V_1 \to V_2 \to V_2^* \to V_1^*$$

where the three maps are given by g, h and $-g^*$ respectively. The scalar product on $V_1 \oplus V_2 \oplus V_2^* \oplus V_1^*$ is given by

$$\langle (v_1, v_2, u_2, u_1), (v_1', v_2', u_2', u_1') \rangle = u_1'(v_1) - u_1(v_1') + u_2'(v_2) - u_2(v_2').$$

Note that $h^* = h$. In this case, symplectic quiver representations of S are $\mathrm{GL}(V_1) \times \mathrm{GL}(V_2)$-orbits in $\mathrm{Hom}(V_1, V_2) \oplus S^2V_2^*$.

The following proposition gives the interpretation of symplectic (resp. orthogonal) representations of symmetric quivers as generalized quivers introduced in Section 1 in the case when G is the symplectic (resp. orthogonal) group.

Proposition 2.3. (a) *To any generalized quiver (G, R, V) with dimension vector, with $G = \mathrm{O}_n$ being the orthogonal group, we can associate a symmetric quiver S with dimension vector in such a way that the generalized* *quiver representations* (G, R, V, Rv) *correspond bijectively to the orthogonal representations of* S *of that dimension.*

(b) *To any generalized quiver* (G, R, V) *with dimension vector, with* $G = \mathrm{SP}_n$ *being the symplectic group, we can associate a symmetric quiver* S *with dimension vector in such a way that the generalized quiver representations* (G, R, V, Rv) *correspond bijectively to the symplectic representations of* S *of that dimension.*

Proof. Let $G = \mathrm{O}_n$. Suppose that R is the centralizer of an abelian closed reductive subgroup H of G. Let W be the natural representation of G (of dimension n) and let $\langle \cdot, \cdot \rangle$ be the nondegenerate symmetric scalar product on W. Let $\chi_1, \ldots, \chi_k$ be the different characters of H appearing in the representation W. Let W_χ be the isotypic component of χ. If χ, μ are two characters of H, then by restricting the scalar product we get a pairing

$$\langle \cdot, \cdot \rangle : W_\chi \times W_\mu \to K.$$

If $h \in H$, then

$$\langle v, w \rangle = \langle hv, hw \rangle = \langle \chi(h)v, \mu(h)w \rangle = \chi(h)\mu(h)\langle v, w \rangle,$$

so the pairing is 0 when the product $\chi\mu$ is not the trivial character, and

$$\langle \cdot, \cdot \rangle : W_\chi \times W_{\chi^{-1}} \to K$$

must be nondegenerate. So W_χ and $W_{\chi^{-1}}$ are dual to each other. In particular, if $\chi = \chi^{-1}$ then the restriction of $\langle \cdot, \cdot \rangle$ to W_χ is nondegenerate and symmetric. Hence we have a decomposition of W into a direct sum

$$W_{\chi_1} \oplus \ldots \oplus W_{\chi_k} \oplus W_{\chi_1^{-1}} \oplus \ldots \oplus W_{\chi_k^{-1}} \oplus W_{\mu_1} \oplus \ldots \oplus W_{\mu_l}$$

where $\chi_1, \ldots, \chi_k, \chi_1^{-1}, \ldots, \chi_k^{-1}, \mu_1, \ldots, \mu_l$ are all different, and μ_i^2 is trivial for all i. If we put $V_i := W_{\chi_i}$ and $W_i := W_{\mu_i}$ then we get

$$W = V_1 \oplus \ldots \oplus V_k \oplus V_1^* \oplus \ldots \oplus V_k^* \oplus W_1 \oplus \ldots \oplus W_l.$$

The group R is exactly the set of all orthogonal maps $W \to W$ which stabilize all V_i's, V_i^*'s and W_i's. This means that

$$R = \mathrm{GL}(V_1) \times \ldots \times \mathrm{GL}(V_k) \times \mathrm{O}(W_1) \times \ldots \times \mathrm{O}(W_l).$$

This, for example, implies that the spaces V_i, W_j are the irreducible representations of R. The adjoint representation of O_n can be identified with $\bigwedge^2(W)$. Its irreducible summands are

$$\bigwedge^2(V_i),\ \bigwedge^2(V_i^*),\ \bigwedge^2(W_i),\ \mathrm{Hom}(V_i, V_j),\ \mathrm{Hom}(V_i, V_j^*),$$
$$\mathrm{Hom}(V_i^*, V_j),\ \mathrm{Hom}(V_i, W_j),\ \mathrm{Hom}(V_i^*, W_j),\ \mathrm{Hom}(W_i, W_j).$$

Note that for example $\mathrm{Hom}(V_i, V_j^*)$ and $\mathrm{Hom}(V_j, V_i^*)$ are the same. The symmetric quiver S is defined as follows.

Write $V = K^s \oplus \bigoplus_{i=1}^t Z_i$ where Z_i are nontrivial irreducible representations of R. We first define the quiver S°. Take open nodes $p_1, \ldots, p_k$, $p_1^*, \ldots, p_k^*$ corresponding to $V_1, \ldots, V_k, V_1^*, \ldots, V_k^*$ respectively and take closed nodes $q_1, \ldots, q_l$ corresponding to $W_1, \ldots, W_l$ respectively. For each summand Z_m we draw the corresponding arrows as follows.

If $Z_m = \mathrm{Hom}(V_i, V_j^*)$,	draw arrows $g_m : p_i \to p_j^*$ and $g_m^* : p_j \to p_i^*$.
If $Z_m = \mathrm{Hom}(V_i^*, V_j)$,	draw arrows $g_m : p_i^* \to p_j$ and $g_m^* : p_j^* \to p_i$.
If $Z_m = \bigwedge^2(V_i)$,	draw an arrow $g_m = g_m^* : p_i^* \to p_i$.
If $Z_m = \bigwedge^2(V_i^*)$,	draw an arrow $g_m = g_m^* : p_i \to p_i^*$.
If $Z_m = \mathrm{Hom}(V_i, W_j)$,	draw arrows $g_m : p_i \to q_j$ and $g_m^* : q_j \to p_i^*$.
If $Z_m = \mathrm{Hom}(V_i^*, W_j)$,	draw arrows $g_m : p_i^* \to q_j$ and $g_m^* : q_j \to p_i$.
If $Z_m = \mathrm{Hom}(W_i, W_j)$,	draw arrows $g_m : q_i \to q_j$ and $g_m^* : q_j \to q_i$ $(i < j)$.
If $Z_m = \bigwedge^2(W_i)$,	draw an arrow $g_m = g_m^* : q_i \to q_i$.
If $Z_m = \mathrm{End}_0(V_i)$,	draw arrows $g_m : V_i \to V_i$ and $g_m^* : V_i^* \to V_i^*$.

