HEREDITARILY INDECOMPOSABLE CONTINUA WITH EXACTLY \( n \) AUTOHOMEOMORPHISMS

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Abstract. The main goal of this paper is to construct, for every \( n, m \in \mathbb{N} \), a hereditarily indecomposable continuum \( X_{nm} \) of dimension \( m \) which has exactly \( n \) autohomeomorphisms.

1. Introduction. All spaces considered are assumed to be metrizable separable. Our terminology follows [6] and [10]. A continuum \( X \) is hereditarily indecomposable, abbreviated HI, if for any two intersecting subcontinua \( K, L \) of \( X \), either \( K \subset L \) or \( L \subset K \). For a continuum \( X \), let \( \mathcal{G}(X) \) denotes the group of all homeomorphisms of \( X \) onto \( X \). A continuum \( X \) is rigid if the identity \( 1_X \) is the only homeomorphism of \( X \) onto \( X \), i.e., \( \mathcal{G}(X) = \{1_X\} \). In [5] H. Cook gave an example of a rigid, 1-dimensional, HI continuum. Recently M. Reńska [18] constructed, for every \( m \in \mathbb{N} \), an HI rigid \( m \)-dimensional Cantor manifold. The main goal of this paper is to prove the following theorem.

1.1. THEOREM. For every \( n \in \mathbb{N} \) and \( m \in \mathbb{N} \cup \{\infty\} \) there exists an HI continuum \( X_{nm} \) such that \( \dim X_{nm} = m \) and the group \( \mathcal{G}(X_{nm}) \) of homeomorphisms of \( X_{nm} \) onto \( X_{nm} \) is a cyclic group of order \( n \).

A homeomorphism \( h : X \to X \) is stable if there exist homeomorphisms \( h_0, h_1, \ldots, h_n \) such that \( h = h_nh_{n-1} \ldots h_1h_0 \) and for every \( i \leq n \) there exists a nonempty open set \( U_i \) such that \( h_i|U_i \) is the identity. The continua \( X_{nm} \) constructed in Theorem 1.1 have the property that the set of stable homeomorphisms of \( X_{nm} \) onto \( X_{nm} \) is degenerate and is not dense in the space \( \mathcal{G}(X_{nm}) \) for \( n > 1 \). The next theorem shows that there exist HI continua with \( 2^{\aleph_0} \) homeomorphisms, each of which is stable (moreover, it is the identity on some open nonempty subset).

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1.2. Theorem. For every \( m \in \mathbb{N} \cup \{\infty\} \) there exists an HI continuum \( Y_m \) with \( \dim Y_m = m \) such that the group \( \mathcal{G}(Y_m) \) of homeomorphisms of \( Y_m \) onto \( Y_m \) has cardinality \( 2^{\aleph_0} \) and there exists a nonempty open subset \( U_m \) of \( Y_m \) such that for every \( h \in \mathcal{G}(Y_m) \), \( h|U_m = 1_{U_m} \).

A space \( X \) is strongly infinite-dimensional (abbreviated SID) if there exists an infinite sequence \( (A_1, B_1), (A_2, B_2), \ldots \) of pairs of disjoint closed subsets of \( X \) such that if \( L_i \) is a partition between \( A_i \) and \( B_i \) in \( X \) for \( i = 1, 2, \ldots \) then \( \bigcap_{i=1}^{\infty} L_i \neq \emptyset \). An SID space \( X \) is hereditarily SID if every subset of \( X \) of positive dimension is SID. An infinite-dimensional continuum \( X \) is a Cantor manifold if all closed sets which disconnect \( X \) are infinite-dimensional. The first hereditarily SID compactum was constructed by Rubin \[19\] (cf. [6, Problem 6.1.G]); while the first example of an SID compactum all of whose nontrivial subcontinua are infinite-dimensional was given earlier by Henderson \[7\]. In \[18\] M. Reńska constructed a rigid HI hereditarily SID Cantor manifold. We will prove the following theorem.

1.3. Theorem. For every \( n \in \mathbb{N} \) there exists an HI continuum \( Z_n \), all of whose nontrivial subcontinua are strongly infinite-dimensional, such that the group \( \mathcal{G}(Z_n) \) of homeomorphisms of \( Z_n \) onto \( Z_n \) is a cyclic group of order \( n \).

