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SOME COMMENTS ON INFINITE BOOLEAN FUNCTIONS

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Abstract. We introduce infinite Boolean functions and investigate some of their properties.

1. Introduction. We use the ordinal notation and let 2 denote the set $\{0,1\}$. We let $2^{\mathbb{Z}}$ be the set of all functions from $\mathbb{Z}$ to 2. We use $2^{<\infty}$ to denote the set of all functions into 2 whose domain is a finite interval in $\mathbb{Z}$. We will use lower case Greek letters η, τ, ν etc. (except σ) to denote elements of $2^{\mathbb{Z}}$ as well as elements of $2^{<\infty}$. If $\tau \in 2^{<\infty}$, then $[\tau]$ is the set of all elements of $2^{\mathbb{Z}}$ which extend τ.

We endow 2 with the discrete topology and $2^{\mathbb{Z}}$ with the product topology generated by the discrete topology on 2. The product space $2^{\mathbb{Z}}$ is sometimes referred to as the *cylinder space* as every basic open set in $2^{\mathbb{Z}}$ has the form $[\tau]$ for some $\tau \in 2^{<\infty}$. The space $2^{\mathbb{Z}}$ is also called the *Cantor space* as $2^{\mathbb{Z}}$ is homeomorphic to the middle 1/3 Cantor set. The left shift on $2^{\mathbb{Z}}$, denoted by σ, is defined by $\sigma(\tau)(k) = \tau(k+1)$ for all $\tau \in 2^{\mathbb{Z}}$. Of course, σ^{-1} is the right shift by one and σ^0 is the identity map. For an integer n, σ^n is defined in the obvious fashion.

If $m \le n$ are integers, then $[m,n]$ is the interval in $\mathbb{Z}$ containing m,n and all integers between. A (finite) *Boolean building block* is simply a function $f : 2^{[m,n]} \to 2$. This f induces a *finite Boolean function* $F : 2^{\mathbb{Z}} \to 2^{\mathbb{Z}}$ as follows: $F(\tau)(i) = f(\sigma^i(\tau)|_{[m,n]})$. We note that F is a continuous function from $2^{\mathbb{Z}}$ into $2^{\mathbb{Z}}$. There are alternate ways to look at the Boolean building block f and the corresponding map F. Let $K \subseteq 2^{\mathbb{Z}}$ and $g_K : 2^{\mathbb{Z}} \to 2^{\mathbb{Z}}$ be defined as $g_K(\tau)(i) = 1$ iff $\sigma^i(\tau) \in K$. Then we have the following.

LEMMA 1. *If F is a (finite) Boolean function, then there is a clopen set K such that $g_K = F$, and conversely, if K is clopen, then g_K is a finite Boolean function.*

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Proof. Suppose F is a finite Boolean function and $f : 2^{[m,n]} \to 2$ gives rise to F. Then $K = \bigcup\{[\tau] : \tau \in 2^{[m,n]}$ and $f(\tau) = 1\}$ is a clopen set and $g_K = F$.

To prove the converse, let K be some clopen set. Then there are integers m, n such that $K = \bigcup_{\tau \in A}[\tau]$ for some set $A \subseteq 2^{[m,n]}$. Let $f : 2^{[m,n]} \to 2$ be such that $f^{-1}(1) = A$. Then the resulting Boolean function F is such that $g_K = F$. ∎

Hence, finite Boolean functions are naturally associated with clopen subsets of $2^{\mathbb{Z}}$. A natural generalization of finite Boolean functions is to consider sets other than clopen. We focus on closed sets K and the corresponding functions g_K.

First, let us note the following:

LEMMA 2. *Let $F : 2^{\mathbb{Z}} \to 2^{\mathbb{Z}}$. Then F commutes with σ if and only if $F = g_K$ for some $K \subset 2^{\mathbb{Z}}$.*

Proof. First, note that $\sigma \circ g_K(\tau)(n) = 1 \Leftrightarrow g_K(\tau)(n+1) = 1 \Leftrightarrow \sigma^{n+1}(\tau) \in K$. Also, $(g_K \circ \sigma(\tau))(n) = 1 \Leftrightarrow g_K(\sigma\tau)(n) = 1 \Leftrightarrow \sigma^n(\sigma\tau) = \sigma^{n+1}(\tau) \in K$. So, $\sigma \circ g_K(\tau)(n) = 1 \Leftrightarrow g_K \circ \sigma(\tau)(n) = 1$. Thus, $\sigma g_K = g_K \sigma$.