($\mathrm{End}_0(V_i) \subset \mathrm{End}(V_i)$ is the subspace of all endomorphisms with zero trace.)

Define an involution of the quiver S° by $\sigma(p_i) = p_i^*$, $\sigma(p_i^*) = p_i$, $\sigma(q_i) = q_i$, $\sigma(g_i) = g_i^*$, $\sigma(g_i^*) = g_i$ for all i. Note that if g is an arrow from a to b, then $\sigma(g)$ is an arrow from $\sigma(b)$ to $\sigma(a)$. It is clear that the pair (S°, σ) defines a symmetric quiver S. The representations of S are in bijection with the generalized quiver representations (O_n, R, V, Rv). This proves (a).

The situation for symplectic quivers is similar. Suppose $(\mathrm{SP}_n, R, V, Rv)$ is a generalized quiver representation. Let W be the standard symplectic representation where $\langle \cdot, \cdot \rangle$ is the nondegenerate skew-symmetric scalar product. Again we can decompose W as follows:

$$W = V_1 \oplus \ldots \oplus V_k \oplus V_1^* \oplus \ldots \oplus V_k^* \oplus W_1 \oplus \ldots \oplus W_l$$

where V_i and V_i^* are dual with respect to $\langle \cdot, \cdot \rangle$ and the restriction of $\langle \cdot, \cdot \rangle$ to W_i is nondegenerate and skew-symmetric. The group R is isomorphic to

$$\mathrm{GL}(V_1) \times \ldots \times \mathrm{GL}(V_k) \times \mathrm{SP}(W_1) \times \ldots \times \mathrm{SP}(W_l).$$

The adjoint representation of SP_n can be identified with $S^2 W$. The irreducible summands are

$$S^2 V_i,\ S^2 V_i^*,\ S^2 W_i,\ \mathrm{Hom}(V_i, V_j),\ \mathrm{Hom}(V_i^*, V_j),$$
$$\mathrm{Hom}(V_i, V_j^*),\ \mathrm{Hom}(V_i, W_j),\ \mathrm{Hom}(V_i^*, W_j),\ \mathrm{Hom}(W_i, W_j).$$

We can define the symplectic quiver associated to (SP_n, R, V) as in the orthogonal case. The proposition is proven. ■

REMARK 2.4. Note that an orthogonal quiver representation (O_n, R, V, Rv) or a symplectic quiver representation $(\mathrm{SP}_n, R, V, Rv)$ is decomposable if and only if the stabilizer of v contains an abelian reductive subgroup of R which is not contained in the center of R (see Lemma 2.3 in [K2]). This also allows one to define decomposable representations for quiver representations for arbitrary reductive groups (see Section 2 of [K2]).

Our next goal is to relate the symplectic and orthogonal indecomposable representations of the symmetric quiver S to the indecomposable representations of the quiver S°.

The general categorical framework for this type of result was provided in the paper by Roĭter [R]. However the proof given there consists mostly of a reference to a rather obscure book on linear algebra. Therefore we include the proofs for the convenience of the reader.

LEMMA 2.5. *Let $P(X) \in K[X]$ be a polynomial with $P(0) \neq 0$. Then there exists a polynomial $R(X) \in K[X]$ such that $R(X)^2 X - 1$ is divisible by $P(X)$.*

Proof. First, we consider the case of $P(X) = (X-\lambda)^n$ by induction on n. If $n = 1$, then we can take $R(X) = 1/\sqrt{\lambda}$. If $R(X)^2 X \equiv 1 \bmod (X-\lambda)^n$, we can write $R(X)^2 X \equiv 1 + c(X-\lambda)^n \bmod (X-\lambda)^{n+1}$ for some $c \in K$. Then $S(X)^2 X \equiv 1 \bmod (X-\lambda)^{n+1}$ with $S(X) = R(X) - c(X-\lambda)^n/(2\lambda R(\lambda))$. In general, we can write

$$P(X) = \alpha(X-\lambda_1)^{n_1} \dots (X-\lambda_r)^{n_r}.$$

For every i, there exists $Q_i(X)$ such that $Q_i(X)^2 X \equiv 1 \bmod (X-\lambda_i)^{n_i}$. By the Chinese remainder theorem, there exists a $Q(X) \in K[X]$ such that $Q(X) \equiv Q_i(X) \bmod (X-\lambda_i)^{n_i}$ for all i. Hence $Q(X)^2 X \equiv 1 \bmod P(X)$. ■

THEOREM 2.6. *Let S be a symmetric quiver and suppose that $(V, \phi, \langle\cdot,\cdot\rangle)$ and $(V', \phi', \langle\cdot,\cdot\rangle')$ are two symplectic or orthogonal representations of S. Then $(V, \phi, \langle\cdot,\cdot\rangle)$ and $(V', \phi', \langle\cdot,\cdot\rangle')$ are isomorphic if and only if (V, ϕ) and (V', ϕ') are isomorphic as representations of the quiver S°.*

Proof. Let $\psi : (V, \phi) \to (V', \phi')$ be an isomorphism. Then $\psi^* : (V', \phi') \to (V, \phi)$ is also an isomorphism and $\psi^* \circ \psi$ is an automorphism of (V, ϕ). Put $\tau = (\psi^* \circ \psi)_\Sigma = \psi_\Sigma^* \circ \psi_\Sigma$, where ψ_Σ^* is the dual of ψ_Σ with respect to $\langle\cdot,\cdot\rangle$ and $\langle\cdot,\cdot\rangle'$. We will modify ψ_Σ by multiplying it on the right by a polynomial $R(\tau)$ such that $w := \psi_\Sigma R(\tau)$ will be an isomorphism $(V, \phi, \langle\cdot,\cdot\rangle) \to (V', \phi', \langle\cdot,\cdot\rangle')$. Let $P(X) \in K[X]$ be the characteristic polynomial of τ, and take $R(X)$ as in Lemma 2.5. Then

$$\omega^*\omega = R(\tau)\psi_\Sigma^*\psi_\Sigma R(\tau) = R(\tau)\tau R(\tau) = \mathrm{id}.$$

It follows that ω decomposes into a collection $\omega_p : V_p \to V_p'$, $p \in S_0$, which is an isomorphism between $(V, \phi, \langle\cdot,\cdot\rangle)$ and $(V', \phi', \langle\cdot,\cdot\rangle')$.

