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THE COMPLETELY SEPARATING INCIDENCE ALGEBRAS OF TAME REPRESENTATION TYPE

BY

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Abstract. We prove that a completely separating incidence algebra of a partially ordered set is of tame representation type if and only if the associated Tits integral quadratic form is weakly non-negative.

1. Introduction. Let K be an algebraically closed field and $Q = (Q_0, Q_1)$ a finite connected quiver without oriented cycles and arrows having the same starting and ending vertex with another path (hence, without multiple arrows). By the *incidence algebra* $A(Q)$ of Q we mean the bound quiver algebra KQ/I, where I is the ideal in the path algebra KQ generated by the commutativity relations, that is, by all elements $\omega_1 - \omega_2$ given by the pairs $\{\omega_1, \omega_2\}$ of paths in Q having the same starting and ending vertices. Clearly, $A(Q)$ is the incidence algebra of a finite poset (partially ordered set) whose Hasse quiver is Q. Assume that the incidence algebra $A(Q)$ is completely separating (for the definition see [D] or below). We are concerned with the problem of deciding when $A(Q)$ is of tame representation type.

By a remark due to J. Tits there is a connection between the representation type of an algebra A and the definiteness of the associated quadratic form q, called the Tits form of A. For $A = A(Q)$, let R be a minimal set of relations which generate I. Then the *Tits form* of A is the integral quadratic form $q : \mathbb{Z}^n \to \mathbb{Z}$ defined by the formula

$$q(x) = \sum_{i\in Q_0} x_i^2 - \sum_{\alpha\in Q_1} x_{s(\alpha)}x_{t(\alpha)} + \sum_{i,j\in Q_0} r_{ij}x_ix_j,$$

where Q_0 (respectively, Q_1) is the set of vertices (respectively, arrows) of Q, $n = |Q_0|$, $s(\alpha)$ and $t(\alpha)$ denote the source and target of an arrow $\alpha \in Q_1$, and r_{ij} is the cardinality of $R \cap KQ(i,j)$, where $KQ(i,j)$ is the vector space spanned by the paths from i to j (see [Bo1]). It is well known (see [P1]) that if $A(Q)$ is of tame representation type, then $q_{A(Q)}$ is weakly non-negative, that is, $q_{A(Q)}(x) \geq 0$ for any $x \in \mathbb{Z}^n$ with non-negative coordinates.

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Our main result is the following theorem:

THEOREM. *Suppose that Q is a poset and $A(Q)$ is the incidence algebra of Q. If $A(Q)$ is completely separating, then the following conditions are equivalent*:

(i) $A(Q)$ *is of tame representation type.*
(ii) *The associated Tits form* $q : \mathbb{Z}^{|Q_0|} \to \mathbb{Z}$ *is weakly non-negative.*
(iii) $A(Q)$ *does not contain a hypercritical algebra as a convex subcategory.*

The above result allows us to study the tameness of our class of algebras by means of Computer Algebra Programs (see [DN]).

Among tame algebras we distinguish the class of *polynomial growth algebras* [Sk1], for which there exists a positive integer m such that the number of one-parameter families of indecomposable modules is bounded, in each dimension d, by d^m. Note that criteria for polynomial growth of completely separating incidence algebras follow from [Sk3]. Therefore, our aim in this paper is to describe the completely separating incidence algebras $A(Q)$ of tame representation type such that $A(Q)$ admits a convex pg-critical subalgebra in the sense of [NS].

2. Preliminaries. Let P be a finite partially ordered set (briefly, poset). The vertices x, y in P are called *neighbours* if either $x \leq y$ or $y \leq x$, and if $x \leq c \leq y$ or $y \leq c \leq x$ then $c \in \{x, y\}$. To any finite poset P we associate the bound quiver $Q(P) = (Q(P)_0, Q(P)_1, I(P))$, where $Q(P)_0 = P$ and $\alpha : x \to y$ is an arrow in $Q(P)_1$ if $x \leq y$ and x, y are neighbours, and $I(P)$ is the ideal of the path K-algebra of $(Q(P_0), Q(P_1))$ generated by the set of all commutativity relations.

