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THE YOUNG INEQUALITY AND THE Δ_2-CONDITION

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Abstract. If $\varphi : [0,\infty) \to [0,\infty)$ is a convex function with $\varphi(0) = 0$ and conjugate function φ^*, the inequality $xy \le \varepsilon\varphi(x)+C_\varepsilon\, \varphi^*(y)$ is shown to hold true for every $\varepsilon \in (0,\infty)$ if and only if φ^* satisfies the Δ_2-condition.

For a given $p \in (1,\infty)$ the Young inequality asserts that

$$xy \le \frac{x^p}{p} + \frac{y^{p'}}{p'}, \qquad (x,y) \in [0,\infty) \times [0,\infty), \tag{1}$$

where $p' := p/(p-1)$ is the conjugate exponent of p. A straightforward consequence of (1), sometimes called the "Young inequality with ε", plays an important role in the analysis of non-linear partial differential equations and reads: for each $\varepsilon \in (0,\infty)$,

$$xy \le \varepsilon\, \frac{x^p}{p} + \varepsilon^{-1/(p-1)}\, \frac{y^{p'}}{p'}, \qquad (x,y) \in [0,\infty) \times [0,\infty). \tag{2}$$

The Young inequality (1) is actually valid for a larger class of convex functions. Indeed, let $\varphi : [0,\infty)\to[0,\infty)$ be a convex function with $\varphi(0)=0$. Introducing the conjugate function φ^* of φ defined by

$$\varphi^*(x) := \sup_{y\in[0,\infty)} \{xy - \varphi(y)\}, \qquad x \in [0,\infty),$$

which is also a non-negative convex function, we have the Young inequality

$$xy \le \varphi(x) + \varphi^*(y), \qquad (x,y) \in [0,\infty) \times [0,\infty). \tag{3}$$

A natural question then is whether an inequality similar to (2) holds true for φ. As we shall see below, the answer is negative in general and it turns out that the availability of such an inequality is equivalent to the fact that φ^* satisfies the Δ_2-condition (a definition of the Δ_2-condition, together with its applications to the theory of Orlicz spaces, may be found in, e.g., [1, 2] and is recalled in assertion (i) of Theorem 1 below). More precisely we have the following result.

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Theorem 1. *Let* $\varphi : [0,\infty) \to [0,\infty)$ *be a convex function with* $\varphi(0)=0$ *and such that* $\varphi^*(x) < \infty$ *for* $x \in [0,\infty)$. *The following assertions are equivalent*:

(i) φ^* *satisfies the* Δ_2-*condition*, *that is*, *there is* $\lambda > 0$ *such that* $\varphi^*(2x) \le \lambda\varphi^*(x)$ *for every* $x \in [0,\infty)$,

(ii) *there exist* $\varepsilon \in (0,1)$ *and* $C_\varepsilon > 0$ *such that*

$$xy \le \varepsilon\varphi(x) + C_\varepsilon\varphi^*(y), \qquad (x,y) \in [0,\infty)\times[0,\infty), \tag{4}$$

(iii) *for each* $\varepsilon \in (0,\infty)$ *there is* $C_\varepsilon > 0$ *such that*

$$xy \le \varepsilon\varphi(x) + C_\varepsilon\varphi^*(y), \qquad (x,y) \in [0,\infty)\times[0,\infty).$$

Clearly the power function $\varphi(x) = x^p/p$ satisfies the first condition of Theorem 1 for $p \in (1,\infty)$. The function $\varphi(x) = e^x - x - 1$ also satisfies condition (i) since $\varphi^*(x) = (1+x)\ln(1+x) - x$ satisfies the Δ_2-condition with $\lambda = 4$. On the other hand, Theorem 1 does not apply to the function

$$\varphi(x) = (1+x)\ln(1+x) - x,$$

since its conjugate $\varphi^*(x) = e^x - x - 1$ does not satisfy the Δ_2-condition [2].

Remark 2. Owing to the Young inequality (3), the inequality (4) always holds true for $\varepsilon \ge 1$. Indeed, if $\varepsilon \ge 1$ we have

$$xy \le \varepsilon\varphi(x) + \varphi^*(y), \qquad (x,y) \in [0,\infty)\times[0,\infty),$$

by (3), whence (4) with $C_\varepsilon = 1$.

Proof of Theorem 1. For $\varepsilon \in (0,\infty)$ and $x \in [0,\infty)$ we put $\varphi_\varepsilon(x) = \varepsilon\varphi(x)$ and notice that the conjugate function φ_ε^* of φ_ε is given by

$$\varphi_\varepsilon^*(x) = \varepsilon\varphi^*(x/\varepsilon), \qquad x \in [0,\infty). \tag{5}$$

(i)⇒(iii). Since (4) always holds true for $\varepsilon \ge 1$ (recall Remark 2) we assume that $\varepsilon \in (0,1)$. We infer from (5) and the Young inequality for φ_ε that

$$xy \le \varepsilon\varphi(x) + \varepsilon\varphi^*(y/\varepsilon), \qquad (x,y) \in [0,\infty)\times[0,\infty).$$

Since φ^* satisfies the Δ_2-condition and $\varepsilon \in (0,1)$ it follows from [1, p. 23] that there is a constant $C'_\varepsilon > 0$ such that

$$\varphi^*(y/\varepsilon) \le C'_\varepsilon\varphi^*(y), \qquad y \in [0,\infty).$$

Consequently,

$$xy \le \varepsilon\varphi(x) + \varepsilon C'_\varepsilon\varphi^*(y), \qquad (x,y) \in [0,\infty)\times[0,\infty),$$

whence (iii).

(ii)⇒(i). Consider $y \in [0,\infty)$. For $x \in [0,\infty)$ we infer from (ii) that

$$xy \le \varepsilon\varphi(x) + C_\varepsilon\varphi^*(y), \qquad \frac{xy}{\varepsilon} - \varphi(x) \le \frac{C_\varepsilon}{\varepsilon}\,\varphi^*(y),$$

from which we readily deduce that

$$\varphi^*(y/\varepsilon) \leq \frac{C_\varepsilon}{\varepsilon}\,\varphi^*(y).$$

Since $1/\varepsilon > 1$ we infer from [1, p. 23] that the above inequality guarantees that φ^* satisfies the Δ_2-condition. ∎

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