Another more general result of this type, but in the context of reductive groups, is described in [M-W-Z, Section 2.1].

Proposition 2.7. *If* $(V, \phi, \langle\cdot,\cdot\rangle)$ *is an indecomposable orthogonal or symplectic representation of the symmetric quiver* S, *then as a representation of the underlying quiver* S°, (V, ϕ) *is indecomposable or isomorphic to the direct sum of* (W, ψ) *and* $(W, \psi)^*$, *where* (W, ψ) *is an indecomposable representation of* S°.

Proof. We prove the lemma for the symplectic object (V, ϕ) since the proof in the orthogonal case is the same. We start with a useful lemma.

Lemma 2.8. *Let* (V, ϕ) *be a symplectic object. Let* (W, ψ) *be a direct summand of* (V, ψ) *in* $\mathrm{Rep}(S^\circ)$ *such that the restriction of* $\langle\cdot,\cdot\rangle$ *to* (W, ψ) *is nondegenerate. Then we have the decomposition*

$$(V, \phi) = (W, \psi) \oplus (W, \psi)^\perp$$

into the orthogonal direct sum of symplectic objects, where $(W, \psi)^\perp$ *denotes the orthogonal complement with respect to the form* $\langle\cdot,\cdot\rangle$.

Proof. The only thing to check is that $(W, \psi)^\perp$ is an object in $\mathrm{Rep}(S^\circ)$. Let $g \in (S^\circ)_1$. Take $u \in W^\perp_{i(g)} \subset V_{i(g)}$. We want to show that $\phi_g(u) \in (W^\perp)_{f(g)}$. To check it we need to know that $\langle \phi_g(u), v\rangle = 0$ for $v \in W_{\sigma(f(g))}$. By definition of a symplectic representation we have $\langle \phi_g(u), v\rangle + \langle u, \phi_{\sigma(g)}(v)\rangle = 0$. But $\phi_{\sigma(g)}(v) \in W_{\sigma(i(g))}$ so $\langle u, \phi_{\sigma(g)}(v)\rangle = 0$. This proves our claim. ∎

Let (W, ψ) be an indecomposable direct summand of (V, ϕ). Let $i : (W, \psi) \to (V, \phi)$ and $p : (V, \phi) \to (W, \psi)$ be the the canonical embedding and projection respectively. Let $\omega : (V, \phi) \to (V, \phi)^*$ be the isomorphism induced by the symplectic form. Consider the composition $i^*\omega i : (W, \psi) \to (W, \psi)^*$. We have two cases.

1) The composition $i^*\omega i$ is an isomorphism. This means that the restriction of the form $\langle\cdot,\cdot\rangle$ to W is nondegenerate and we can write $(V, \phi) = (W, \psi) \oplus (W, \psi)^\perp$ by Lemma 2.8. Since (V, ϕ) is an indecomposable symplectic object, we have $(W, \psi)^\perp = 0$ and we are done.

2) The composition $i^*\omega i$ is not an isomorphism. Consider the map $j : (W, \psi) \oplus (W, \psi)^* \to (V, \phi)$ given in matrix form by $j = (i,\ \omega^{-1}p^*)$. Then the composition $j^*\omega j : (W, \psi) \oplus (W, \psi)^* \to (W, \psi)^* \oplus (W, \psi)$ can be written in matrix form

$$j^*\omega j = \begin{pmatrix} i^*\omega i & \mathrm{id}_{W^*} \\ -\mathrm{id}_W & -p\omega^{-1}p^* \end{pmatrix}.$$

We claim the map $j^*\omega j$ is an isomorphism. To see that, we multiply it on the left with the matrix

$$\begin{pmatrix} \mathrm{id}_{W^*} & i^*\omega i \\ 0 & \mathrm{id}_W \end{pmatrix}.$$

The resulting matrix is

$$\begin{pmatrix} 0 & \mathrm{id}_{W^*} - i^*\omega i p\omega^{-1}p^* \\ -\mathrm{id}_W & -p\omega^{-1}p^* \end{pmatrix}.$$

The map $i^*\omega i p\omega^{-1}p^*$ is an endomorphism of (W,ψ) which is not an isomorphism, and therefore nilpotent (since (W,ψ) is indecomposable its ring of endomorphisms is local, say by [B-D, Lemma 6.47]). This means that $\mathrm{id}_{W^*} - i^*\omega i p\omega^{-1}p^*$ is invertible, which proves that $j^*\omega j$ is invertible.

It follows that j is an inclusion and that the restriction of the symplectic form $\langle\cdot,\cdot\rangle$ to $(W,\psi)\oplus(W,\psi)^*$ (given by $j^*\omega j$) is nondegenerate. Now we can decompose (V,ϕ) as the orthogonal sum of $(W,\psi)\oplus(W,\psi)^*$ and its orthogonal complement by Lemma 2.8. Since (V,ϕ) is indecomposable as a symplectic object, we must have $(V,\phi)=(W,\psi)\oplus(W,\psi)^*$. The proof of Proposition 2.7 is complete. ∎

We can summarize the above result by saying that if $(V,\phi,\langle\cdot,\cdot\rangle)$ is an indecomposable orthogonal or symplectic representation of a symmetric quiver, then there are three possibilities for the S° representation (V,ϕ):

(a) (V,ϕ) is indecomposable;

(b) $(V,\phi)=(W,\psi)\oplus(W,\psi)$ and (W,ψ) is indecomposable such that $(W,\psi)\cong(W,\psi)^*$ ((V,ϕ) *ramifies*);

(c) $(V,\phi)=(W,\psi)\oplus(W,\psi)^*$ and (W,ψ) and $(W,\psi)^*$ are not isomorphic ((V,ϕ) *splits*).