Let (Q, I) be a bound quiver without oriented cycles and arrows having the same starting and ending vertex with another path. Assume that the admissible ideal I in the path algebra KQ is generated by all elements $\omega_1 - \omega_2$ given by the pairs $\{\omega_1, \omega_2\}$ of paths in Q having the same starting and ending vertices. We say that for two vertices x, y of Q we have $x \leq y$ if there is a path in Q from x to y.

From now on, we denote a poset and its associated bound quiver by the same letter.

For a poset Q we denote by $\omega(Q)$ the *width* of Q, that is, the greatest number of pairwise incomparable points.

For a poset Q we denote by $\mathrm{rep}_K(Q)$ the category of K-linear representations of Q. The objects of $\mathrm{rep}_K(Q)$ are systems (V_x, f_α), where V_x is a finite-dimensional vector space for any $x \in Q_0$ and $f_\alpha : V_x \to V_y$ is a linear homomorphism for any arrow $\alpha : x \to y$ in Q_1, and for two paths ω_1, ω_2 in Q starting at the same point x and ending at the same point

y, the compositions f_{ω_1}, f_{ω_2} of the k-linear homomorphisms f_α for the arrows α lying on the path ω_1 (resp. ω_2) are equal as homomorphisms from V_x to V_y. A family (h_x) of K-linear homomorphisms $h_x : V_x \to W_x$, for $x \in Q_0$, is a morphism from (V_x, f_α) to (W_x, g_α) in $\mathrm{rep}_K(Q)$ if for any arrow $\alpha : x \to y$ in Q_1 we have $h_y \circ f_\alpha = g_\alpha \circ h_x$. As usual, we identify $\mathrm{rep}_K(Q)$ with the category of modules over the incidence algebra $A(Q)$ (because $A(Q)$ is the bound quiver algebra of the poset Q; see [GR] and [S2, Chapter 14]).

Following [Lo], for given arrow $\alpha : a \to b$ in Q_1 we associate to the poset Q a poset Q' obtained from Q by contracting the arrow α to the vertex $a = b$. That is, we take the poset $Q' = Q \setminus \{a, b\} \cup \{\overline{\{a, b\}}\}$ together with the inequalities: $x \le y$ (in Q') if $\{x, y\} \subseteq Q \setminus \{a, b\}$ and $x \le y$ in Q; $x \le \overline{\{a, b\}}$ in Q' if either $x \le a$ or $x \le b$ in Q; and $\overline{\{a, b\}} \le y$ if either $a \le y$ or $b \le y$ in Q. We call the poset Q' a *contraction* of Q at α.

The following example illustrates well what it is changed after the contraction at α:

$$Q = \begin{tikzcd} 5 \arrow[rr] & & 2 \arrow[rr] & & 8 \\ & & & 7 \arrow[ur] & \\ 3 \arrow[uu] \arrow[r] & 4 \arrow[r] & 1 \arrow[uu, "\alpha"] \arrow[ur] \arrow[rr] & & 6 \arrow[uu] \end{tikzcd} \qquad Q' = \begin{tikzcd} & 5 \arrow[dr] & & 7 \arrow[dr] & \\ 3 \arrow[ur] \arrow[dr] & & 0 \arrow[ur] \arrow[dr] & & 8 \\ & 4 \arrow[ur] & & 6 \arrow[ur] & \end{tikzcd} \qquad (0 = \{1, 2\})$$

LEMMA 2.1. (a) *If the incidence algebra $A(Q)$ of a poset Q is of tame representation type and Q' is a subposet of Q, then the algebra $A(Q')$ is of tame representation type.*

(b) *If Q' is a convex subposet of Q and the Tits form of $A(Q)$ is weakly non-negative then so is the Tits form of $A(Q')$.*

