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SELFINJECTIVE ALGEBRAS OF TUBULAR TYPE

BY

JERZY BIAŁKOWSKI and ANDRZEJ SKOWROŃSKI (Toruń)

Abstract. We classify all tame selfinjective algebras having simply connected Galois coverings and the stable Auslander–Reiten quivers consisting of stable tubes. Moreover, the classification of nondomestic polynomial growth standard selfinjective algebras is completed.

Introduction. Throughout, by an *algebra* we mean a basic connected finite-dimensional associative K-algebra with an identity over a (fixed) algebraically closed field K. By a *module* over an algebra A we mean a right A-module of finite K-dimension. An algebra A with A_A injective is called *selfinjective*. An important class of selfinjective algebras is formed by the algebras of the form $\widehat{B}/G$ where $\widehat{B}$ is the repetitive algebra [20] (locally finite-dimensional, without identity) of an algebra B and G is an admissible group of K-linear automorphisms of $\widehat{B}$.

From Drozd's remarkable Tame and Wild Theorem [10] the class of algebras may be divided into two disjoint classes. One class consists of tame algebras for which the indecomposable modules occur, in each dimension d, in a finite number of discrete and a finite number of one-parameter families. The second class is formed by the wild algebras whose representation theory is as complicated as the study of finite-dimensional K-vector spaces together with two noncommuting endomorphisms, the classification of which is a well known unsolved problem. Hence, we can hope to classify the modules only for tame algebras. Frequently, tame algebras are deformations of tame algebras which admit simply connected Galois coverings. This is the case for all representation-finite algebras (see [5], [6]).

In this paper we are concerned with the problem of describing (tame) selfinjective algebras all of whose indecomposable nonprojective modules are periodic with respect to the action of the Auslander–Reiten translation. This class of algebras contains all representation-finite selfinjective algebras which have been completely classified almost 20 years ago (see [7], [20], [27], [28]). Moreover, it is known that every indecomposable nonprojective

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module over a selfinjective algebra A is periodic if and only if the Auslander–Reiten quiver Γ_A of A is cyclic, that is, all its vertices lie on oriented cycles (see [3], [35]). Further, for a representation-infinite selfinjective algebra A, this is equivalent to the fact that the stable Auslander–Reiten quiver Γ_A^s of A consists only of stable tubes. It is known (see [11], [12]) that, for a representation-infinite block B of the group algebra KG of a finite group G, all indecomposable nonprojective B-modules are periodic if and only if K is of characteristic 2 and the defect group of B is a generalized quaternion group, and these blocks are representation-infinite tame of nonpolynomial growth (with the stable tubes of rank at most 2). Another class of tame selfinjective algebras with stable Auslander–Reiten quiver consisting only of stable tubes is formed by all nondomestic polynomial growth selfinjective algebras having simply connected Galois coverings studied in [30], and these are the orbit algebras $\widehat{B}/G$ given by Ringel's [29] tubular algebras B and admissible infinite cyclic groups G of K-linear automorphisms of $\widehat{B}$.

Finally, we mention that there are also wild selfinjective algebras (even with simply connected Galois coverings) for which the stable Auslander–Reiten quiver consists only of stable tubes. Namely, as was discovered by Schofield, the Gelfand–Ponomarev [16] preprojective algebras of Dynkin graphs are selfinjective algebras with stable Auslander–Reiten quivers consisting only of stable tubes of ranks dividing 6 (see also [2] for another approach and relationship with hypersurface singularities, and [14] for relationship with Hochschild cohomology).

The aim of this paper is to classify all tame selfinjective algebras which admit a simply connected Galois covering and their stable Auslander–Reiten quiver consists only of stable tubes. We prove (Theorem 3.1) that this class of algebras coincides with the class of algebras of the form $\widehat{B}/G$, where B is a tubular algebra and G is an admissible infinite cyclic group G of K-linear automorphisms of $\widehat{B}$, and hence with the class of all nondomestic standard selfinjective algebras of polynomial growth investigated in [23], [30]. A classification of algebras $\widehat{B}/G$, with B tubular of type $(2,3,6)$, has been done recently in [21], invoking the derived category of coherent sheaves on the corresponding weighted projective line over K. Moreover, this can also be done easily (see Theorem 4.2) in the tubular case $(2,2,2,2)$, because there are only 10 one-parameter families of such algebras [30]. The second objective of the paper is to give a complete classification (Theorems 5.2 and 6.2) of algebras $\widehat{B}/G$ for the remaining two tubular cases $(3,3,3)$ and $(2,4,4)$, where a large number of algebras is involved. This is done with the help of computer.

For basic background on the representation theory of algebras we refer to the books [3], [29], and on selfinjective algebras to [11], [34].

1. Selfinjective algebras with simply connected Galois coverings. Following [5], by a *locally bounded category* we mean a K-category R which is isomorphic to the factor category KQ/I, where Q is a locally finite quiver and I is an admissible ideal in the path category KQ of Q. Recall that the objects of a locally bounded category $R = KQ/I$ are given by the vertices of Q, and the morphism spaces $R(x, y)$ are the quotients of the spaces $KQ(x, y)$ generated by all paths in Q from x to y modulo the subspaces $I \cap KQ(x, y)$. We shall consider an algebra A as a locally bounded category with finitely many objects, called briefly a *finite category.* For a locally bounded category R, we denote by $\operatorname{mod} R$ the category of all finite-dimensional right R-modules, by Γ_R the *Auslander–Reiten quiver* of R, and by τ_R the *Auslander–Reiten translation* $D\operatorname{Tr}$ on $\operatorname{mod} R$. We shall not distinguish between an indecomposable module from $\operatorname{mod} R$ and the vertex of Γ_R corresponding to it. Following [10], a finite bounded category R is said to be *tame* if, for any dimension d, there exists a finite number of $K[x]$-R-bimodules M_i, $1 \le i \le n_d$, which are finitely generated and free as left $K[x]$-modules, and all but a finite number of isomorphism classes of indecomposable (right) R-modules of dimension d are of the form $K[x]/(x - \lambda) \otimes_{K[x]} M_i$ for some $\lambda \in K$ and some i. Moreover, R is said to be of *polynomial growth* if there exists a natural number m such that for any $d \ge 1$ the least number of $K[x]$-R-bimodules satisfying the above conditions for d is bounded by d^m (see [31]). Finally, an arbitrary locally bounded category R is said to be *tame* (respectively, of *polynomial growth*) if so is every finite full subcategory of R (see [8]).