Proposition 2.9. *Suppose that S is a symmetric quiver, and (V,ϕ) is an indecomposable representation of S°. Then exactly one of the following statements is true*:

(a) *(V,ϕ) is not isomorphic to $(V,\phi)^*$.*

(b) *There exists a symmetric scalar product $\langle\cdot,\cdot\rangle$ on V_Σ such that $(V,\phi,\langle\cdot,\cdot\rangle)$ is an orthogonal representation of S.*

(c) *There exists a skew-symmetric scalar product $\langle\cdot,\cdot\rangle$ on V_Σ such that $(V,\phi,\langle\cdot,\cdot\rangle)$ is a symplectic representation of S.*

Proof. If (a) is true, then we have already seen that (b) and (c) are not true. Suppose that $\psi:(V,\phi)\to(V,\phi)^*$ is an isomorphism of representations of S°. Then $(\psi^*)^{-1}\circ\psi$ is an automorphism of (V,ϕ). Put $\tau:=(\psi_\Sigma^*)^{-1}\circ\psi_\Sigma$, which is an automorphism of V_Σ. Let $V_\Sigma^{(\lambda)}$ be the generalized eigenspace of eigenvalue λ in V_Σ. The decomposition

$$V_\Sigma=\bigoplus_\lambda V_\Sigma^{(\lambda)}$$

corresponds to a decomposition of the quiver representation (V,ϕ). Since (V,ϕ) is indecomposable, we must have $V_\Sigma=V_\Sigma^{(\lambda)}$ for some $\lambda\in K$. So

$\psi_\Sigma - \lambda\psi_\Sigma^*$ is not bijective, and therefore $(\psi_\Sigma - \lambda\psi_\Sigma^*)^* = \psi_\Sigma^* - \lambda\psi_\Sigma$ is not bijective. Let $v \neq 0$ be in the kernel of $\psi_\Sigma^* - \lambda\psi_\Sigma$. Then v is an eigenvector of τ with an eigenvalue λ^{-1}. It follows that $\lambda = \lambda^{-1}$, so $\lambda = 1$ or $\lambda = -1$. So either $\psi + \psi^*$ or $\psi - \psi^*$ is bijective. If $\psi + \psi^*$ is bijective, then we define a scalar product on V_Σ by $\langle v, w\rangle := \langle(\psi+\psi^*)v, w\rangle'$ where $\langle\cdot,\cdot\rangle'$ is the canonical pairing $V_\Sigma^* \times V_\Sigma \to K$. It is easy to check that $(V, \phi, \langle\cdot,\cdot\rangle)$ is an orthogonal representation of S. If $\psi - \psi^*$ is bijective, then we define a scalar product on V_Σ by $\langle v, w\rangle := \langle(\psi - \psi^*)v, w\rangle'$. It is easy to check that $(V, \phi, \langle\cdot,\cdot\rangle)$ is a symplectic representation. Statements (b) and (c) cannot be both true. If (V, ϕ) has an orthogonal structure, then there exists an isomorphism $\psi_1 : (V, \phi) \to (V, \phi)^*$ such that $\psi_1^* = \psi_1$. If (V, ϕ) has a symplectic structure, then there exists an isomorphism $\phi_2 : (V, \phi) \to (V, \phi)^*$ such that $\phi_2^* = -\phi_2$. Let $\psi = \psi_1 + \psi_2$. Now $\psi + \psi^* = 2\phi_1$ and $\psi - \psi^* = 2\psi_2$ are both invertible. This is a contradiction. ∎

Remark 2.10. Note that if (V, ϕ) is a representation of S°, then $(V, \phi) \oplus (V, \phi)^*$ can be made into an orthogonal or symplectic representation of S. Define $\langle\cdot,\cdot\rangle$ on $V_\Sigma \oplus V_\Sigma^*$ by

$$\langle(v_1, w_1), (v_2, w_2)\rangle = w_2(v_1) + w_1(v_2).$$

It is easy to see that $(W, \psi, \langle\cdot,\cdot\rangle)$ is an orthogonal representation of S, where $(W, \psi) = (V, \phi) \oplus (V, \phi)^*$. If we define $\langle\cdot,\cdot\rangle$ on $V_\Sigma \oplus V_\Sigma^*$ by

$$\langle(v_1, w_1), (v_2, w_2)\rangle = w_2(v_1) - w_1(v_2),$$

then $(W, \psi, \langle\cdot,\cdot\rangle)$ is a symplectic representation of S.

Remark 2.11. Define an equivalence relation $\sim$ on the set of indecomposable representations of S° by $(V, \phi) \sim (V', \phi')$ if and only if (V, ϕ) is isomorphic to (V', ϕ') or its dual. Then there is a 1-1 correspondence between $\sim$-equivalence classes of indecomposable S°-representations and indecomposable orthogonal S-representations. Namely, if (V, ϕ) and $(V, \phi)^*$ are not isomorphic, then $(V, \phi) \oplus (V, \phi)^*$ carries a unique structure of an indecomposable orthogonal S-representation. If (V, ϕ) and $(V, \phi)^*$ are isomorphic, then either (V, ϕ) or $(V, \phi) \oplus (V, \phi)$ carries a unique structure of indecomposable orthogonal representation of S. Moreover, all indecomposable orthogonal representations are constructed in this way. A similar statement is also true for symplectic representations.

3. Finite type symmetric quivers. In this section we give a classification of finite type symmetric quivers. The proofs are based on Theorem 2.6. The remaining proofs are standard so we skip them and just list all the cases with the classification of indecomposable representations.

Definition. A symmetric quiver is said to be of *finite type* if it has only finitely many indecomposable orthogonal (resp. symplectic) representations up to isomorphism.