Proof. The statement (a) is known since Q' is a full subcategory of Q (see [DS1, Lemma 6]). We have (b) since for a convex subposet Q' the Tits quadratic form $q_A(Q')$ is a restriction of $q_A(Q)$. ■

LEMMA 2.2. (a) *If the incidence algebra $A(Q)$ of a poset Q is of tame representation type and Q' is an iterated contraction of Q, then $A(Q')$ is of tame representation type.*

(b) *If Q' is an iterated contraction of Q and the Tits form of $A(Q)$ is weakly non-negative then so is the Tits form of $A(Q')$.*

Proof. It is enough to prove the assertion for a contraction of one arrow. The statement (a) is obvious since the category $\mathrm{rep}_K(Q')$ is equivalent to the subcategory of $\mathrm{rep}_K(Q)$ consisting of the representations $V = (V_x, f_\alpha)$ in which f_α is an isomorphism.

(b) Suppose q' (resp. q) denotes the quadratic form associated to $A(Q')$ (resp. $A(Q)$) and let $y \in \mathbb{Z}^{n-1}$ be such that $q'(y) < 0$. We may assume that for the arrow α (which is contracted in Q') we have $s(\alpha) = 1$, $t(\alpha) = 2$, where $1, 2 \in Q_0 = \{1, 2, \ldots, n\}$. Denote the vertex $\overline{\{1,2\}}$ of Q' by 0 and take $x \in \mathbb{Z}^n$ with coordinates $x_1 = y_0$, $x_2 = y_0$, and $x_i = y_i$ for $i \geq 3$. We are going to prove that $q(x) < 0$.

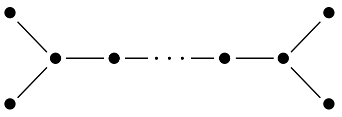
Since $x_k = y_k$ for $k = 3, \ldots, n$, and $r_{ij} = r'_{ij}$ for $\{i, j\} \cap \{1, 2\} = \emptyset$, we have

$$\begin{aligned} q(x) - q'(y) = x_1^2 &+ x_2^2 - x_1x_2 \\ &- y_0^2 - \sum (x_s x_1 + x_2 x_t) + \sum (\varepsilon_s y_s y_0 + \eta_t y_0 y_t) \\ &+ \sum (r_{s1} x_s x_1 + r_{2t} x_2 x_t) - \sum (r'_{s1} y_s y_0 + r'_{2t} y_0 y_t) \end{aligned}$$

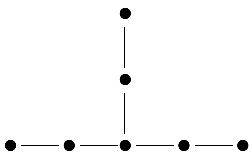
where s is a predecessor of 1 in Q (t is a successor of 2 in Q), and $\varepsilon_s = 0$ (resp. $\eta_t = 0$) if there exist at least two paths from s to 2 (resp. from 1 to t), whereas $\varepsilon_s = 1$ (resp. $\eta_t = 1$) otherwise. In the above formula ε_s, η_s appear because if we contract an arrow from the path ω of length 2 such that there exists another path ν in Q with $s(\nu) = s(\omega)$, $t(\nu) = t(\omega)$, then (see the above example) in Q' there is one path (with source $s(\omega)$ and target $t(\omega)$) less (in Q' the path ω is omitted). One can prove the inequalities $r'_{ij} \leq r_{ij} \leq r'_{ij} + 1$ for any pair $\{i, j\}$ (where r'_{ij} is the cardinality of the corresponding set for Q'). Suppose $r_{ij} = r'_{ij}$ for any pair $\{i, j\}$. Since $y_0 = x_1 = x_2$, we have $q(x) = q'(y)$, and $q(x) < 0$. One can see that $r_{ij} = r'_{ij} + 1$ if and only if either (a) $j = 2$, i is a predecessor of 1 and there is a path from i to 2 different from $i \to 1 \to 2$, or (b) $i = 1$, j is a successor of 2 and there is a path from 1 to j different from $1 \to 2 \to j$. Suppose $r_{ij} = r'_{ij} + 1$ and Q has property (a). One can see that $\varepsilon_i = 0$. Similarly for (b) we have $\eta_j = 0$ for the successor of 2. Hence $q(x) = q'(y)$. ■