A group G of K-linear automorphisms of a locally bounded category R is said to be *admissible* if its action on the objects of R is free and has finitely many orbits. Then the finite bounded category (algebra) R/G is defined and there is a *Galois covering* functor $F : R \to R/G$ which assigns to each object x of R its G-orbit Gx (see [15]). We denote by

$$F_\lambda : \operatorname{mod} R \to \operatorname{mod} R/G$$

the *push-down functor* induced by the covering $F : R \to R/G$ (see [5]). It is well known that if G is torsion-free then the push-down functor F_λ preserves indecomposability of modules and Auslander–Reiten sequences (see [15]). A locally bounded category R is called *simply connected* [1] if it is triangular (its quiver has no oriented cycles) and for any presentation $R \xrightarrow{\sim} KQ/I$ of R as a bound quiver category, the fundamental group $\Pi_1(Q, I)$ of (Q, I), defined in [17], [22], is trivial. It has been proved in [31] that a triangular locally bounded category R is simply connected if and only if each Galois covering of R is trivial.

The *repetitive category* [20] of a locally bounded category R is the selfinjective locally bounded category $\widehat{R}$ whose objects are the pairs $(n, x) = x_n$,

$x \in R$, $n \in \mathbb{Z}$ (the set of all integers) and $\widehat{R}(x_n, y_n) = \{n\} \times R(x, y)$, $\widehat{R}(x_n, y_{n+1}) = \{n\} \times DR(y, x)$, and $\widehat{R}(x_p, y_q) = 0$ if $q \neq p, p+1$, where DV denotes the dual space $\operatorname{Hom}_K(V, K)$. We denote by $\nu_{\widehat{R}}$ the *Nakayama automorphism* of $\widehat{R}$, which assigns to each object $x_n = (n, x)$ the object $x_{n+1} = (n+1, x)$. Observe that if R is finite, then the infinite cyclic group $(\nu_{\widehat{R}})$ generated by $\nu_{\widehat{R}}$ is admissible and $\widehat{R}/(\nu_{\widehat{R}})$ is isomorphic to the *trivial extension*

$$T(R) = R \ltimes D(R)$$

of R by its minimal injective cogenerator $D(R) = \operatorname{Hom}_K(R, K)$. Further, we note that a locally bounded category R is simply connected if and only if its repetitive category $\widehat{R}$ is simply connected. A K-linear automorphism φ of a repetitive category $\widehat{R}$ is said to be *positive* if, for any object x_n of $\widehat{R}$, we have $\varphi(x_n) = y_m$ for some object y of R and some $m \geq n$. Moreover, φ is said to be *rigid* if, for any object x_n of $\widehat{R}$, we have $\varphi(x_n) = y_n$ for some object y of R. We refer to [25] for some results on the structure of K-linear automorphisms of repetitive categories, and to [33] for results on the presentations of selfinjective algebras A in the form $A \cong \widehat{B}/(\varphi\nu_{\widehat{B}})$ with B a triangular algebra (triangular finite bounded category) and φ a positive K-linear automorphism of $\widehat{B}$.

For a selfinjective locally bounded category R, we denote by Γ_R^{s} the *stable Auslander–Reiten quiver* of R, obtained from the Auslander–Reiten quiver Γ_R by removing all projective modules and arrows attached to them. A component of Γ_R (respectively, Γ_R^{s}) of the form $\mathbb{Z}\mathbb{A}_\infty/(\tau^r)$, $r \geq 1$, is said to be a *stable tube of rank* r. Therefore, a stable tube in Γ_R (respectively, Γ_R^{s}) consists of τ_R-periodic indecomposable R-modules having period r. Finally, since R is selfinjective, we have $\tau_R = \Omega_R^2 \circ \mathcal{N}_R$ where Ω_R is Heller's syzygy operator and

$$\mathcal{N}_R : \operatorname{mod} R \to \operatorname{mod} R$$

is the equivalence induced by the Nakayama automorphism ν_R of R (see [3, IV.3.7] and [34]).

The following theorem proved in [32] gives a characterization of all tame selfinjective algebras which admit simply connected Galois coverings.

THEOREM 1.1. *A selfinjective algebra A is tame and admits a simply connected Galois covering if and only if $A \cong \widehat{B}/G$, where B is a simply connected locally bounded category such that the Euler form χ_C of any finite convex subcategory C of B is nonnegative, and G is an admissible torsion-free group of K-linear automorphisms of $\widehat{B}$.*

Recall that a full subcategory Λ of a locally bounded category $R = KQ/I$ is called *convex* provided $\Lambda = KQ'/I'$ for a convex subquiver Q' of Q and $I' = I \cap KQ'$. Moreover, the *Euler form* χ_R of a finite triangular bounded category R is the integral quadratic form $\chi_R : K_0(R) \to \mathbb{Z}$ on the Grothendieck group $K_0(R) \cong \mathbb{Z}^m$ (m = the number of objects in R) given by $\chi_R(x) = xC_R^{-t}x^t$, $x \in K_0(R)$, where C_R is the *Cartan matrix* $(\dim_K R(a,b))_{a,b\in R}$ of R.

2. Selfinjective algebras of tubular type. Following [29], by a *tubular algebra* we mean a tubular extension (equivalently, tubular coextension) B of a tame concealed algebra C of tubular type $(2,2,2,2)$, $(3,3,3)$, $(2,4,4)$, or $(2,3,6)$. Then the rank of the Grothendieck group $K_0(B)$ of B is equal to 6, 8, 9, or 10, respectively. By a *selfinjective algebra of tubular type* we mean an algebra of the form $\widehat{B}/G$, where B is a tubular algebra and G is an admissible group of K-linear automorphisms of $\widehat{B}$. We shall exhibit here basic facts on the repetitive categories of tubular algebras and selfinjective algebras of tubular type, established in [23] and [30], needed in our further considerations.

Let B be an algebra and $e_1,\dots,e_n$ be a complete set of primitive orthogonal idempotents of B such that $1 = e_1 + \dots + e_n$. Denote by Q_B the (Gabriel) quiver of B with the set of vertices $\{1,\dots,n\}$ corresponding to the set $e_1,\dots,e_n$. For each vertex $i \in Q_B$, denote by $P_B(i)$ the indecomposable projective B-module e_iB and by $I_B(i)$ the indecomposable injective B-module $D(Be_i)$. Then, for a sink $i \in Q_B$, the *reflection* S_i^+B *of* B *at* i is the quotient of the one-point extension $B[I_B(i)]$ by the two-sided ideal generated by e_i. The quiver $\sigma_i^+Q_B$ of S_i^+B is called the *reflection of* Q_B *at* i. Observe that the sink i of Q_B is replaced in $\sigma_i^+Q_B$ by a source i'. Moreover, we have

$$\widehat{B} \cong \widehat{S_i^+B}.$$

A *reflection sequence of sinks* is a sequence $i_1,\dots,i_t$ of vertices of Q_B such that i_s is a sink of $\sigma_{i_{s-1}}^+ \dots \sigma_{i_1}^+ Q_B$ for $1 \le s \le t$ (see [20, (2.8)]). We have the following fact, proved in [23, Section 4], describing the relationship between tubular algebras with isomorphic repetitive algebras.