In our constructions we apply some ideas of [3], [18] and [16].

It is an interesting question whether the spaces \( X_{nm} \) and \( Y_m \) satisfying the conditions of Theorems 1.1 and 1.2 can be \( m \)-dimensional Cantor manifolds (for \( m > 1 \)) and whether the spaces \( Z_n \) from Theorem 1.3 can be infinite-dimensional Cantor manifolds.

2. Preliminaries. The first HI continuum, now called the pseudo-arc, was constructed by B. Knaster \[9\] in 1922. The pseudo-arc, which will be denoted by \( P \), is an HI one-dimensional chainable continuum (unique, up to homeomorphism); and it is the only (up to homeomorphism) nondegenerate, homogeneous, chainable continuum. The pseudo-arc \( P \) is also hereditarily equivalent, i.e., every nontrivial subcontinuum of \( P \) is homeomorphic to \( P \) (cf. [10, §48, X], or [13]).

The first examples of HI continua of arbitrary dimension \( n \), where \( n \in \{2, 3, \ldots, \infty\} \), were constructed by R. H. Bing \[2\].

The composant of a point \( x \) in a continuum \( X \) is the union of all proper subcontinua of \( X \) containing \( x \). If \( X \) is a nontrivial HI continuum, then (see [10, §48, VI])

(a) every composant of \( X \) is a connected \( F_\sigma \)-subset of \( X \), both dense and boundary in \( X \),
(b) different composants of \( X \) are disjoint, and
(c) (Mazurkiewicz’s theorem) \( X \) has continuum many different composants.

A subcontinuum \( K \) of a continuum \( X \) is \textit{terminal} if every subcontinuum of \( X \) which intersects both \( K \) and its complement must contain \( K \). A continuous mapping from a continuum \( X \) onto \( Y \) is called \textit{atomic} if every fiber of \( f \) is a terminal subcontinuum of \( X \).

In our constructions we will apply the method of condensation of singularities, which goes back to Anderson and Choquet [1]. Namely, we will need the following construction, based on the technique of Maćkowiak [14], [15] and described in detail in [17] (cf. also [4]).

\[ \text{2.1. Theorem. Let } X \text{ be a continuum, } \{ Z_i : i \in \mathbb{N} \} \text{ a sequence of compacta, } \{ A_i : i \in \mathbb{N} \} \text{ a sequence of 0-dimensional compact disjoint subsets of } X, \text{ and suppose each } Z_i \text{ admits a continuous map onto } A_i \text{ with connected fibers. Then there exist a continuum } L(X, Z_i, A_i) \text{ and an atomic mapping } p : L(X, Z_i, A_i) \to X \text{ such that } \]
\[ \begin{align*}
\text{(i) } & p|p^{-1}(X \setminus \bigcup_{i=1}^{\infty} A_i) : p^{-1}(X \setminus \bigcup_{i=1}^{\infty} A_i) \to X \setminus \bigcup_{i=1}^{\infty} A_i \text{ is a homeomorphism,} \\
\text{(ii) } & p^{-1}(X \setminus \bigcup_{i=1}^{\infty} A_i) \text{ is dense in } L(X, Z_i, A_i), \\
\text{(iii) } & p^{-1}(A_i) \text{ is homeomorphic to } Z_i \text{ for every } i \in \mathbb{N} \text{ (hence } p^{-1}(a) \text{ is homeomorphic to a component of } Z_i \text{ if } a \in A_i), \\
\text{(iv) } & \text{if } n \text{ and } m \text{ are natural numbers such that } \dim X \leq n \text{ and } \dim Z_i \leq m \text{ for every } i \in \mathbb{N} \text{ then } \dim L(X, Z_i, A_i) \leq \max(n, m), \\
\text{(v) } & \text{if } C(x) \text{ is the composant of } x \text{ in } L(X, Z_i, A_i) \text{ then } C(x) = p^{-1}(C(p(x))), \text{ where } C(p(x)) \text{ is the composant of } p(x) \text{ in } X. 
\end{align*} \]

The existence of the space \( L(X, Z_i, A_i) \) which admits an atomic mapping \( p : L(X, Z_i, A_i) \to X \) with properties (i)–(iv) follows from [17, Theorem 3.2] and property (v) follows from the atomicity of \( p \) (see [17, Lemma 2.8]).