We know that a hereditary algebra is simply connected (see [Sk2] for the definition) if and only if it is the path algebra of a tree. Let $\varDelta$ be a finite connected quiver whose underlying graph $\overline{\varDelta}$ is a tree, and $H = K\varDelta$. Then it is well known that H is representation-infinite and tame if and only if $\overline{\varDelta}$ is one of the Euclidean graphs

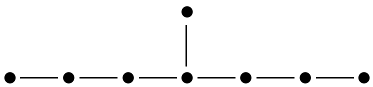
$\widetilde{\mathbb{D}}_n$:



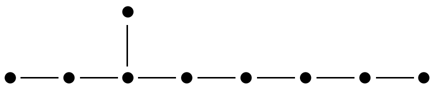
$\widetilde{\mathbb{E}}_6$:



$\widetilde{\mathbb{E}}_7$:



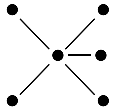
$\widetilde{\mathbb{E}}_8$:



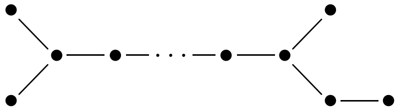
where $\widetilde{\mathbb{D}}_n$ has $n+1$ vertices, $n \geq 4$.

Hence, $H = K\varDelta$ is wild if and only if $\bar{\varDelta}$ contains one of the following graphs:

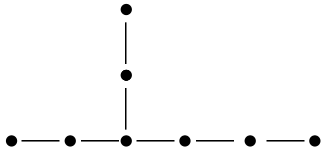
T_5 :



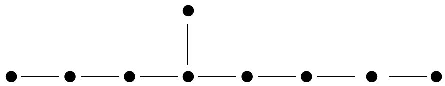
$\widetilde{\widetilde{\mathbb{D}}}_n$:



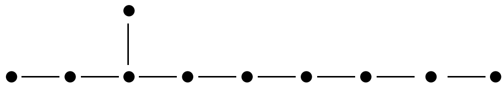
$\widetilde{\widetilde{\mathbb{E}}}_6$:



$\widetilde{\widetilde{\mathbb{E}}}_7$:



$\widetilde{\widetilde{\mathbb{E}}}_8$:

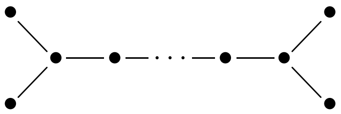


where $\widetilde{\widetilde{\mathbb{D}}}_n$ has $n+2$ vertices, $4 \leq n \leq 8$.

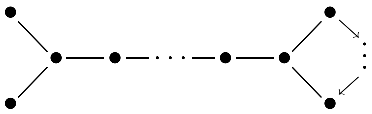
Assume that $H = K\varDelta$ is representation-infinite ($\bar{\varDelta}$ is not a Dynkin graph) and T is a preprojective tilting H-module, that is, $\mathrm{Ext}^1_H(T,T) = 0$ and T is a direct sum of $n = |\varDelta_0|$ pairwise non-isomorphic indecomposable H-modules lying in the τ_H-orbits of projective modules. Then $C = \mathrm{End}_H(T)$ is called a *concealed algebra* of type $\bar{\varDelta}$. It is known that $\mathrm{gl.dim}\, C \leq 2$ and C has the same representation type as H (see [HR]). A concealed algebra of type $\bar{\varDelta} = \widetilde{\mathbb{D}}_n$, $\widetilde{\mathbb{E}}_6$, $\widetilde{\mathbb{E}}_7$, $\widetilde{\mathbb{E}}_8$ (resp. $\bar{\varDelta} = T_5$, $\widetilde{\widetilde{\mathbb{D}}}_n$, $\widetilde{\widetilde{\mathbb{E}}}_6$, $\widetilde{\widetilde{\mathbb{E}}}_7$, $\widetilde{\widetilde{\mathbb{E}}}_8$) is said to be *critical* (resp. *hypercritical*). The critical (resp. hypercritical) algebras have been completely classified in [Bo2], [HV] (resp. [U], see also [L] and [Wi]). It is known [Bo2] that a simply connected algebra A is representation-finite if and only if A does not contain a critical convex subcategory. It is expected (see [P1]) that a simply connected algebra A is tame if and only if A does not contain a hypercritical convex subcategory.