THEOREM 2.1. *Let B be a tubular algebra with Q_B having n vertices. There is a sequence of natural numbers $1 \le t_1 < \dots < t_{r+1} = n$, uniquely determined by B, and a reflection sequence of sinks $i_1,\dots,i_{t_1}, i_{t_1+1},\dots,i_{t_r}$, $i_{t_r+1},\dots,i_n$ in Q_B such that*:

(a) $S_{i_n}^+ \dots S_{i_1}^+ B \cong \nu_{\widehat{B}}(B) \cong B$.

(b) $S_{i_{t_j}}^+ \dots S_{i_1}^+ B$, $1 \le j \le r$, *are tubular algebras of the same tubular type as B.*

(c) *Every tubular algebra D with $\widehat{D} \cong \widehat{B}$ is isomorphic to $S_{i_{t_j}}^+ \dots S_{i_1}^+ B$ for some $1 \le j \le r+1$.*

Following [30], the tubular algebra B is said to be *normal* if the tubular algebras $S^+_{i_{t_j}} \dots S^+_{i_1} B$, $1 \le j \le r+1$, are pairwise nonisomorphic, or equivalently, $B \not\cong S^+_{i_{t_j}} \dots S^+_{i_1} B$ for any $1 \le j \le r$. Otherwise, B is said to be *exceptional*. It follows from [30, Section 3] that B is exceptional if and only if there exists an automorphism φ of $\widehat{B}$ such that $\varphi^d = \varrho \nu_{\widehat{B}}$ for some $d \ge 2$ and a rigid automorphism ϱ of $\widehat{B}$ induced by an automorphism of B.

The following proposition gives a general description of admissible groups of K-linear automorphisms of repetitive categories of tubular algebras (see [30, (3.9)]).

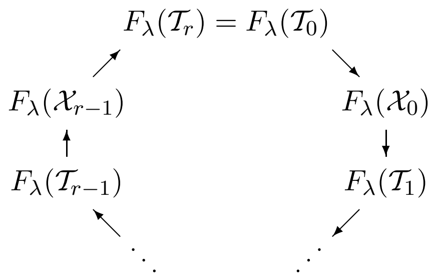
Proposition 2.2. *Let B be a tubular algebra and G an admissible group of K-linear automorphisms of $\widehat{B}$. Then G is an infinite cyclic group generated by an automorphism $\sigma\varphi^t_{\widehat{B}}$ for some $t \ge 1$, where σ is a rigid automorphism of $\widehat{B}$ and $\varphi_{\widehat{B}}$ is a K-linear automorphism of $\widehat{B}$ such that $\varphi^d_{\widehat{B}} = \varrho\nu_{\widehat{B}}$ for some $d \ge 1$ and a rigid automorphism ϱ of $\widehat{B}$. Moreover, if B is normal, we may take $\varphi_{\widehat{B}} = \nu_{\widehat{B}}$.*

We also note that, for any tubular algebra B and an admissible group G of K-linear automorphisms of $\widehat{B}$, $F : \widehat{B} \to \widehat{B}/G$ is a simply connected Galois covering of $\widehat{B}/G$, because B simply connected implies $\widehat{B}$ simply connected.

We end this section with the description of the structure of the Auslander–Reiten quivers of selfinjective algebras of tubular type. Let B be a tubular algebra of tubular type $n_B = (n_\lambda)_{\lambda \in \mathbb{P}_1(K)}$ consisting of positive integers n_λ, $\lambda \in \mathbb{P}_1(K)$, and all but finitely many equal to 1. We shall write instead of $(n_\lambda)_{\lambda \in \mathbb{P}_1(K)}$ the finite sequence consisting of all n_λ which are larger than 1, and arranged in nondecreasing order. Then n_B is one of the types $(2,2,2,2)$, $(3,3,3)$, $(2,4,4)$, or $(2,3,6)$. It follows from [23, Section 3] that the Auslander–Reiten quiver $\Gamma_{\widehat{B}}$ of $\widehat{B}$ is of the form

$$\Gamma_{\widehat{B}} = \bigvee_{p \in \mathbb{Z}} \mathcal{T}_p \vee \mathcal{X}_p$$

where, for each $p \in \mathbb{Z}$, $\mathcal{T}_p$ is a nonstable $\mathbb{P}_1(K)$-family of quasi-tubes (in the sense of [31, (1.2)] whose stable part $\mathcal{T}^{\mathrm{s}}_p$ is a $\mathbb{P}_1(K)$-family of stable tubes of tubular type n_B, $\mathcal{X}_p = \bigvee_{\gamma \in \mathbb{Q}^p_{p+1}} \mathcal{T}_\gamma$, $\mathbb{Q}^p_{p+1} = \mathbb{Q} \cap (p, p+1)$, and, for each $\gamma \in \mathbb{Q}^p_{p+1}$, $\mathcal{T}_\gamma$ is a $\mathbb{P}_1(K)$-family of stable tubes of tubular type n_B. Further, there exists $s \ge 3$ such that $\nu_{\widehat{B}}(\mathcal{T}_p) = \mathcal{T}_{p+s}$ and $\nu_{\widehat{B}}(\mathcal{X}_p) = \mathcal{X}_{p+s}$ for all $p \in \mathbb{Z}$. In particular, the stable Auslander–Reiten quiver $\Gamma^{\mathrm{s}}_{\widehat{B}}$ of $\widehat{B}$ consists of the rational family of $\mathbb{P}_1(K)$-families of stable tubes, all of them of tubular type n_B. Let G be an admissible group of K-linear automorphisms of $\widehat{B}$ and $A = \widehat{B}/G$ the associated selfinjective algebra (of tubular type n_B). Since G is, by Proposition 2.2, infinite cyclic (hence torsion-free), the push-down functor $F_\lambda : \operatorname{mod} \widehat{B} \to \operatorname{mod} \widehat{B}/G = \operatorname{mod} A$ associated with the Galois covering $F : \widehat{B} \to \widehat{B}/G = A$ preserves indecomposable modules and Auslander–Reiten sequences [15, Section 3]. Moreover, $\widehat{B}$ is locally support-finite [23, Section 3], and hence invoking the main result of [9] we conclude that $F_\lambda : \operatorname{mod} \widehat{B} \to \operatorname{mod} A$ is dense. As a consequence, the Auslander–Reiten quiver Γ_A of A is the orbit quiver $\Gamma_{\widehat{B}}/G$, and so it is obtained from $\Gamma_{\widehat{B}}$ by identifying (via F_λ) $\mathcal{T}_p$ with $\mathcal{T}_{p+r}$ and $\mathcal{X}_p$ with $\mathcal{X}_{p+r}$ for some $r \geq 1$ and all $p \in \mathbb{Z}$. Thus Γ_A has the following "clock structure":