Now, suppose $F\sigma = \sigma F$. Let $K = F^{-1}(\pi_0^{-1}(1))$, where π_0 is the projection of $2^{\mathbb{Z}}$ onto the 0th entry. Then $F(\omega)(n) = 1 \Leftrightarrow \sigma^n F(\omega)(0) = 1 \Leftrightarrow F\sigma^n(\omega)(0) = 1 \Leftrightarrow (g_K(\sigma^n(\omega))(0) = 1 \Leftrightarrow (\sigma^n g_K(\omega))(0) = 1 \Leftrightarrow g_K(\omega)(n) = 1$. Thus, $F = g_K$. ∎

Thus, as is well known, we may also characterize the finite Boolean functions as follows:

LEMMA 3. *Let $F : 2^{\mathbb{Z}} \to 2^{\mathbb{Z}}$. Then F commutes with σ and is continuous if and only if F is a finite Boolean function.*

An important notion in the study of Boolean functions is that of shift invariance. We say $A \subseteq 2^{\mathbb{Z}}$ is *shift invariant* if $\sigma(A) = A$.

We begin by studying the range of g_K. The possible range of a finite Boolean function has been completely characterized by Boyle [1].

Let us fix some notation. We use $\underline{0}, \underline{1}$ to denote the zero sequence and the one sequence, respectively. For a subset $K \subseteq 2^{\mathbb{Z}}$, we let $R(K) = g_K(2^{\mathbb{Z}})$ denote the range of g_K. Note that $\phi(R(K)) = R(2^{\mathbb{Z}} \setminus K)$, where ϕ is the homeomorphism which flips each entry of an element of $2^{\mathbb{Z}}$. The following theorem characterizes the range of general Boolean functions K.

THEOREM 1. *Suppose $A \subseteq 2^{\mathbb{Z}}$. Then $A = R(K)$ for some set $K \subseteq 2^{\mathbb{Z}}$ iff A is shift invariant and either $\underline{0}$ or $\underline{1}$ belongs to A.*

Proof. It is clear that if $A = R(K)$ for some K, then A is shift invariant. Also, note that $g_K(\underline{0})$ must be either $\underline{0}$ or $\underline{1}$. For the converse, let A be shift invariant and suppose $\underline{0} \in A$; let $K = A \cap \{\tau : \tau(0) = 1\} = A \cap \pi_0^{-1}(1)$. Then

$g_K(\tau) = \underline{0}$ whenever $\tau \notin A$, and $g_K(\tau) = \tau$ otherwise. If A is shift invariant and $\underline{1} \in A$, then $A' = \phi(A)$ is shift invariant and $\underline{0} \in A'$. Let $R(K') = A'$; then $R(2^{\mathbb{Z}} \setminus K') = A$. ∎

Now we like to investigate what can be said if K is closed or in general a Borel set.

THEOREM 2. *If K is closed, then g_K is of Borel class* 1, *i.e., for each open set $U \in 2^{\mathbb{Z}}$, $g_K^{-1}(U)$ is F_σ.*

Proof. Let $(e_m, \ldots, e_n)$ be a basic open set. Then

$$g_K^{-1}((e_m, \ldots, e_n)) = \bigcap_{i=m}^{n} g_K^{-1}(\pi_i^{-1}(e_i)) = \bigcap_{i=m}^{n} \sigma^{-i}(K^{e_i}),$$

where $K^0 = 2^{\mathbb{Z}} \setminus K$ and $K^1 = K$. So if K is closed, then $g_K^{-1}((e_m, \ldots, e_n))$ is the intersection of an open set and a closed set and $g_K^{-1}(U)$ is an F_σ set, for any open set U. ∎

More generally, if K is of Borel class α in the above theorem, then the argument shows that g_K is of Borel class $\alpha + 1$.

COROLLARY 1. *If $K \subseteq 2^{\mathbb{Z}}$ is a Borel set, then $R(K)$ is analytic.*

We would like next to see how complicated $R(K)$ can be when K ranges over arbitrary closed sets. From one of the results which we will prove later, it will follow that there is a closed set K for which $R(K)$ is not Borel. However, we would first like to introduce an intermediate process inspired by infinite Boolean functions. This process is interesting in its own right as it is reminiscent of Suslin's operation and it characterizes analytic sets in $2^{\mathbb{Z}}$ modulo countable sets.