We will also need the following auxiliary facts.

\[ \text{2.2. Lemma. For every } m \in \mathbb{N} \text{ there exists an infinite family of pairwise nonhomeomorphic HI } m\text{-dimensional Cantor manifolds.} \]

For \( m = 1 \) such a family of cardinality \( 2^{\aleph_0} \) was constructed by R. H. Bing [3]. For \( m = 2, 3, \ldots \), the existence of such a family follows, for example, from the following lemma proved by M. Reńka in [18]: for every \( m\)–dimensional HI continuum \( K \) there exists an HI \( m\)–dimensional Cantor manifold \( M \) such that \( K \) is not embeddable into \( M \). Indeed, let \( K \) and \( M \) be two \( m\)–dimensional Cantor manifolds such that \( K \) does not embed in \( M \) and let \( a_1, a_2, \ldots \) be points of \( M \) such that \( a_k \) and \( a_l \) belong to different composants of \( M \) if \( k \neq l \). Then \( K_j = L(M, Z_i, A_i) \), where \( Z_i = K \) and \( A_i = \{ a_i \} \) for \( i = 1, \ldots, j \) and \( Z_i = \emptyset = A_i \) for \( i > j \), is an \( m\)–dimensional Cantor manifold exactly \( j \) of whose composants do not embed in \( M \) (see Theorem 2.1).
Thus $K_j$ is not homeomorphic to $K_l$ for $j \neq l$. Let us add that, as proved in [18], there also exists a family of cardinality $2^{n_0}$ consisting of topologically different HI rigid $m$-dimensional Cantor manifolds.

2.3. Lemma (H. Cook [5]). There exists a one-dimensional HI continuum no two of whose nondegenerate subcontinua are homeomorphic.

2.4. Lemma. There exists an infinite family of topologically different HI hereditarily SID Cantor manifolds.

The existence of such a family follows from Corollary 4.3 of [16] stating that for every hereditarily SID compactum $K$ there exists an HI hereditarily SID Cantor manifold which does not embed in $K$. Moreover, as proved in [16], there exists such a family of cardinality $2^{n_0}$.

2.5. Lemma (W. Lewis [12]). For every $n \in \mathbb{N}$ there exists a homeomorphism $r$ of the pseudo-arc of period $n$. Moreover, for each $n \in \mathbb{N}$ there exists an embedding of the pseudo-arc in the plane such that $r$ is the restriction of a period $n$ rotation of the plane.

2.6. Lemma. Let $U$ be an open subset of the pseudo-arc $P$ such that $P \setminus \overline{U} \neq \emptyset$. Then there exists a family $\{h_t : t \in T\}$ of homeomorphisms of $P$ onto $P$, where $|T| = 2^{n_0}$, such that $h_{t'} \neq h_t$ if $t' \neq t$ and $h_t|U = 1_U$ for every $t$.

This lemma follows immediately from Theorem 8 in [11], stating that if $p$ and $q$ are distinct points of $P \setminus U$, where $U$ is open in $P$, such that the subcontinuum $M$ irreducible between $p$ and $q$ does not intersect $\text{cl}(U)$, then there is a homeomorphism $h : P \rightarrow P$ with $h(p) = q$ and $h|U = 1_U$ (cf. also [8, Theorem]).

2.7. Lemma. Let $p : X \rightarrow Y$ and $\tilde{p} : \tilde{X} \rightarrow Y$ be mappings between continua such that $p$ is atomic and for every $y \in Y$ with $\tilde{p}^{-1}(y)$ nondegenerate there exists an open neighborhood $U$ of $y$ in $Y$, a homeomorphism $h$ of $\tilde{p}^{-1}(\overline{U})$ onto a subset of $X$ and a homeomorphism $g$ of $ph(\tilde{p}^{-1}(\overline{U}))$ onto $\overline{U}$ such that $\tilde{p}(x) = gph(x)$ for every $x \in \tilde{X}$. Then $\tilde{p}$ is atomic.

Proof. Take $y \in Y$ such that $\tilde{p}^{-1}(y)$ is nondegenerate and let $U$, $h$ and $g$ be as above. Let $L$ be any continuum in $\tilde{X}$ such that $L \cap \tilde{p}^{-1}(y) \neq \emptyset \neq L \setminus \tilde{p}^{-1}(y)$. We will show that $L \supset \tilde{p}^{-1}(y)$.