3. Tame selfinjective algebras with tubular stable components. The main aim of this section is to prove the following theorem.

THEOREM 3.1. *For an algebra A the following two conditions are equivalent*:

(i) *A is a selfinjective algebra of tubular type.*

(ii) *A is tame*, *selfinjective*, *admits a simply connected Galois covering and the stable Auslander–Reiten quiver Γ_A^{s} of A consists only of stable tubes.*

Proof. The implication (i)⇒(ii) has already been established in the previous section. Assume that A satisfies condition (ii). Applying Theorem 1.1 we conclude that there is a Galois covering $F : R \to R/G = A$ where R is a simply connected selfinjective locally bounded category of the form $\widehat{B}$, for a simply connected locally bounded category B such that the Euler quadratic form of every finite convex subcategory of B is nonnegative, and G is a torsion-free group of K-linear automorphisms of $\widehat{B}$. We shall prove that B is a tubular algebra, and consequently $A = \widehat{B}/G$ is selfinjective of tubular type, as required. Since Γ_A^{s} consists of stable tubes, we conclude that A is representation-infinite. Assume now that A is of polynomial growth. Applying the main result of [30] we then conclude that either B is tubular or $A \cong \widehat{D}/H$ where D is a tilted algebra of Euclidean type and H is an admissible infinite cyclic group of K-linear automorphisms of $\widehat{D}$. But in the latter case, Γ_A^{s} admits (see [30, Section 2]) a nonperiodic component of the form $\mathbb{Z}\Delta$, for a Euclidean quiver Δ, and this contradicts our assumption on Γ_A^{s}.

Hence B is tubular. Suppose now that A is not of polynomial growth. We still have two cases to consider. Assume first that every full finite subcategory of $R = \widehat{B}$ is representation-finite. Then, by the main theorem of [26], we infer that B is an infinite locally bounded gentle tree K-category and $A = \widehat{B}/G$ is special biserial. Moreover, since A is not of polynomial growth, applying [13, Theorem 2.2], we deduce that Γ_A^{s} has a component of the form $\mathbb{Z}\mathbb{A}_\infty^\infty$, which contradicts our assumption on Γ_A^{s}. Finally, assume that R contains a representation-infinite full finite subcategory, and A is not of polynomial growth. Then, by [30, Theorem 1.5], R is also not of polynomial growth. Applying now arguments as in the proof of Proposition 4.3 in [30] (see also [32]) we conclude that B contains a finite convex simply connected category Λ which is a generalized pg-critical algebra, that is, is given by one of the 31 frames of pg-critical algebras or one of the algebras $(r1)$–$(r6)$ presented in [24, Section 3]. It follows from [24, Theorem 6.1] that Γ_Λ admits an Auslander–Reiten sequence of the form

$$0 \to M \to E_1 \oplus E_2 \oplus E_3 \to N \to 0$$

with E_1, E_2, E_3 indecomposable, and the full translation subquiver of Γ_Λ formed by all successors of M (in Γ_Λ) is of the form $(-\mathbb{N})\mathbb{D}_\infty$. Moreover, M, E_1, E_2, E_3 and N are nonprojective indecomposable R-modules, because Λ is also a finite convex subcategory of $R = \widehat{B}$. Further, it is known that the Auslander–Reiten sequence in $\operatorname{mod} R$ with left term M (or right term N) is an Auslander–Reiten sequence in $\operatorname{mod} D$ for a finite convex subcategory D of R containing Λ. On the other hand, our assumption that A is tame implies that R, and hence Λ, is tame [9, Proposition 2]. Invoking now the fact that D can be obtained from Λ by a sequence of one-point extensions and coextensions, and the formula [29, p. 88] on the Auslander–Reiten sequences for one-point extensions (or coextensions), we conclude that

$$0 \to M \to E_1 \oplus E_2 \oplus E_3 \to N \to 0$$

is also an Auslander–Reiten sequence in $\operatorname{mod} R$. Finally, since the group G is torsion-free, applying the push-down functor $F_\lambda : \operatorname{mod} R \to \operatorname{mod} R/G = \operatorname{mod} A$ associated with the Galois covering $F : R \to R/G = A$, we obtain an Auslander–Reiten sequence

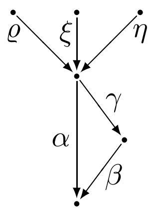
$$0 \to F_\lambda(M) \to F_\lambda(E_1) \oplus F_\lambda(E_2) \oplus F_\lambda(E_3) \to F_\lambda(N) \to 0$$

in $\operatorname{mod} A$, where the modules $F_\lambda(E_1)$, $F_\lambda(E_2)$ and $F_\lambda(N)$ are indecomposable and nonprojective. This contradicts our assumption that Γ_A^{s} consists only of stable tubes. Therefore, (ii) implies (i). ■

As a direct consequence of the above theorem and results presented in Sections 1 and 2 we obtain the following fact.

Corollary 3.2. *Let A be a tame selfinjective algebra having a simply connected Galois covering and the stable Auslander–Reiten quiver Γ_A^{s} consisting of stable tubes. Then all tubes in Γ_A^{s} have rank at most* 6.