Let $\{K_n\}_{n\in\mathbb{Z}}$ be a sequence of closed subsets of $2^{\mathbb{Z}}$. Then $g_{\{K_n\}} : 2^{\mathbb{Z}} \to 2^{\mathbb{Z}}$ is defined by $g_{\{K_n\}}(\tau)(i) = 1$ iff $\tau \in K_i$. The function $g_{\{K_n\}}$ was first defined and studied by Szpilrajn-Marczewski in another context [4], [5]. Thus, our function g_K is $g_{\{K_n\}}$ where $K_n = \sigma^{-n}(K)$. We let $R(\{K_n\}) = g_{\{K_n\}}(2^{\mathbb{Z}})$.

THEOREM 3. *Let $A \subseteq 2^{\mathbb{Z}}$ be analytic. Then there is a sequence $\{K_n\}_{n\in\mathbb{Z}}$ of closed subsets of $2^{\mathbb{Z}}$ such that $R(\{K_n\}) = A \cup C$ where C is some countable set.*

Proof. Let $S \subseteq 2^{\mathbb{Z}}$ be a countable dense subset of $2^{\mathbb{Z}}$. Then $Q = 2^{\mathbb{Z}} \setminus S$ is homeomorphic to $\mathbb{N}^{\mathbb{N}}$. As A is analytic, we may obtain a continuous mapping f from Q onto A. Let $L_n = \{\tau \in 2^{\mathbb{Z}} : \tau(n) = 1\}$, $K'_n = f^{-1}(L_n)$ and K_n be the closure of K'_n in $2^{\mathbb{Z}}$. We note that L_n is a clopen subset of $2^{\mathbb{Z}}$, K'_n is closed relative to Q and K_n is a closed set which contains only countably many points which do not belong to K'_n. We want to show that $R(\{K_n\}) = A \cup C$ where C is some countable set. To this end, let $\tau \in Q$. Then, for all i, we see that

$$g_{\{K_n\}}(\tau)(i) = 1 \Leftrightarrow \tau \in K_i \Leftrightarrow \tau \in K'_i \Leftrightarrow f(\tau) \in L_i \Leftrightarrow f(\tau)(i) = 1.$$

Hence, $g_{\{K_n\}}(\tau) = f(\tau)$ for all $\tau \in Q$. Since S is countable, we have $R(\{K_n\}) = g_{\{K_n\}}(2^{\mathbb{Z}}) = A \cup C$ where $C = g_{\{K_n\}}(S)$. ■

EXAMPLE 1. We note that the set C in Theorem 3 cannot be eliminated. For example, let $A = 2^{\mathbb{Z}} \setminus \{\underline{1}\}$ and suppose $g_{\{K_n\}}(2^{\mathbb{Z}}) \supset A$. Then for every integer p there is some point x_p such that $g_{\{K_n\}}(x_p) = 1^p\underline{0}$, where $1^p\underline{0}$ is the sequence whose entries are 0 for indices greater than p and are 1 otherwise. But this implies there is some point x such that $x \in \bigcap_{n=-\infty}^{\infty} K_n$. Thus, $R(\{K_n\}) = 2^{\mathbb{Z}}$.

Now we define the notion of shiftwise disjointness, which will be used frequently. We say that $\tau \in 2^{\mathbb{Z}}$ is *periodic* if there is some $n \neq 0$ such that $\sigma^n(\tau) = \tau$. We say that a set $K \subseteq 2^{\mathbb{Z}}$ is *shiftwise disjoint* if for all $\tau \in K$,

(1) τ is not periodic, and
(2) $\operatorname{orbit}(\tau) \cap K = \{\tau\}$ where $\operatorname{orbit}(\tau) = \{\sigma^n(\tau) : n \in \mathbb{Z}\}$.

We remark here that if P is shiftwise disjoint, then $\sigma^n(P) \cap \sigma^m(P) = \emptyset$ for all $n \neq m$.

Define $\eta \sim \tau$ if they have the same orbit. Note that $\sim$ is an equivalence relation, and in fact it is an F_σ equivalence relation as $\{(\eta, \sigma^n(\eta)) : \eta \in 2^{\mathbb{Z}}\}$ is a closed set for all n.

The following is a rather powerful result of Silver. We do not need the full strength of it; however, it is convenient. The reader is referred to [3] for details.

THEOREM 4 (Silver). *Suppose X is a complete, separable metric space and R is a coanalytic equivalence relation on X. Then either R has countably many equivalence classes or there is a perfect set P which meets each equivalence class in no more than one element.*

Using Silver's theorem, we can obtain a perfect set which intersects each orbit class in no more than one point. As the set of periodic points is countable, and each uncountable Borel set contains a perfect set, we can obtain a perfect set P which is shiftwise disjoint.