(a) First consider the case when $L \subset \tilde{p}^{-1}(\overline{U})$. Then $h(L)$ is a continuum in $h(\tilde{p}^{-1}(\overline{U}))$ such that $h(L) \cap (g \circ p)^{-1}(y) \neq \emptyset \neq h(L) \setminus (g \circ p)^{-1}(y)$. Since $g \circ p$ is atomic as the composition of an atomic map and a homeomorphism, we have $h(L) \supset (g \circ p)^{-1}(y)$. Thus $L \supset h^{-1}p^{-1}g^{-1}(y) = \tilde{p}^{-1}(y)$.

(b) Suppose now that $L \not\subset \tilde{p}^{-1}(\overline{U})$. Then $L \cap \tilde{p}^{-1}(\overline{U})$ intersects the boundary of $\tilde{p}^{-1}(\overline{U})$ in $\tilde{X}$. Let $y_0 \in L \cap \tilde{p}^{-1}(y)$. Then the component $K$ of
Let \( L \cap \tilde{p}^{-1}(\overline{U}) \) containing \( y_0 \) intersects the boundary of \( L \cap \tilde{p}^{-1}(\overline{U}) \) in \( L \) (by Janiszewski’s theorem, see [10, §47, III]). Thus \( K \setminus \tilde{p}^{-1}(y) \neq \emptyset \neq K \setminus \tilde{p}^{-1}(y) \) and \( K \subset \tilde{p}^{-1}(\overline{U}) \), hence, by case (a), \( K \supset K \), that finishes the proof.

3. The proofs

Proof of Theorem 1.1. Fix \( n, m \in \mathbb{N} \). By Lemma 2.5, there exist a pseudo-arc \( P \subset \mathbb{R}^2 \) and a homeomorphism \( r : P \to P \) of period \( n \), which is the restriction of a period \( n \) rotation of the plane around \((0, 0)\) (so the point \((0, 0) \in P \) is a fixed point of \( r \)). Let \( V_0 = \{(x_1, x_2) \in \mathbb{R}^2 : x_1 = \lambda \cos \alpha \) and \( x_2 = \lambda \sin \alpha \) for some \( 0 < \lambda < \infty \) and \( 0 < \alpha < 2\pi/n \}\), \( P_0 = P \cap V_0 \) and \( P_k = r^k(P_0) \) for \( k = 0, 1, \ldots, n - 1 \). Then every \( P_k \) is open in \( P \) and \( P \setminus \bigcup_{k=0}^{n-1} P_k \) is a 0-dimensional boundary subset of \( P \).

Let \( A_0 = \{a_1, a_2, \ldots\} \) be a countable dense subset of \( P_0 \) such that \( a_i \) and \( a_j \) are in the same composant of \( P \) if and only if \( i = j \). Put \( A_k = r^k(A_0) \) for \( k = 0, 1, \ldots, n - 1 \) and let \( A = \bigcup_{k=0}^{n-1} A_k \). The set \( A \) is dense in \( P \). Since a homeomorphic image of a composant of \( P \) is a composant of \( P \), every composant of \( P \) contains at most \( n \) points of \( A \).

Consider now three cases. If \( 2 \leq m < \infty \) then let \( K_1, K_2, \ldots \) be a sequence of topologically different \( HI \) \( m \)-dimensional Cantor manifolds (see Lemma 2.2). If \( m = 1 \) then, by Cook’s Lemma 2.3, one can find a sequence \( K_1, K_2, \ldots \) of \( HI \) one-dimensional continua such that: if \( L \) and \( L' \) are two different nondegenerate subcontinua of \( K_1 \) and \( K_j \) respectively, where \( i, j \in \mathbb{N} \), then \( L \) and \( L' \) are not homeomorphic; in particular, no subcontinuum of any \( K_i \) is homeomorphic to the pseudo-arc. If \( m = \infty \) then let \( K_i \) be any \( i \)-dimensional \( HI \) Cantor manifold, for \( i = 1, 2, \ldots \).