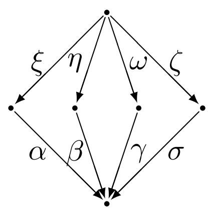
4. Selfinjective algebras of type $(2,2,2,2)$**.** We give a complete description of all selfinjective algebras of tubular type $(2,2,2,2)$. Consider the following family of bound quiver algebras (see [30, (3.3)]):



$\alpha\varrho = 0, \quad \alpha\xi = \beta\gamma\xi$

$\alpha\eta = \lambda\beta\gamma\eta$

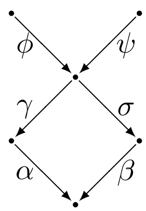
$B_1(\lambda)$



$\alpha\xi + \beta\eta + \gamma\omega = 0$

$\alpha\xi + \lambda\beta\eta + \sigma\zeta = 0$

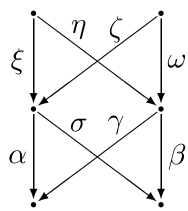
$B_2(\lambda)$



$\alpha\gamma\phi = \beta\sigma\phi$

$\alpha\gamma\psi = \lambda\beta\sigma\psi$

$B_3(\lambda)$



$\alpha\xi = \gamma\eta, \quad \alpha\zeta = \gamma\omega$

$\sigma\xi = \beta\eta, \quad \sigma\zeta = \lambda\beta\omega$

$B_4(\lambda)$

where $\lambda \in K^0 = K \setminus \{0,1\}$.

Theorem 4.1. *Let B be a tubular algebra of type $(2,2,2,2)$. Then*:

(a) *$\widehat{B}$ is isomorphic to one of the repetitive algebras $\widehat{B_1(\lambda)}$, $\widehat{B_2(\lambda)}$, $\widehat{B_3(\lambda)}$, $\widehat{B_4(\lambda)}$ or $\lambda \in K^0$.*

(b) *$\widehat{B}$ has a nontrivial rigid automorphism if and only if $\widehat{B}$ is isomorphic to one of the algebras*
  - (i) *$\widehat{B_2(\lambda)}$, $\widehat{B_3(\lambda)}$ or $\widehat{B_4(\lambda)}$, $\lambda \in K^0$,*
  - (ii) *$\widehat{B_1(-1)}$, if $\operatorname{char} K \neq 2$,*
  - (iii) *$\widehat{B_1(-\varepsilon)}$, where ε is a primitive 3-root of 1, if $\operatorname{char} K \neq 3$.*

(c) *B is exceptional if and only if B is isomorphic to $B_3(\lambda)$, $B_3(\lambda)^{\mathrm{op}}$ or $B_4(\lambda)$, $\lambda \in K^0$.*

Proof. The statement (a) follows from [30, (3.3)] where all 10 one-parameter families of tubular algebras of type $(2,2,2,2)$, and their reflection sequences, are presented. The statement (b) is proved by a direct checking. ∎

We also mention that for a fixed $i \in \{1,\dots,4\}$ and any $\lambda \in K^0$ there are only finitely many $\mu \in K^0$ such that $B_i(\lambda) \cong B_i(\mu)$ (see [18]).

Then we obtain the following complete classification (see [30, Theorem 1.5]) of all selfinjective algebras of tubular type $(2,2,2,2)$.

THEOREM 4.2. *Let A be a selfinjective algebra. Then A is of tubular type $(2,2,2,2)$ if and only if A is isomorphic to one of the algebras*:

(a) $\widehat{B}/(\nu_{\widehat{B}}^m)$, *where B is one of the tubular algebras $\widehat{B_1(\lambda)}$, $\widehat{B_2(\lambda)}$, $\widehat{B_3(\lambda)}$ or $\widehat{B_4(\lambda)}$, $\lambda \in K^0$, and m is a positive integer.*

(b) $\widehat{B}/(\varrho\,\nu_{\widehat{B}}^m)$, *where B is one of the algebras $B_2(\lambda)$, $B_3(\lambda)$ or $B_4(\lambda)$, $\lambda \in K^0$, ϱ is a rigid automorphism of $\widehat{B}$ induced by the corresponding automorphism of B of order 2, and m is a positive integer.*

(c) $\widehat{B}/(\varrho\,\nu_{\widehat{B}}^m)$, *for* $\operatorname{char} K \neq 2$, *where $B = B_1(-1)$, ϱ is a rigid automorphism of $\widehat{B}$ induced by the corresponding automorphism of B of order 2, and m is a positive integer.*

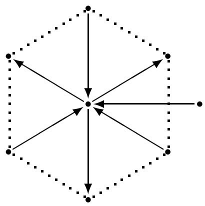
(d) $\widehat{B}/(\varrho\,\nu_{\widehat{B}}^m)$, *for* $\operatorname{char} K \neq 3$, *where B is one of the algebras $B_1(-\varepsilon)$, $B_2(-\varepsilon)$, ε is a primitive 3-root of 1, ϱ is a rigid automorphism of $\widehat{B}$ induced by the corresponding automorphism of B of order 3, and m is a positive integer.*

(e) $\widehat{B}/(\varrho\,\nu_{\widehat{B}}^m)$, *for* $\operatorname{char} K \neq 2$, *where B is one of the algebras $B_2(-1)$ or $B_4(-1)$, ϱ is a rigid automorphism of $\widehat{B}$ induced by the corresponding automorphism of B of order 4, and m is a positive integer.*

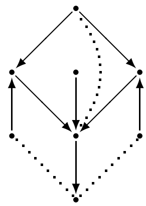
(f) $\widehat{B}/(\varphi^m)$ *or* $\widehat{B}/(\varrho\,\varphi^m)$, *where $B = B_3(\lambda)$ for some $\lambda \in K^0$, φ is a 2-root of $\nu_{\widehat{B}}$, ϱ is a rigid automorphism of $\widehat{B}$ induced by the corresponding automorphism of B of order 2, and m is an odd natural number.*

(g) $\widehat{B}/(\varphi^m)$ *or* $\widehat{B}/(\varrho\,\varphi^m)$, *where $B = B_4(\lambda)$ for some $\lambda \in K^0$, φ is a 3-root of $\nu_{\widehat{B}}$, ϱ is a rigid automorphism of $\widehat{B}$ induced by the corresponding automorphism of B of order 2 or (for* $\operatorname{char} K \neq 2$ *and $\lambda = -1$) of order 4, and m is a natural number not divisible by 3.*

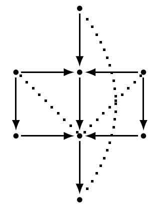
5. Selfinjective algebras of type $(3,3,3)$**.** We give a complete description of all selfinjective algebras of tubular type $(3,3,3)$. Consider the following family of bound quiver algebras (where a dotted line means that the sum of the paths indicated by it is zero):



B_1



B_2



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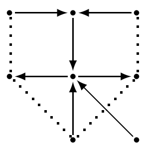
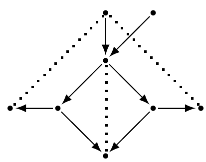
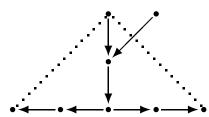