Theorem 3.1. *A symmetric quiver S is of finite type if and only if the underlying quiver S° is of Dynkin type.*

Definition. If $S = (S_0, S_1, i, f, \sigma)$ and $S' = (S_0', S_1', i', f', \sigma')$ are two symmetric quivers, then we define the disjoint union $S \amalg S'$ as

$$(S_0 \amalg S_0', S_1 \amalg S_1', i'', f'', \sigma'')$$

where the restrictions of i'' to S_1 and S_1' are i, i' respectively, the restrictions of f'' to S_1 and S_1' are f, f' respectively, and the restrictions of σ'' to $S_0 \amalg S_1$ and $S_0' \amalg S_1'$ are σ and σ' respectively. We call a symmetric quiver S *irreducible* if S is not the disjoint union of two nontrivial symmetric quivers.

If S is an irreducible symmetric quiver, then there are two possibilities. Either S° is connected, or S° has two connected components which are interchanged by σ.

Proposition 3.2. *Suppose that S is an irreducible symmetric quiver such that S° has two connected components. Then S° is isomorphic to the disjoint union of Γ and Γ' where $\Gamma = (\Gamma_0, \Gamma_1, i, f)$ is a quiver, and $\Gamma' = (\Gamma_0, \Gamma_1, f, i)$ is its dual* (*reversing all arrows of Γ*)*. The automorphism σ of S corresponds to interchanging the vertices and arrows of Γ with the vertices and arrows of Γ'. There is a correspondence between quiver representations of Γ and orthogonal* (*resp. symplectic*) *representations of S.*

Proposition 3.3. *If S is a symmetric quiver of finite type with S° connected, then S° must be a quiver of Dynkin type A_n.*

Sketch of proof. One can easily see that for quivers of Dynkin type D_n, $E_{6,7,8}$ there is no orientation of arrows that admits an involution reversing the orientation of arrows. Indeed, for D_n, $n > 4$, the involution would have to fix the long arm, so it could not invert the orientations of the arrows there. For D_4 the involution would have to fix at least one of three edges (so it cannot reverse the orientation of the corresponding arrow). For E_7, E_8 there is no nontrivial involution of the underlying graph. Finally, for E_6 the underlying Dynkin graph has one nontrivial involution but it fixes the short arm, so it cannot invert the orientation of the corresponding arrow. ∎

This means that there are two types of symmetric quivers of finite type:

$$A_n^{\mathrm{odd}} : \circ \to \circ \to \ldots \to \circ \to \bullet \to \circ \to \circ \to \ldots \to \circ$$

with $(A_n^{\mathrm{odd}})^0 = A_{2n-1}$ (and arbitrary orientation of the arrows reversed by the involution), and

$$A_n^{\mathrm{even}} : \circ \to \circ \to \ldots \to \circ \to \circ$$

with $(A_n^{\mathrm{even}})^0 = A_{2n}$ (and arbitrary orientation of the arrows reversed by the involution).

In the next few statements we give the structure of indecomposable orthogonal and symplectic representations for symmetric quivers of finite type. In some cases there is a natural bijection between symplectic (orthogonal) representations of such a quiver and the positive roots in certain root systems. This extends the usual Gabriel correspondence for quivers.

We start with some general remarks. Denote by ∇ the duality $V \mapsto V^*$ on the category $\mathrm{Rep}(S^\circ)$. This is a contravariant exact endofunctor on $\mathrm{Rep}(S^\circ)$. Its relation to the Coxeter functors (see [B-G-P], [A-P-R] for the definition of C^+ and C^-) is expressed as follows.

Proposition 3.4. *The functor ∇ commutes with the Coxeter functors, i.e.*

$$\nabla C^+ = C^- \nabla, \qquad \nabla C^- = C^+ \nabla.$$

Proof. The functor ∇ takes projective modules to injective ones and vice versa. It is a contravariant exact functor, so we have

$$\mathrm{Hom}_{\mathrm{Rep}(S^\circ)}(V, W) = \mathrm{Hom}_{\mathrm{Rep}(S^\circ)}(\nabla(W), \nabla(V)),$$
$$\mathrm{Ext}^1_{\mathrm{Rep}(S^\circ)}(V, W) = \mathrm{Ext}^1_{\mathrm{Rep}(S^\circ)}(\nabla(W), \nabla(V)).$$

It follows that ∇ preserves almost split sequences (and inverts their homomorphisms). The assertion follows. ■

Assume that S° is of finite type. The functor ∇ defines an arrow inverting symmetry of the whole Auslander–Reiten quiver of S°. The indecomposable orthogonal (resp. symplectic) representations are therefore of three kinds.

(a) representations $V \oplus \nabla(V)$, where V is not selfdual,

(b) representations V where $V = \nabla(V)$ and V is symplectic (resp. orthogonal),

(c) ramified representations $V \oplus V$ where $V = \nabla(V)$, but V is not symplectic (resp. orthogonal).

Example 3.5. Consider the symmetric quiver

$$A_3^{\mathrm{odd}} : \circ \to \circ \to \bullet \to \circ \to \circ$$

An orthogonal representation of A_3^{odd} is a collection of two vector spaces V_1, V_2 and an orthogonal space W together with linear maps $V_1 \to V_2$ and $V_2 \to W$. First, we examine the underlying quiver

$$A_5 = (A_3^{\mathrm{odd}})^\circ : \circ \to \circ \to \circ \to \circ \to \circ$$

The indecomposable representations of A_5 are the modules $M_{i,j}$ $(1 \le i \le j \le 5)$ where $M_{i,j}$ is the representation

$$0 \to \dots \to 0 \to \underset{i}{K} \to \dots \to \underset{j}{K} \to 0 \to \dots \to 0$$

and all maps between the one-dimensional spaces are identities. For the indecomposable module $M_{i,j}$ with nonsymmetric dimension, the corresponding symplectic or orthogonal indecomposable is the sum $M_{i,j} \oplus \nabla(M_{i,j})$. The modules $M_{1,5}, M_{2,4}, M_{3,3}$ with symmetric dimensions are easily seen to be orthogonal.