Let \( M_{nm} = L(P, K_i, \{a_i\}) \) be an \( HI \) continuum and \( p : M_{nm} \to P \) be an atomic mapping satisfying the conditions of Theorem 2.1. We can assume additionally that \( M_{nm} \subset P \times I^\infty \), where \( I = [0, 1] \), and that \( p \) is the restriction of the projection of \( P \times I^\infty \) onto \( P \). Moreover, we can assume that \( p^{-1}(y) = (y, (0, 0, \ldots)) \) for every \( y \in P \setminus P_0 \).

Indeed, assume that \( M_{nm} \subset I^\infty \) and \( p : M_{nm} \to P \) is as in Theorem 2.1, and for \( x, y \in \mathbb{R}^2 \) let \( g(x, y) = \min(g_e(x, y), 1) \), where \( g_e \) the Euclidean metric in the plane. Put \( f(x) = (p(x), g(p(x), \mathbb{R}^2 \setminus V_0) \cdot x) \) for \( x \in M_{nm} \). Then \( f \) is continuous and one-to-one, hence it is a homeomorphism of \( M_{nm} \) onto \( f(M_{nm}) \subset P \times I^\infty \). Thus we can replace \( M_{nm} \) by \( f(M_{nm}) \) and \( p \) by the restriction of the projection of \( P \times I^\infty \) onto \( P \).

Let \( \overline{\pi}(y, t) = (r(y), t) \) for \((y, t) \in P \times I^\infty \). For \( k = 0, 1, \ldots, n - 1 \), let \( \tilde{P}_k = \pi^k(p^{-1}(\overline{P}_0)) \) and \( \tilde{X}_{nm} = \bigcup_{k=0}^{n-1} \tilde{P}_k \).

Let \( \tilde{p} : X_{nm} \to P \) be the restriction of the projection of \( P \times I^\infty \) onto \( P \). The mapping \( \tilde{r} = \pi|X_{nm} \) is a period \( n \) homeomorphism of \( X_{nm} \) onto \( X_{nm} \),
which is the restriction of the product of $r$ and the identity. Thus,

\[(1)\quad \tilde{p} \circ r^k = r^k \circ \tilde{p} \quad \text{for every } k = 1, \ldots, n - 1,\]

and

\[(2)\quad p(x) = \tilde{p}(x) \quad \text{for } x \in \tilde{P}_0.\]

The map $\tilde{p}$ is atomic by Lemma 2.7. Indeed, we apply Lemma 2.7 to $\tilde{X} = X_{nm}$, $X = M_{nm}$, $Y = P$ and let $\tilde{p} : \tilde{X}_{nm} \to P$ and $p : M_{nm} \to P$ be as above. For every $y \in P$ such that $\tilde{p}^{-1}(y)$ is nondegenerate there exists $k \in \{0, 1, \ldots, n - 1\}$ such that $y \in r^k(P_0)$. If we put $U = \tilde{p}^{-1}(r^k(P_0))$, $h(x) = (\tilde{r}^k)^{-1}(x)$ for $x \in U$ and $g = r^k$ then $\text{gph}(y) = r^k(\tilde{p}(\tilde{r}^k)^{-1}(y)) = \tilde{p}(y)$ by (1) and (2). Thus the assumptions of Lemma 2.7 are satisfied. It follows that $\tilde{p}$ is atomic.

The space $X_{nm}$ is an HI continuum, being the preimage of an HI continuum under the atomic mapping $\tilde{p}$ with HI fibers (see [15]). By Theorem 2.1(iv), $\dim M_{nm} = m$, so $\dim X_{nm} = m$ by the sum theorem.

Note that $\tilde{p}[\tilde{p}^{-1}(P \setminus A) : \tilde{p}^{-1}(P \setminus A) \to P \setminus A$ is one-to-one, so $\tilde{p}^{-1}(P \setminus A)$ is a one-dimensional set homeomorphic to $P \setminus A$.