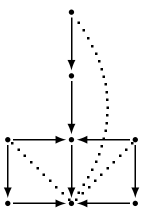
B_4



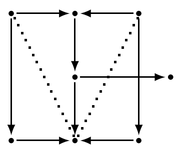
B_5



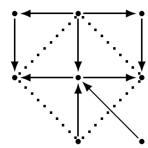
B_6



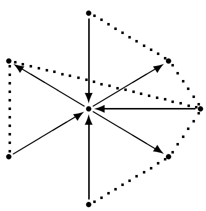
B_7



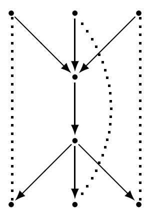
B_8



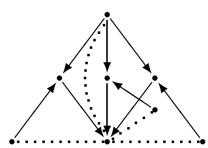
B_9



B_{10}



B_{11}



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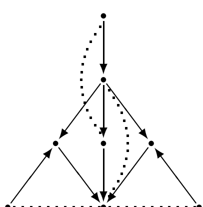
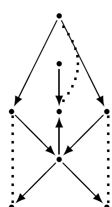
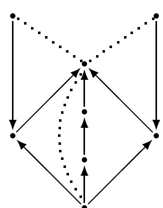


B_{13}



B_{14}



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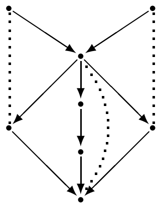
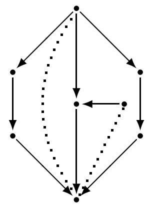
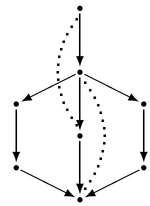


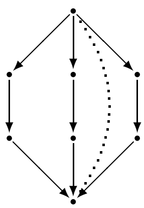
B_{16}



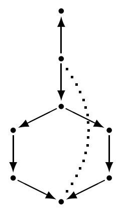
B_{17}



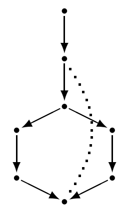
B_{18}



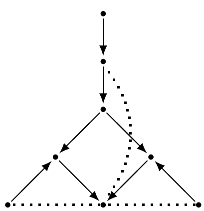
B_{19}



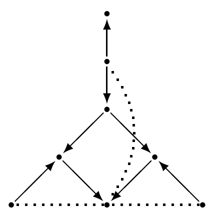
B_{20}



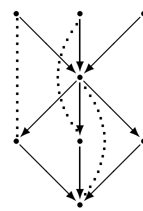
B_{21}



B_{22}



B_{23}



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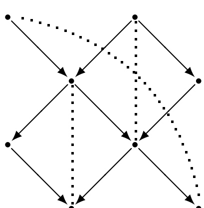
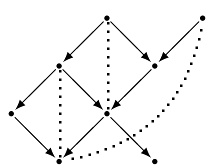
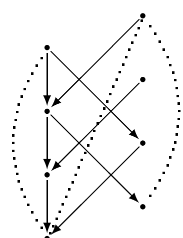


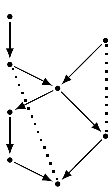
B_{25}



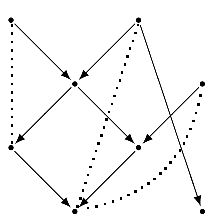
B_{26}



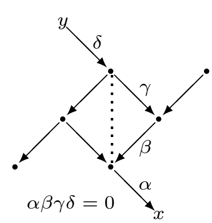
B_{27}



B_{28}



B_{29}



B'_{26}

We note that the algebras $B_1, \ldots, B_{29}$ are pairwise nonisomorphic, $S_y^- B'_{26} \cong B_{26}$ and $S_x^+ B'_{26} \cong B_{26}^{\mathrm{op}}$.

Then we have the following theorem.

THEOREM 5.1. (a) $B_1, \ldots, B_{29}, B_1^{\mathrm{op}}, \ldots, B_9^{\mathrm{op}}$ *are tubular algebras of type* $(3, 3, 3)$.

(b) *The repetitive algebras* $\widehat{B}_1, \ldots, \widehat{B}_{24}, \widehat{B}_1^{\mathrm{op}}, \ldots, \widehat{B}_9^{\mathrm{op}}$ *form a complete family of pairwise nonisomorphic repetitive algebras of tubular type* $(3, 3, 3)$ *having a nontrivial rigid automorphism.*

(c) B_{24}, B_{24}^{op}, B_{25}, B_{26}, B'_{26}, B_{26}^{op}, B_{27}, B_{27}^{op}, B_{28}, B_{28}^{op}, B_{29}, *and* B_{29}^{op} *are* (*up to isomorphism*) *the only exceptional tubular algebras of type* $(3, 3, 3)$.

Proof. This is done with the help of a computer program calculating:

- all tubular algebras of type $(3, 3, 3)$, using the Bongartz–Happel–Vossieck list [4], [19] of tame concealed algebras and tubular extensions of such algebras in the sense of [29],
- the reflection equivalence classes of tubular algebras of type $(3, 3, 3)$,
- exceptional tubular algebras of type $(3, 3, 3)$,
- nontrivial rigid automorphisms of repetitive algebras from pairwise nonequivalent reflection classes of tubular algebras of type $(3, 3, 3)$.

For details concerning these calculations we refer to the home page of the first named author (http://www.mat.uni.torun.pl/~jb/en/research/tubular/). ■

We note that there are 49 (pairwise nonisomorphic) reflection sequences of algebras of tubular type $(3, 3, 3)$; 33 of them have a nontrivial rigid automorphism, 6 are exceptional (contain exceptional algebras), and one has both these properties.

Then we obtain the following complete classification (see [30, Theorem 1.5]) of all selfinjective algebras of tubular type $(3, 3, 3)$.