Now we list all triples $(\dim V_1, \dim V_2, \dim W)$ such that there is an indecomposable orthogonal representation $V_1 \to V_2 \to W \to V_2^* \to V_1^*$:

$$\begin{array}{ccc} (1,0,0) & (0,1,0) & (0,0,1) \\ (1,1,0) & (0,1,2) & (1,1,2) \\ (0,1,1) & (1,2,2) & (1,1,1) \end{array}$$

Note that these vectors correspond to the positive roots of the root system of type B_3, in the following way. The triple (c_1, c_2, c_3) corresponds to $c_1\alpha_1 + c_2\alpha_2 + c_3\alpha_3$ which is a positive root in the root system C_3 (where $\alpha_1, \alpha_2, \alpha_3$ are the simple roots, and α_3 is a short root).

Consider the symplectic representations of A_3^{odd}. Clearly, no indecomposable representation (V, ϕ) of A_5 with symmetric dimension can have the structure of a symplectic representation because W is forced to be a one-dimensional symplectic space, which is impossible. So all indecomposable symplectic representations of S are of the form $V \oplus \nabla(V)$ where V is an indecomposable representation of A_5. We list all triples $(\dim V_1, \dim V_2, \dim W)$ where an indecomposable symplectic representation exists:

$$\begin{array}{ccc} (1,0,0) & (0,1,0) & (0,0,2) \\ (1,1,0) & (0,1,2) & (1,1,2) \\ (0,2,2) & (1,2,2) & (2,2,2) \end{array}$$

Note that these are all triples of the form $(c_1, c_2, 2c_3)$ such that $c_1\alpha_1 + c_2\alpha_2 + c_3\alpha_3$ is a positive root in the root system C_3 (where $\alpha_1, \alpha_2, \alpha_3$ are the simple roots, and α_3 is a long root).

These considerations generalize easily to the symmetric quiver A_n^{odd} for arbitrary n.

Proposition 3.6. *Let* $S = A_n^{\mathrm{odd}}$.

(a) *The dimension vectors of the indecomposable orthogonal representations of S correspond naturally to the positive roots of the root system of type B_n. To the dimension vector $(c_1, \ldots, c_n)$ we associate the root $c_1\alpha_1 + \ldots + c_n\alpha_n$ where $\alpha_1, \ldots, \alpha_n$ are the simple roots, with α_n being the short root. In every such dimension there is exactly one orthogonal indecomposable representation which is an open orbit with respect to* $\mathrm{GL}(V_1) \times \ldots \times \mathrm{GL}(V_n) \times \mathrm{O}(W)$.

(b) *The dimension vectors of the indecomposable symplectic representations of S correspond naturally to the positive roots of the root system of type C_n. To the dimension vector $(c_1, \ldots, c_{n-1}, 2c_n)$ we associate the* *root* $c_1\alpha_1 + \ldots + c_n\alpha_n$ *where* $\alpha_1, \ldots, \alpha_n$ *are the simple roots, with* α_n *being the long root. In every such dimension there is exactly one symplectic indecomposable representation which is an open orbit with respect to* $\mathrm{GL}(V_1) \times \ldots \times \mathrm{GL}(V_n) \times \mathrm{SP}(W)$.

Proof. All the statements except the existence of a unique open orbit follow from the above discussion. To see this last fact we observe that for any action of a connected algebraic group on an irreducible variety the closure of an orbit is a union of this orbit and other orbits of smaller dimension. This implies that for the action with finitely many orbits the union of orbits of dimension $\leq s$ is Zariski closed for every s. Taking s equal to the dimension of our representation we get the existence of a unique open orbit. ∎

EXAMPLE 3.7. Consider the symmetric quiver

$$S = A_3^{\text{even}} : \circ \to \circ \to \circ \to \circ \to \circ \to \circ$$

A symplectic representation of S is a set of vector spaces V_1, V_2, V_3 together with linear maps $V_1 \to V_2$, $V_2 \to V_3$ and a symmetric linear map $V_3 \to V_3^*$ (which is a symmetric bilinear form on V_3).

Again, from the indecomposable representations $M_{i,j}$ of S° we can construct the indecomposable symplectic representations of S. The indecomposables $M_{i,j}$ with symmetric dimension vectors now admit a symplectic structure (the map in the middle has rank one, so it cannot be skew-symmetric; that is why the dimension vector $(2,2,2)$ is not on our list). The triples $(\dim V_1, \dim V_2, \dim V_3)$ where indecomposables exist are:

$$\begin{array}{ccc} (1,0,0) & (0,1,0) & (0,0,1) \\ (1,1,0) & (0,1,1) & (1,1,1) \\ (0,1,2) & (1,1,2) & (1,2,2) \end{array}$$

For $(0,0,1)$, $(0,1,1)$ and $(1,1,1)$ there are two indecomposables, and for the other dimension vectors there is a unique indecomposable. There are 12 indecomposables in total. In one of the two indecomposables for the dimension vector $(0,0,1)$, the symmetric map $V_3 \to V_3^*$ is an isomorphism, and in the other one it is 0.

The indecomposable orthogonal representations of S have the dimension vectors

$$\begin{array}{cccc} (1,0,0) & (0,1,0) & (0,0,1) & (1,1,0) \\ (0,1,1) & (0,0,2) & (1,1,1) & (0,1,2) \\ (1,1,2) & (0,2,2) & (1,2,2) & (2,2,2) \end{array}$$

with one indecomposable occurring in every dimension.

These considerations generalize easily to an arbitrary quiver A_n^{even}.

PROPOSITION 3.8. *Let S be a symmetric quiver of type A_n^{even}.*

(a) *There are $(n+1)n$ indecomposable orthogonal representations for S. Each occurs in a different dimension. The list of dimensions consists of all n-tuples that correspond to the dimensions of indecomposables for the system A_n, and all nondecreasing n-tuples whose last term is* 2.