From [HV] we know that there are only four families of critical algebras of type $\widetilde{\mathbb{D}}_n$, given by the following quivers:

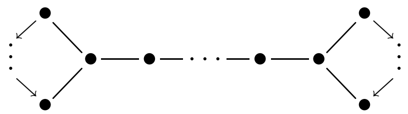
(1)



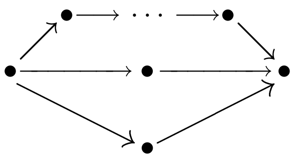
(2)



(3)



(4)



where $\bullet\!-\!\bullet$ means $\bullet\!\to\!\bullet$ or $\bullet\!\leftarrow\!\bullet$, the algebras of types (2) and (3) are bound by the commutativity relations, and those of type (4) by the vanishing of the sum of paths from the unique source to the unique sink (those of type (4) are not incidence algebras).

Following [NS], by a *pg-critical algebra* we mean an algebra A satisfying the following conditions:

(i) A is one of the matrix algebras

$$B[M] = \begin{bmatrix} K & M \\ 0 & B \end{bmatrix}, \quad B[N,t] = \begin{bmatrix} K & K & \dots & K & K & K & N \\ & K & \dots & K & K & K & 0 \\ & & \ddots & \vdots & \vdots & \vdots & \vdots \\ & & & K & K & K & 0 \\ & & & & K & 0 & 0 \\ & \text{\Large 0} & & & & K & 0 \\ & & & & & & B \end{bmatrix}$$

where B is a representation-infinite tilted algebra of Euclidean type $\widetilde{\mathbb{D}}_n$, $n \geq 4$, with a complete slice in the preinjective component of the Auslander–Reiten quiver $\varGamma_B$, M (resp. N) is an indecomposable regular B-module of regular length 2 (resp. length 1) lying in a tube with $n-2$ rays, and $t+1$ ($t \geq 2$) is the number of isoclasses of simple $B[N,t]$-modules which are not B-modules.

(ii) Every proper convex subcategory of A is of polynomial growth.

It is known that every pg-critical algebra is tame but it is not of polynomial growth [NS].

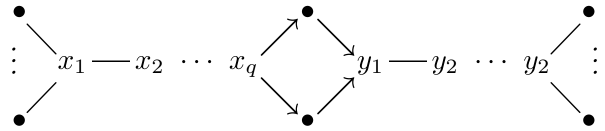
Let $A = KQ/I$ be a bound quiver algebra. Following [D] we call a module V *thin* if $\dim_K V_x \leq 1$ for any vertex x. We have the following trivial fact:

REMARK 2.3. Suppose B is an incidence algebra of a poset and $A = B[V]$ ($A = [V]B$) denotes a one-point (co-) extension of the algebra B by the module V. If A is an incidence algebra (for the large poset), then the module V is thin.

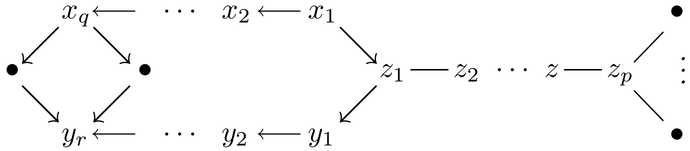
We know that among the four families of concealed algebras of type $\widetilde{\mathbb{D}}_n$ there are three families of incidence algebras of posets. Since for concealed incidence algebras from our three families the simple regular thin modules and the indecomposable regular thin modules of regular length 2 are known [NS], one can give a classification of pg-critical incidence algebras. In [Sk2] a bound quiver algebra KQ/I is called *strongly simply connected* if for each convex full subquiver Q' of Q the associated algebra KQ'/I' is simply connected. All strongly simply connected tame algebras minimal of non-polynomial growth are listed in [NS]. In particular, the following is proved in [NS, Theorem 3.2]:

Lemma 2.4. *Let $A = A(Q)$ be a pg-critical incidence algebra of a poset Q. Then Q or Q^{op} is of one of the forms*

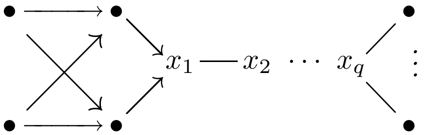
(2.4a)



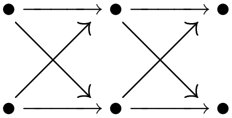
(2.4b)



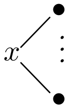
(2.4c)



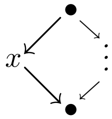
(2.4d)



where $p, q, r \geq 1$ and



denotes either a tree $\bullet - x - \bullet$ with any orientation of edges or a poset

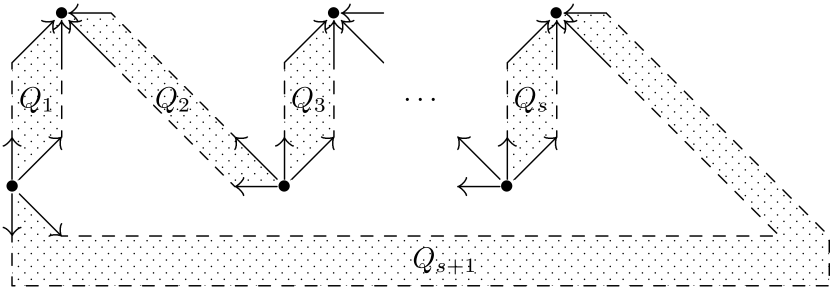


with one commutativity relation.

Let $A = KQ/I$ be the incidence algebra of a finite poset Q. For every $x \in Q$ we denote by P_x the indecomposable projective A-module associated with x. The module P_x is said to have a *separated radical* if the supports of any two non-isomorphic indecomposable direct summands of $\operatorname{rad} P_x$ are contained in different connected components of the subposet Q_x of Q obtained by deleting all those points y such that there is a path with source y and target x. If all the indecomposable projective A-modules have separated radical, then A is said to satisfy the *separating condition* [BLS]. The incidence algebra A is called *completely separating* if for any convex subposet Q' the associated incidence algebra KQ'/I' also satisfies the separating condition [D].

Consider a poset of the form

(2.5)



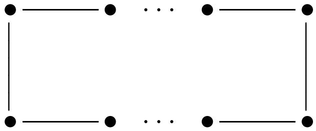
(where $s \geq 3$) such that the poset Q_i has one minimal and one maximal point for any $1 \leq i \leq s+1$, and the points $x \in Q_i$, $y \in Q_j$ are incomparable for $i \neq j$ except the case $\{x, y\} \cap Q_i \cap Q_j \neq \emptyset$.

Lemma 2.6. *The incidence algebra $A = A(Q)$ of a poset Q is completely separating if and only if Q contains no convex subposet of the form* (2.5).

This fact is proved in [D, Theorem 4.3], where (2.5) is called a *crown.*

We denote by $\widetilde{A}_{m,n}$ the poset of the form

$\widetilde{A}_{m,n} =$



(without commutativity relations), where $\bullet\!\!-\!\!\bullet$ means either $\bullet\!\to\!\bullet$ or $\bullet\!\leftarrow\!\bullet$ and $\widetilde{A}_{m,n}$ has $m+n$ vertices and $m+n$ arrows; m of them have clockwise orientation and n counterclockwise orientation.

Hence, if $\widetilde{A}_{m,n}$ has at least two minimal (or equivalently, at least two maximal) vertices, then it is a crown.