THEOREM 5.2. *Let A be a selfinjective algebra. Then A is of tubular type* $(3, 3, 3)$ *if and only if A is isomorphic to one of the algebras*:

(a) $\widehat{B}/(\nu_{\widehat{B}}^m)$, *where B is a tubular algebra of type* $(3, 3, 3)$ *and m is a positive integer.*

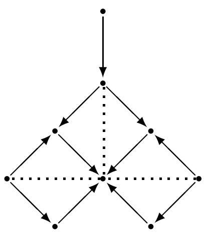
(b) $\widehat{B}/(\varrho\nu_{\widehat{B}}^m)$, *where B is one of the algebras* $B_1, \ldots, B_{24}, B_1^{\mathrm{op}}, \ldots, B_9^{\mathrm{op}}$, *$\varrho$ is a rigid automorphism of $\widehat{B}$ induced by the corresponding automorphism of B of order* 2 *or* 3, *and m is a positive integer.*

(c) $\widehat{B}/(\varphi^m)$, *where B is one of the algebras* $B_{25}, \ldots, B_{29}$, *φ is the* 2*-root of $\nu_{\widehat{B}}$, and m is an odd natural number.*

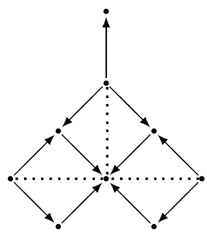
(d) $\widehat{B}/(\varphi^m)$, *where B is one of the algebras* B_{25}, B_{27}, *φ is the* 4*-root of $\nu_{\widehat{B}}$, and m is an odd natural number.*

(e) $\widehat{B}/(\varphi^m)$ *or* $\widehat{B}/(\varrho\varphi^m)$, *where* $B = B_{24}$, φ *is a* 2*-root of* $\nu_{\widehat{B}}$, ϱ *is a rigid automorphism of* $\widehat{B}$ *induced by the corresponding automorphism of* B *of order* 2 *or* 3, *and* m *is an odd natural number.*

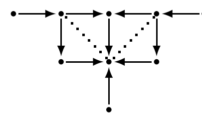
6. Selfinjective algebras of type $(2, 4, 4)$**.** We give a complete description of all selfinjective algebras of tubular type $(2, 4, 4)$. Consider the following family of bound quiver algebras:



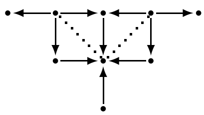
B_1



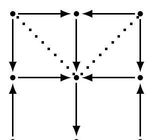
B_2



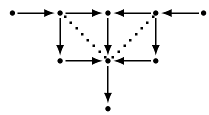
B_3



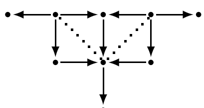
B_4



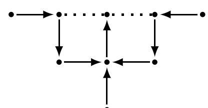
B_5



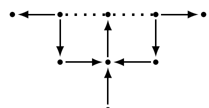
B_6



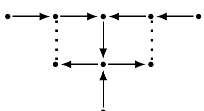
B_7



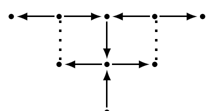
B_8



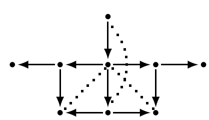
B_9



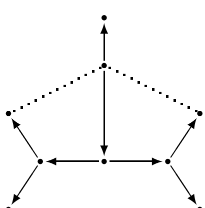
B_{10}



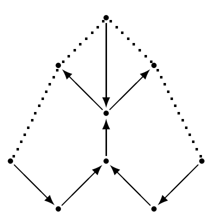
B_{11}



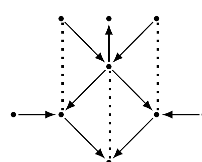
B_{12}



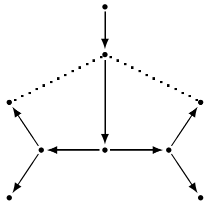
B_{13}



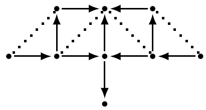
B_{14}



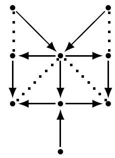
B_{15}



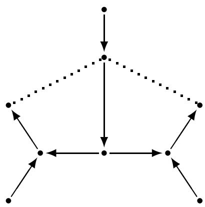
B_{16}



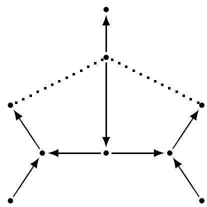
B_{17}



B_{18}



B_{19}



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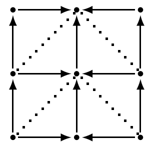

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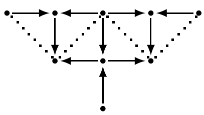

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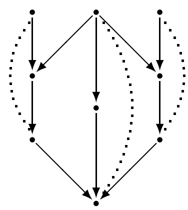

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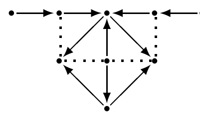
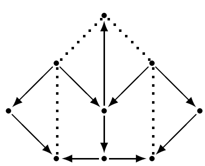


B_{24}



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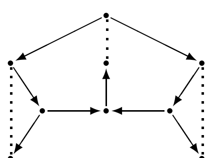
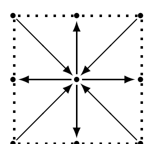


B_{26}



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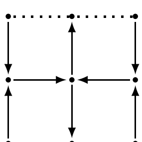
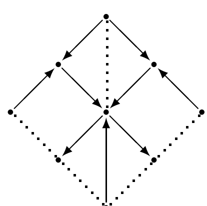


B_{28}



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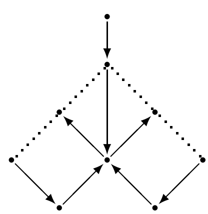
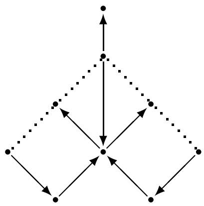


B_{30}



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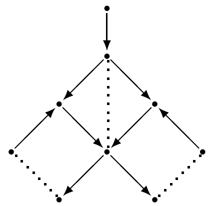
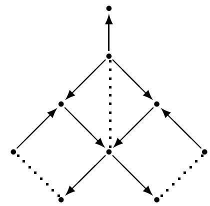


B_{32}



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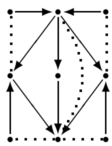
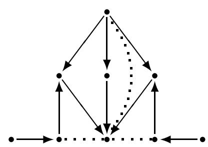


B_{34}



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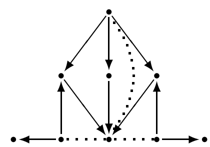

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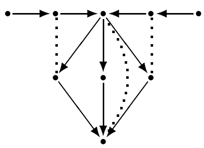

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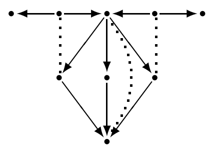

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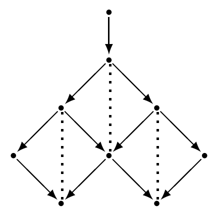