On the other hand, if $t \in A$, then $t = r^k(a_i)$ for some $i \in \mathbb{N}$ and $k \in \{0, 1, \ldots, n - 1\}$; hence $\tilde{p}^{-1}(t)$ is homeomorphic to $K_i$. Note that by Theorem 2.1(v) every composant of $X_{nm}$ is the preimage under $\tilde{p}$ of a composant of $P$, hence it is the union of a one-dimensional subset homeomorphic to $P$ with finitely many points removed and of at most $n$ disjoint $m$-dimensional Cantor manifolds homeomorphic to some $K_i$. In particular, if $t \in A$ then $\tilde{p}^{-1}(t)$ is a maximal $m$-dimensional Cantor manifold which is a proper subset of $X_{nm}$ and homeomorphic to some $K_i$ (that is, there is no $m$-dimensional Cantor manifold contained in a certain composant, homeomorphic to $\tilde{p}^{-1}(t)$ and containing $\tilde{p}^{-1}(t)$). For $m > 1$ this follows from the fact that $\tilde{p}^{-1}(t)$ is obviously a maximal $m$-dimensional Cantor manifold which is a proper subset of $X_{nm}$. For $m = 1$, $\tilde{p}^{-1}(t)$ is a maximal proper subcontinuum of $X_{nm}$ homeomorphic to $\tilde{p}^{-1}(t)$, since no subcontinuum of $\tilde{p}^{-1}(t)$ embeds in $P$.

We will show that $1_{X_{nm}} = \tilde{r}_0, \tilde{r}, \tilde{r}^2, \ldots, \tilde{r}^{n-1}$ are the only homeomorphisms of $X_{nm}$ onto $X_{nm}$, so $\mathcal{G}(X_{nm})$ is a cyclic group of order $n$.

Let $h$ be an arbitrary homeomorphism of $X_{nm}$ onto $X_{nm}$. The image under $h$ of a composant $C$ of $X_{nm}$ is a composant of $X_{nm}$. Since $h$ must map maximal $m$-dimensional Cantor manifolds lying in $C$ and homeomorphic to $K_i$ onto maximal $m$-dimensional Cantor manifolds in $h(C)$ homeomorphic to $K_i$, and since no two different $K_i$’s are homeomorphic, we have

\[(3)\quad \text{for every } t \in A, h(\tilde{p}^{-1}(t)) = \tilde{r}^k(\tilde{p}^{-1}(t)) \quad \text{for some } k \in \{0, 1, \ldots, n - 1\}.\]

Thus the mapping $\overline{h} : P \to P$, where $\overline{h}(t) = \tilde{p}(h(\tilde{r}^{-1}(t)))$, is well defined. From the upper semicontinuity of $\tilde{p}^{-1}$ it follows that $\overline{h}$ is continuous (in fact,
it is a homeomorphism). By (3) we have

\[(4) \quad \text{for every } t \in A, \bar{h}(t) = r^k(t) \text{ for some } k \in \{0, 1, \ldots, n-1\}.\]

For every \(k \in \{0, 1, \ldots, n-1\}\) let \(D_k = \{t \in P : \bar{h}(t) = r^k(t)\}\). It is easy to see that every \(D_k\) is closed and \(D_k \cap D_l = \{(0,0)\}\) for every \(k \neq l\). By (4) the set \(\bigcup_{k=0}^{n-1} D_k\) is dense in \(P\), hence we have \(P = \bigcup_{k=0}^{n-1} D_k\). Since \(P\) is connected, every \(D_k\) is connected. From indecomposability of \(P\) we have \(P = D_{k_0}\) for some \(k_0\), so \(\bar{h} = r^{k_0}\). Since \(\bar{p}|\bar{p}^{-1}(P \setminus A)\) is one-to-one, \(h\) coincides with \(\bar{r}^{k_0}\) on \(\bar{p}^{-1}(P \setminus A)\). Since the latter set is dense in \(X_{nm}\), \(h\) coincides with \(\bar{r}^{k_0}\) on the whole space. This ends the proof.

**Proof of Theorem 1.2.** From the proof of Theorem 1.1 it follows that if \(n > 1\) then the continuum \(M_{nm}\) and the mapping \(p : M_{nm} \to P\) obtained during the construction of \(X_{nm}\) have the following properties:

(a) \(M_{nm}\) is HI and \(\dim M_{nm} = m\),

(b) if \(U_m = p^{-1}(P_0)\), then \(U_m\) is an open subset of \(M_{nm}\) such that every homeomorphism of \(M_{nm}\) onto \(M_{nm}\) is the identity on \(U_m\),

(c) if \(V_m = p^{-1}(P \setminus \bar{P}_0)\), then \(V_m\) is an open nonempty subset of \(M_{nm}\) such that \(\bar{V}_n \cup \bar{U}_n = M_{nm}\) and \(p : V_n \to P \setminus \bar{P}_0\) is a homeomorphism (recall that \(p^{-1}(y) = (y, (0,0,\ldots))\) for all \(y \in P \setminus P_0\).