In fact, given any uncountable Borel set B, we can get a perfect set $P \subseteq B$ which is shiftwise disjoint. To do this simply apply Silver's theorem to the Borel equivalence relation R defined by

$$R = \{(\eta, \tau) : \eta \sim \tau \text{ and } \operatorname{orbit}(\eta) \cap B \neq \emptyset\} \cup \Big\{(\eta, \tau) : \eta, \tau \in 2^{\mathbb{Z}} \setminus \bigcup_{n \in \mathbb{Z}} \sigma_n(B)\Big\}.$$

Let P be the set given by Silver's Theorem and let $P' \subseteq P$ be a perfect set which misses the periodic points. Then P' is a shiftwise disjoint perfect set which is a subset of B.

We remark that for this particular equivalence relation it is easy to obtain such perfect sets directly without using Silver's theorem.

LEMMA 4. *Let $T = \{\tau : \text{at most one coordinate of } \tau \text{ is nonzero}\}$. There is a shiftwise disjoint perfect set P such that*

- $P \cap T = \emptyset$ *and hence* $\sigma^n(P) \cap T = \emptyset$ *for all* n,
- $\overline{\bigcup_{n\in\mathbb{Z}} \sigma^n(P)} = \bigcup_{n\in\mathbb{Z}} \sigma^n(P) \cup T$,

Proof. Consider the uncountable Borel set $B = \{\tau : \tau(i) = 0$ unless $i = k!$ for some $k \geq 0\}$. Apply the discussion before this lemma to this set $B \setminus T$ to obtain a shiftwise disjoint perfect set $P \subseteq B \setminus T$. This has the desired properties. ∎

THEOREM 5. *Suppose $A \subseteq 2^{\mathbb{Z}}$ is shift invariant and $\underline{0} \in A$.*

(1) *If A is analytic, then there is a closed set K such that $R(K) = A \cup C$ where C is some countable set. If K is allowed to be a G_δ set, we can choose a set K so that $R(K) = A$.*

(2) *If A is an F_σ, then we can choose a closed set K so that $R(K) = A$.*

Proof. We first deal with the case when A is analytic and the set K to be constructed is closed. We proceed as in the proof of Theorem 3. Let P be the set of Lemma 4. Let $Q \subseteq P$ be such that Q is homeomorphic to $\mathbb{N}^{\mathbb{N}}$ and $P \setminus Q$ is countable. Let $Q_n = \sigma^n(Q)$, and $P_n = \sigma^n(P)$. Let h_0 be a continuous mapping from Q_0 onto A. For each n, define $h_n : Q_n \to A$ by $h_n(\tau) = \sigma^n(h_0(\sigma^{-n}(\tau)))$, for each $\tau \in Q_n$. Note that each h_n is a continuous function mapping Q_n onto A. Let $L = \{\tau : \tau(0) = 1\}$. Then L is closed and $K'_n = h_n^{-1}(L)$ is closed relative to Q_n for all n. Each set $K_n = \overline{K'_n}$ is closed in $2^{\mathbb{Z}}$ and $K_n \setminus K'_n$ is countable. Let $K = \overline{\bigcup_n K_n}$. Then $K = \bigcup_n K'_n \cup S \cup T'$, where S is a subset of the countable set $\bigcup_n (P_n \setminus Q_n)$ and $T' \subset T$, where T is the countable set from Lemma 4. Let $\tau \in 2^{\mathbb{Z}}$. If $\tau \notin \overline{\bigcup_n P_n}$, then $g_K(\tau) = \underline{0}$ as $\overline{\bigcup_n P_n}$ is a shift invariant set. The set $\overline{\bigcup_n P_n} \setminus \bigcup_n Q_n$ is countable so we let C be its image under g_K. Finally, consider $\tau \in \bigcup_n Q_n$. There is a unique m such that $\tau \in Q_m$ and there is a unique $\gamma \in Q$ such that $\tau = \sigma^m(\gamma)$. We have

$$\begin{aligned} g_K(\tau)(i) = 1 \;&\Leftrightarrow\; \sigma^i(\tau) \in K \;\Leftrightarrow\; \sigma^i(\tau) \in \bigcup_n K_n \;\Leftrightarrow\; \sigma^i(\tau) \in \bigcup_n K'_n \\ &\Leftrightarrow\; \sigma^i(\tau) \in K'_{m+i} \quad \text{for some } m \\ &\Leftrightarrow\; h_{m+i}(\sigma^i(\tau)) \in L \;\Leftrightarrow\; h_{m+i}(\sigma^i(\tau))(0) = 1 \\ &\Leftrightarrow\; h_{m+i}(\sigma^{m+i}(\gamma))(0) = 1 \;\Leftrightarrow\; h_0(\gamma)(m+i) = 1. \end{aligned}$$

Therefore, $g_K(\tau) = \sigma^m(h_0(\gamma)) \in A$. Hence, $R(K) = A \cup C$ when A is analytic, as required.