(b) *There are $(n+1)n$ indecomposable symplectic representations for S. The list of dimensions consists of all n-tuples that correspond to the dimensions of indecomposables for the system A_n, and all nondecreasing n-tuples whose last term is* 2, *and which contain at least one term equal to* 1. *The dimensions* $(0,\dots,0,1,\dots,1)$ *contain two indecomposables*, *all the other dimensions contain one indecomposable.*

4. Symmetric quivers of tame type

DEFINITION. A symmetric quiver S is said to be *tame* if it is not of finite type, but in every dimension the orthogonal (resp. symplectic) indecomposable modules occur in families of dimension ≤ 1. More precisely one requires that for any dimension vector α there are finitely many morphisms $f_1(\alpha),\dots,f_{N(\alpha)}(\alpha)$ from K^1 to $\mathrm{Rep}(Q,\alpha)$ such that the images of all but finitely many points under $f_i(\alpha)$ are indecomposable orthogonal (resp. symplectic) representations of Q, and all but finitely many indecomposable orthogonal (resp. symplectic) representations of dimension α are obtained in this way.

The definition does not depend on whether we use orthogonal or symplectic representations because from Theorem 2.6 and standard results on classification of tame quivers we get

THEOREM 4.1. *A symmetric quiver S with S° connected is tame if and only if the underlying quiver S° is the extended Dynkin diagram.*

Now it is very easy to classify the irreducible symmetric tame quivers. If S° has two connected components, then each component has to be isomorphic to an extended Dynkin quiver. For tame symmetric quivers with S° connected we have the following possibilities.

PROPOSITION 4.2. *Let S be an irreducible symmetric quiver. Then one of the two cases occurs*:

1. *S_0 is the union of two connected components* (*dual to each other*), *each of which is an extended Dynkin quiver.*

2. *S_0 is connected*, *of one of the following six types*: (*In each of the cases below except* (d) *the involution is the reflection in a central horizontal line. In case* (d) *the involution is a central symmetry.*)

(a) $S = \widehat{A}_n^{\mathrm{odd}}$, $S^\circ = \widehat{A}_{2n}$, *for* $n \geq 1$, *with arbitrary orientation of the arrows that is reversed under the involution* (*below we have* $n = 3$):

$$\begin{array}{ccccc} & & \bullet & & \\ & \nearrow & & \searrow & \\ \circ & & & & \circ \\ \downarrow & & & & \uparrow \\ \circ & & & & \circ \\ \uparrow & & & & \downarrow \\ \circ & & \longrightarrow & & \circ \end{array}$$

(b) $S = \widehat{A}_{n,1}^{\mathrm{even}}$, $S^\circ = \widehat{A}_{2n-1}$, *for* $n \geq 1$, *with arbitrary orientation of the arrows that is reversed under the involution* (*below we have* $n = 4$):

$$\begin{array}{ccccc} & & \bullet & & \\ & \nearrow & & \searrow & \\ \circ & & & & \circ \\ \downarrow & & & & \uparrow \\ \circ & & & & \circ \\ \uparrow & & & & \downarrow \\ \circ & & & & \circ \\ & \searrow & & \nearrow & \\ & & \bullet & & \end{array}$$

(c) $S = \widehat{A}_{n,2}^{\mathrm{even}}$, $S^\circ = \widehat{A}_{2n+1}$, *for* $n \geq 1$, *with arbitrary orientation of the arrows that is reversed under the involution* (*below we have* $n = 3$):

$$\begin{array}{ccc} \circ & \longleftarrow & \circ \\ \downarrow & & \uparrow \\ \circ & & \circ \\ \downarrow & & \uparrow \\ \circ & & \circ \\ \uparrow & & \downarrow \\ \circ & \longrightarrow & \circ \end{array}$$

(d) $S = \widehat{A}_{n,3}^{\mathrm{even}}$, $S^\circ = \widehat{A}_{2n+1}$, *for* $n \geq 1$, *with arbitrary orientation of the arrows that is reversed under the involution* (*below we have* $n = 3$):

$$\begin{tikzcd}
& \circ & \\
\circ \arrow[ur] & & \circ \arrow[ul] \\
\circ \arrow[u] & \cdot & \circ \arrow[u] \\
\circ \arrow[u] & & \circ \arrow[u] \\
& \circ \arrow[ul] \arrow[ur] &
\end{tikzcd}$$

(e) $S = \widehat{D}_n^{\mathrm{odd}}$, $S^\circ = \widehat{D}_{2n}$, *for* $n \geq 2$, *with arbitrary orientation of the arrows that is reversed under the involution* (*below we have* $n = 3$):

$$\begin{tikzcd}
\circ \arrow[dr] & & & & \circ \\
& \circ \arrow[r] & \bullet \arrow[r] & \circ \arrow[ur] \arrow[dr] & \\
\circ \arrow[ur] & & & & \circ
\end{tikzcd}$$

(f) $S = \widehat{D}_n^{\mathrm{even}}$, $S^\circ = \widehat{D}_{2n+1}$, *for* $n \geq 2$, *with arbitrary orientation of the arrows that is reversed under the involution* (*below we have* $n = 2$):

$$\begin{tikzcd}
\circ \arrow[dr] & & & \circ \\
& \circ \arrow[r] & \circ \arrow[ur] \arrow[dr] & \\
\circ \arrow[ur] & & & \circ
\end{tikzcd}$$

The cases above are classified according to the way the involution acts on vertices. In each case there are several possible orientations (all orientations which are reversed by the involution).

We end the section by giving the one-dimensional families of indecomposable symplectic and orthogonal representations of all types of tame symmetric quivers.

Let us recall that for the extended Dynkin quiver the one-dimensional families of indecomposables occur only in dimensions $p\underline{h}$ where $\underline{h}$ is the dimension vector corresponding to the basic imaginary root. For the quiver of type $\widehat{A}_n$ this dimension is given by $\underline{h}(v) = 1$ for all vertices $v \in S_0$. For the quiver of type $\widehat{D}_n$ we have $\underline{h}(v) = 2$ unless v is one of the four vertices on the boundary, for which $\underline{h}(v) = 1$. Therefore it is enough to see whether the general representatives of these families are selfdual in the orthogonal or symplectic sense. We denote by $M(\lambda, p\underline{h})$ the module from the family of indecomposables in dimension $p\underline{h}$ corresponding to the parameter λ. Their canonical form is given in [D-R].

In order to understand one-dimensional families of indecomposables, it suffices by Proposition 2.7 to investigate the duality on the families $M(\lambda, p\underline{h})$.