Set \(Y_m = M_{nm}\), where \(n\) is any fixed natural number \(> 1\). Then \(Y_m\) satisfies the conditions of Theorem 1.2. Indeed, by (a) and (b) it suffices to show that \(Y_m\) has continuum many different autohomeomorphisms. Applying Lemma 2.6 for \(U = P_0\) we obtain a family \(\{h_t : t \in T\}\) of different homeomorphisms of \(P\) onto \(P\), where \(|T| = 2^\aleph_0\), such that \(h_t|P_0 = 1_{P_0}\). Define \(\tilde{h}_t : Y_m \to Y_m\) in the following way: if \(x \in \bar{V}_m\), then \(\tilde{h}_t(x) = p^{-1}h_t p(x)\) and if \(x \in \bar{U}_m\) then \(\tilde{h}_t(x) = x\). It is easy to see that \(p^{-1}h_t p(x) = x\) for \(x \in \bar{V}_m \cup \bar{U}_m\). It follows that \(\tilde{h}_t\) is a homeomorphism of \(Y_m\) onto \(Y_m\). Since \(p|p^{-1}(K)\) is one-to-one and \(h_{t'} \neq h_t\) for \(t' \neq t\), we have \(\tilde{h}_{t'} \neq \tilde{h}_t\) for \(t' \neq t\). This ends the proof.

**Proof of Theorem 1.3.** We use the idea and notation of the proof of Theorem 1.1. First we divide \(P_0\) into two dense 0-dimensional subsets \(P'_0\) and \(P'_1\) such that \(P'_0\) is the union of countably many disjoint sets \(F_1, F_2, \ldots\) closed in \(P\). Then we choose a countable dense subset \(A_0 = \{a_1, a_2, \ldots\}\) of \(P'_1\). Now, let \(B_1, B_2, \ldots\) be a sequence of closed disjoint 0-dimensional subsets of \(P\) defined by \(B_{2i-1} = F_i\) and \(B_{2i} = \{a_i\}\) for \(i = 1, 2, \ldots\). Let \(K_1, K_2, \ldots\) be a sequence of topologically different HI hereditarily SID Cantor manifolds (see Lemma 2.4).

Let \(M = L(P, K_i \times B_i, B_i)\) be an HI continuum and \(p : M \to P\) be an atomic mapping satisfying the conditions of Theorem 2.1. As in the proof of Theorem 1.1, we can assume additionally that \(M \subset P \times I^\infty\), \(p\) is the
restriction of the projection of $P \times I^{\infty}$ onto $P$ and $p^{-1}(y) = (y, (0, 0, \ldots))$
for every $y \in P \setminus P_0$.

Let $\bar{r}$ be the product of $r$ and the identity. For $k = 0, 1, \ldots, n - 1$, put \( \tilde{P}_k = \bar{r}^k(p^{-1}(P_0))\) and \( Z_n = \bigcup_{k=0}^{n-1} \tilde{P}_k \).

Let $\tilde{p} : Z_n \to P$ be the restriction of the projection of $P \times I^{\infty}$ onto $P$.

The map $\tilde{p}$ is atomic by Lemma 2.7. The mapping $\tilde{r} = \bar{r}|Z_m$ is a period $n$
homeomorphism of $Z_n$ onto $Z_n$, which is the restriction of the product of $r$ and the identity. Since $\tilde{p}$ is an atomic mapping with HI fibers onto an HI
continuum, $Z_n$ is HI.

To prove that all nontrivial subcontinua of $Z_n$ are SID, take any non-
trivial continuum $L$ contained in $Z_n$. Note that $B = \bigcup_{k=1}^{n-1} r^k(\bigcup_{i=1}^{\infty} B_i)$ is a
0-dimensional subset of $P$ such that $P \setminus B$ is 0-dimensional and $\tilde{p}|B \setminus \tilde{p}^{-1}(P \setminus B)$
is one-to-one. Thus $\tilde{p}^{-1}(P \setminus B)$ is a 0-dimensional set homeomorphic to $P \setminus B$.
It follows that $L$ must intersect one of the sets $\tilde{p}^{-1}(b)$ for $b \in B$. If $L \subset \tilde{p}^{-1}(b)$
for some $b \in B$, then $L$ is SID, since $\tilde{p}^{-1}(b)$ is homeomorphic to an HI hered-
itarily SID Cantor manifold. If, for some $b \in B$, $L$ intersects both $\tilde{p}^{-1}(b)$
and its complement, then $L \cap \tilde{p}^{-1}(b)$, by the atomicity of $\tilde{p}$. It follows that
$L$ is SID.