We can always obtain a G_δ set K for which $R(K) = A$. We follow the argument as above except we take $K = \bigcup K'_n$. To see the that K is G_δ, we use properties of Lemma 4. It is clear that each K'_n is an absolute G_δ set since it is a closed subset of $\mathbb{N}^{\mathbb{N}}$. Note that $K = (\bigcap_{n\in\mathbb{Z}} R_n) \setminus T$ where $R_n = K'_n \cup \bigcup_{m\in\mathbb{Z},\, m\neq n} P_n \cup T$. That R_n is G_δ follows from the fact that $R_n = \overline{\bigcup_{n\in\mathbb{Z}} P_n} \cap K'_n$.

Now, we want to show that if A is an F_σ, then we can find a closed set K so that $R(K) = A$. Note that if A is a closed set, then we can get a continuous map h_0 from P_0 onto A and we proceed as in the analytic case and obtain $R(K) = A$. For the general F_σ case, we need to do a little more work. We obtain a sequence of pairwise disjoint sets $P^1, P^2, \ldots$ such that

- each P^n is a set of the type of Lemma 4,
- $\bigcup_n P^n$ is shiftwise disjoint, and
- $\overline{\bigcup_n P^n} = \bigcup_n P^n \cup \{\underline{0}\}$.

Now, we may obtain a function h_0 from $\overline{\bigcup_n P^n}$ onto A which is continuous everywhere except possibly at $\underline{0}$. Proceeding again as in the analytic case, we get a closed set K such that $R(K) = A$. ∎

Our next example shows that there is a shift invariant G_δ set A for which there is no closed set K such that $R(K) = A \cup F$, where F is some finite set.

EXAMPLE 2. *Let A be the set which consists of all points which are neither periodic points nor end in a sequence of 1's. Then A is a shift invariant G_δ set for which there is no closed set K such that $R(K) = A \cup F$ for some finite set F.*

Proof. To obtain a contradiction, assume that there is some closed set K such that $R(K) = A \cup F$ for some finite set F. According to Theorem 2, g_K is of Borel class 1 and therefore, for each closed set $M, g_K^{-1}(M)$ is a G_δ set. Also, since periodic points map to periodic points under g_K, we see that $g_K^{-1}(F)$ is a dense G_δ subset of $2^{\mathbb{Z}}$. Let $\bigcup P_n$ be its complement where the sets P_n are closed. Since A is a dense G_δ subset of $2^{\mathbb{Z}}$, by the Baire category theorem, we can obtain an integer m such that $g_K(P_m)$ is dense in some $[\omega]$, where $\omega \in 2^{[-l,l]}$. Since $g_K(P_m)$ is dense in $[\omega]$, we may obtain a sequence $\{\tau_n\}$ in P_m such that $g_K(\tau_n)(i) = 1$ for $l < |i| < l + n$. As P_m is compact, we may assume that $\{\tau_n\}$ is convergent and converges to some $\tau \in P_m$. Since K is closed, we find that if $g_K(\tau_n)(i) = 1$ for all $n > n_0$, then $g_K(\tau)(i) = 1$. Therefore, $g_K(\tau)(i) = 1$ for all $|i| > l$. However, this implies that $g_K(P_m) \cap F \neq \emptyset$, a contradiction. ∎

EXAMPLE 3. *There is a closed set K such that $R(K)$ is analytic and non-Borel.*

Proof. Let P be a perfect set which is shiftwise disjoint and let $Q \subseteq P$ be an analytic set which is non-Borel. Consider $A = \{\underline{0}, \underline{1}\} \cup \bigcup_{n\in\mathbb{Z}} \sigma^n(Q)$. The set A is non-Borel as it is the countable union of non-Borel sets which are separated by disjoint closed sets. By Theorem 5, we know that there is a closed set K such that $R(K) = A \cup C$. ∎

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