This can be done case by case and the results are as follows.

PROPOSITION 4.3. *Let S be a symmetric tame quiver. The dual of the module $M(\lambda, p\underline{h})$ is as follows*:

(a) $S = \widehat{A}_n^{\mathrm{odd}}$, $S^\circ = \widehat{A}_{2n}$, *for* $n \geq 1$, $\nabla(M(\lambda, p\underline{h})) = M(-\lambda, p\underline{h})$,

(b) $S = \widehat{A}_{n,1}^{\mathrm{even}}$, $S^\circ = \widehat{A}_{2n-1}$, *for* $n \geq 1$, $\nabla(M(\lambda, p\underline{h})) = M(\lambda, p\underline{h})$, *and all modules* $M(\lambda, p\underline{h})$ *have an orthogonal structure*,

(c) $S = \widehat{A}_{n,2}^{\mathrm{even}}$, $S^\circ = \widehat{A}_{2n+1}$, *for* $n \geq 1$, $\nabla(M(\lambda, p\underline{h})) = M(\lambda, p\underline{h})$, *and all modules* $M(\lambda, p\underline{h})$ *have a symplectic structure*,

(d) $S = \widehat{A}_{n,3}^{\mathrm{even}}$, $S^\circ = \widehat{A}_{2n+1}$, *for* $n \geq 1$, *Then* $\nabla(M(\lambda, p\underline{h})) = M(\lambda^{-1}, p\underline{h})$,

(e) $S = \widehat{D}_n^{\mathrm{odd}}$, $S^\circ = \widehat{D}_{2n}$, *for* $n \geq 2$, $\nabla(M(\lambda, p\underline{h})) = M(\lambda, p\underline{h})$, *and all modules* $M(\lambda, p\underline{h})$ *have an orthogonal structure*,

(f) $S = \widehat{D}_n^{\mathrm{even}}$, $S^\circ = \widehat{D}_{2n+1}$, *for* $n \geq 2$, $\nabla(M(\lambda, p\underline{h})) = M(\lambda, p\underline{h})$, *and all modules* $M(\lambda, p\underline{h})$ *have a symplectic structure.*

One concludes that the structure of one-dimensional families of indecomposable modules for tame symmetric quivers is as follows.

THEOREM 4.4. *Let S be a symmetric tame quiver. Then the infinite families of symplectic and orthogonal representations of S are as follows*:

1) *The families of symplectic representations occur in dimensions $p\underline{h}$ (p arbitrary) for S of types $\widehat{A}_{n,2}^{\mathrm{even}}, \widehat{D}_n^{\mathrm{even}}$. For the types $\widehat{A}_n^{\mathrm{odd}}, \widehat{A}_{n,1}^{\mathrm{even}}, \widehat{A}_{n,3}^{\mathrm{even}}, \widehat{D}_n^{\mathrm{odd}}$ there are infinite families in dimensions $p\underline{h}$ (p even). In each indicated dimension there is one infinite family of indecomposables*, *parametrized by a projective line with a finite number of points removed.*

2) *The families of orthogonal representations occur in dimensions $p\underline{h}$ (p arbitrary) for S of types $\widehat{A}_{n,1}^{\mathrm{even}}$, $\widehat{D}_n^{\mathrm{odd}}$. For the types $\widehat{A}_n^{\mathrm{odd}}$, $\widehat{A}_{n,2}^{\mathrm{even}}$, $\widehat{A}_{n,3}^{\mathrm{even}}$, $\widehat{D}_n^{\mathrm{even}}$ there are infinite families in dimensions $p\underline{h}$ (p even). In each indicated dimension there is one infinite family of indecomposables*, *parametrized by a projective line with a finite number of points removed.*

5. Applications. In this section we discuss the relation of symmetric quivers to other classification problems.

Quivers with contravariant involutions were considered by Sergeĭchuk ([S1]–[S3]) and by Kruglyak ([Kr]) in the eighties. Their investigations were based on Roĭter's note [R] which contains a lemma similar to Theorem 2.6 but the main part of the proof refers to a rather obscure book by Mal'tsev.

Sergeĭchuk investigated quivers with free involutions, i.e. he assumed that an involution cannot fix a vertex or an arrow of the underlying quiver. Under this assumption he classified in [S1] the finite type and tame quivers. That classification gives exactly the types on our lists that do not include closed nodes. The other cases on our list do not appear in [S1]. In that framework they would appear as quivers with relations, but the classification of finite type and tame quivers for quivers with relations is a very difficult problem.

Many natural linear algebra problems related to classical groups can be stated naturally in terms of symmetric quivers.

EXAMPLE 5.1. Consider a vector space F with a symmetric (or skew symmetric) nondegenerate bilinear form $\langle\cdot,\cdot\rangle$. Let G be the group of symmetries of the form $\langle\cdot,\cdot\rangle$ (i.e. the orthogonal or symplectic group). The homogeneous spaces for G can be expressed in terms of the spaces of isotropic flags in F. So the question of classifying the G-orbits on a multiple isotropic flag variety is a natural and important one. From the point of view of symmetric quivers the isotropic flags in F correspond to the (orthogonal or symplectic) representations of the symmetric quiver

$$\circ\xrightarrow{a_1}\circ\rightarrow\ldots\rightarrow\circ\xrightarrow{a_k}\bullet\xrightarrow{\sigma a_k}\circ\rightarrow\circ\rightarrow\ldots\xrightarrow{\sigma a_1}\circ$$

with the relation $(\sigma a_k)a_k = 0$. Similarly one can treat multiple flag varieties, using the quiver given by multiple paths through the middle closed node with a zero relation along each path. The approach to this classification problem presented in [M-W-Z] could be entirely rephrased in terms of symmetric quivers.

From the point of view of [S1]–[S3] the same classification problem would appear in a much more cumbersome way. In fact the latter approach hides the connection with representations of symplectic and orthogonal groups.

REMARK 5.2. Sergeĭchuk also considered a more general notion of “or-schemes” which include bilinear forms on vector spaces. Kruglyak gave a general classification of finite type and tame quivers with involution in the case of unitary groups. This kind of matrix problems can be treated using our approach starting with the unitary group.

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