To prove that \( 1, \tilde{r}, \tilde{r}^2, \ldots, \tilde{r}^{n-1} \) are the only homeomorphisms of $Z_n$ onto
$Z_n$, we modify the reasoning in the proof of Theorem 1.1. First we note that
a subcontinuum $Z$ of $Z_n$ is a maximal infinite-dimensional Cantor manifold
in $Z_n$ if and only if it is equal to $\tilde{p}^{-1}(b)$ for some $b \in B$. Indeed, if $b \in B$, then
$\tilde{p}^{-1}(b)$ is homeomorphic to some $K_i$. Moreover, if $Z \subset Z_n$ is a continuum
such that $\tilde{p}(Z)$ contains two different points $x$ and $y$, then one can find a
partition $L$ between $x$ and $y$ in $P$ disjoint from $B$ (see [6, Theorem 1.5.13]).
Then $\tilde{p}^{-1}(L) \cap Z$ is a partition of $Z$ homeomorphic to a subset of $L$, hence it
is one-dimensional. This implies that $Z$ is not an infinite-dimensional Cantor
manifold.

Let $h$ be a homeomorphism of $Z_n$ onto $Z_n$. Then, for every $b \in B$, $h$ maps
$\tilde{p}^{-1}(b)$ onto a maximal infinite-dimensional Cantor manifold in $Z_n$, so there exists $b' \in B$ such that $h(\tilde{p}^{-1}(b)) = \tilde{p}^{-1}(b')$. Moreover, since every $\tilde{p}^{-1}(a_i)$
is homeomorphic to $K_{2i}$ and no two different $K_i$’s are homeomorphic, for
every $i \in \mathbb{N}$ and $l \in \{0, 1, \ldots, n - 1\}$ we have
\[
(5) \quad h(\tilde{r}^l(\tilde{p}^{-1}(a_i))) = \tilde{r}^s(\tilde{p}^{-1}(a_i)) \quad \text{for some } s \in \{0, 1, \ldots, n - 1\}.
\]

It follows that the induced continuous mapping $\bar{h} : P \to P$, where $\bar{h} = \tilde{p}(h(\tilde{p}^{-1}(t)))$, has the property:
\[
(6) \quad \text{for every } t \in \bigcup_{k=0}^{n-1} r^k(A_0), \ \bar{h}(t) = r^k(t) \text{ for some } k \in \{0, 1, \ldots, n - 1\}.
\]

Next, as in the proof of Theorem 1.1, we put $D_k = \{t \in P : h(t) = r^k(t)\}$
for $k \in \{0, 1, \ldots, n - 1\}$ and show that $P = D_{k_0}$ for some $k_0$, which implies
that $\overline{h} = r^{k_0}$. Since $\overline{p}^{-1}(P \setminus B)$ is one-to-one, $h$ coincides with $\overline{r}^{k_0}$ on $\overline{p}^{-1}(P \setminus B)$. Since the latter set is dense in $Z_n$, $h$ coincides with $\overline{r}^{k_0}$ on the whole $Z_n$.

3.1. Remark. Note that there exist continuum many topologically different spaces $X_{nm}$ (respectively, $Y_m$, $Z_n$) satisfying the conditions of Theorem 1.1 (resp., 1.2, 1.3). Indeed, if we replace in the proof of Theorem 1.1 (resp., 1.2, 1.3) $K_1$ by a continuum $K'_1$ nonhomeomorphic to any of $K_1, K_2, \ldots$, then we obtain a topologically different continuum. Since we can choose $K'_1$ from an appropriate family of cardinality $2^\infty$ satisfying the conditions of Lemma 2.2, 2.3 or 2.4 (see Sec. 2), we can obtain in this way continuum many nonhomeomorphic continua.

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