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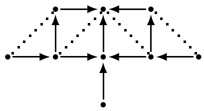

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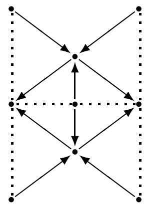
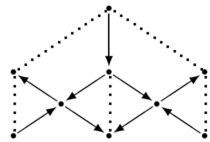
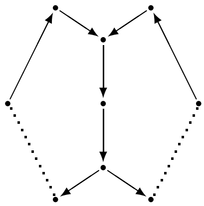


B_{41}



B_{42}



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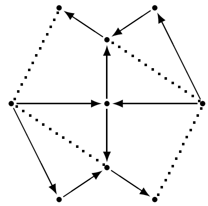

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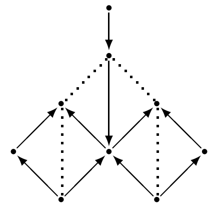
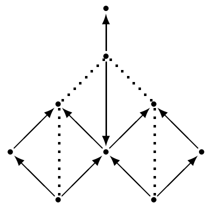


B_{45}



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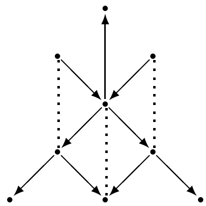

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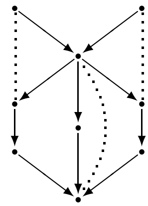
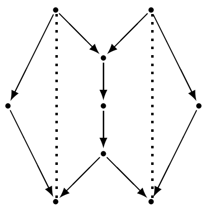
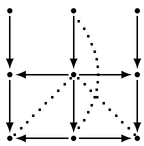


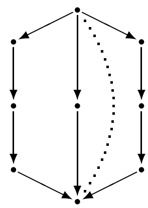
B_{48}



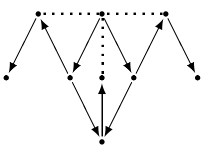
B_{49}



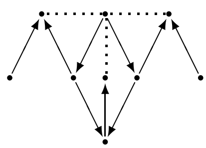
B_{50}



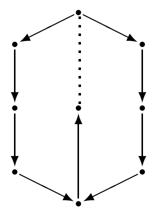
B_{51}



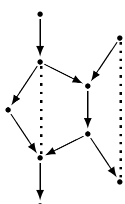
B_{52}



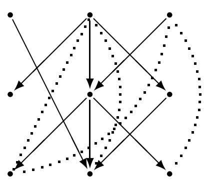
B_{53}



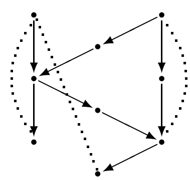
B_{54}



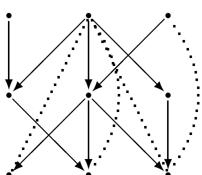
B_{55}



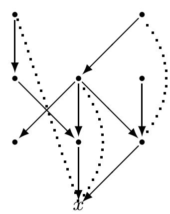
B_{56}



B_{57}



B_{58}



B'_{58}

We note that the algebras $B_1, \ldots, B_{58}$ are pairwise nonisomorphic and $S_x^+ B'_{58} \cong B_{58}$.

Then we have the following theorem.

THEOREM 6.1. (a) $B_1, \dots, B_{58}, B_1^{\mathrm{op}}, \dots, B_{26}^{\mathrm{op}}, B_{58}^{\mathrm{op}}$ *are tubular algebras of type* $(2,4,4)$.

(b) *The repetitive algebras* $\widehat{B}_1, \dots, \widehat{B}_{54}, \widehat{B}_1^{\mathrm{op}}, \dots, \widehat{B}_{26}^{\mathrm{op}}$ *form a complete family of pairwise nonisomorphic repetitive algebras of tubular type* $(2,4,4)$ *having a nontrivial rigid automorphism.*

(c) B_{55}, B_{56}, B_{56}^{op}, B_{57}, B_{57}^{op}, B_{58}, B'_{58}, B_{58}^{op}, $B_{58}^{\prime\mathrm{op}}$ *are* (*up to isomorphism*) *the only exceptional tubular algebras of type* $(2,4,4)$.

Proof. This is done with the help of a computer program calculating:

- all tubular algebras of type $(2,4,4)$, using the Bongartz–Happel–Vossieck list [4], [19] of tame concealed algebras and tubular extensions of such algebras in the sense of [29],
- the reflection equivalence classes of tubular algebras of type $(2,4,4)$,
- exceptional tubular algebras of type $(2,4,4)$,
- nontrivial rigid automorphisms of repetitive algebras from pairwise nonequivalent reflection classes of tubular algebras of type $(2,4,4)$.

For details, we again refer to the home page of the first named author. ∎

We note that there are 454 reflection sequences of algebras of tubular type $(2,4,4)$; 80 of them have a nontrivial rigid automorphism, 5 are exceptional, and none has both these properties.

Then we obtain the following complete classification (see [30, Theorem 1.5]) of all selfinjective algebras of tubular type $(2,4,4)$.

THEOREM 6.2. *Let* A *be a selfinjective algebra. Then* A *is of tubular type* $(2,4,4)$ *if and only if* A *is isomorphic to one of the algebras*:

(a) $\widehat{B}/(\nu_{\widehat{B}}^m)$, *where* B *is a tubular algebra of type* $(2,4,4)$ *and* m *is a positive integer.*

(b) $\widehat{B}/(\varrho\nu_{\widehat{B}}^m)$, *where* B *is one of the algebras* $B_1, \dots, B_{54}, B_1^{\mathrm{op}}, \dots, B_{26}^{\mathrm{op}}$, ϱ *is a rigid automorphism of* $\widehat{B}$ *induced by the corresponding automorphism of* B *of order* 2, *and* m *is a positive integer.*

(c) $\widehat{B}/(\varrho\nu_{\widehat{B}}^m)$, *where* $B = B_{27}$, ϱ *is a rigid automorphism of* $\widehat{B}$ *induced by the corresponding automorphism of* B *of order* 4, *and* m *is a positive integer.*

(d) $\widehat{B}/(\varphi^m)$, *where* B *is one of the algebras* B_{55}, B_{56}, B_{57}, B_{58}, B_{58}^{op}, φ *is the* 3*-root of* $\nu_{\widehat{B}}$, *and* m *is a natural number not divisible by* 3.

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Faculty of Mathematics and Computer Science
Nicholas Copernicus University
Chopina 12/18
87-100 Toruń, Poland
E-mail: jb@mat.uni.torun.pl
skowron@mat.uni.torun.pl

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