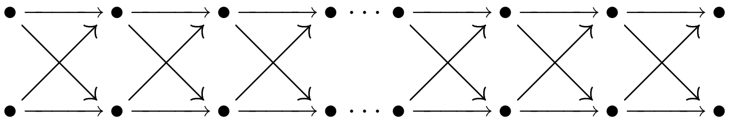
Observe that the pg-critical algebras (2.4c,d) are not completely separating.

We have the obvious

Lemma 2.7. *The incidence algebra $A(Q)$ of a poset Q is strongly simply connected if and only if it is completely separating.*

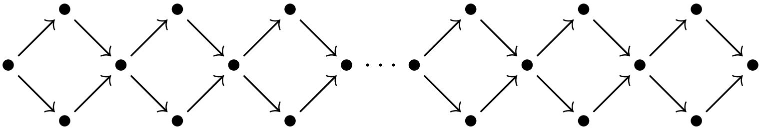
For any natural number n we consider the poset

(2.8) G_n :



having $2(n+1)$ points, and the poset

(2.9) F_n :



having n commutative squares.

We have the following obvious fact:

LEMMA 2.10. (a) *Every G_n is a subposet of some F_m, and every F_n is a subposet of some G_m.*

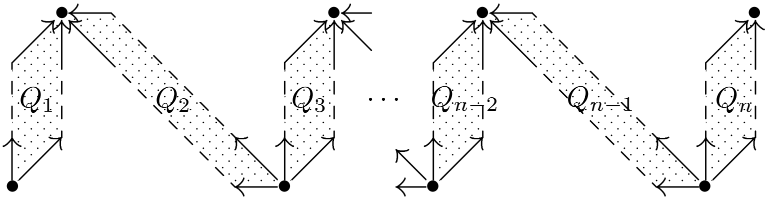
(b) *A poset Q is a subposet of G_n or F_n for some n if and only if Q is of width at most* 2 *and does not contain as a full subposet the* (*disconnected*) *poset*

$$\bullet \longrightarrow \bullet \qquad \bullet \tag{2.11}$$

We consider subposets of (2.9) which are completely separating, that is, are of width 2 and contain neither (2.11) nor G_2 as a convex subposet.

By a *viper* we mean a convex, connected subposet of

(2.12)



where each Q_i is a subposet of some F_n. Note that a viper is completely separating if and only if all the Q_i are completely separating.

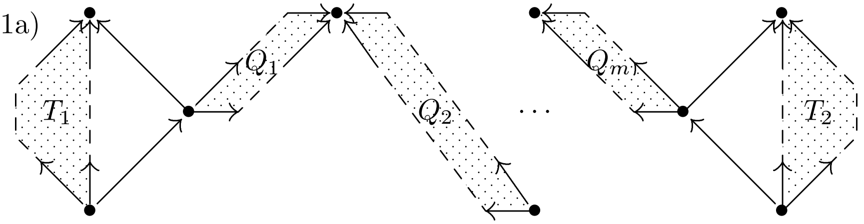
If a poset is a viper and T is either Q_1 or Q_n (in the notation of (2.12)), then T can have two minimal (or two maximal) points. One can see that a viper is a proper convex subposet of a crown.

3. Families of non-polynomial growth posets.

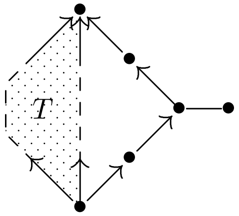
For algebras of polynomial growth, our Theorem is already proved [Sk3, Theorem 4.1 together with Corollary 4.2]. In the case of algebras of non-polynomial growth we have to consider only the iterated one-point extensions or coextensions (see [R2]) of pg-critical completely separating algebras (see Lemma 2.3) whose Gabriel quivers are posets of the form (2.4a) or (2.4b).

We consider the incidence algebras of posets from the following families:

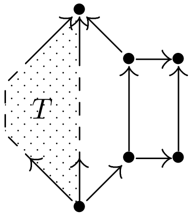
(3.1a)



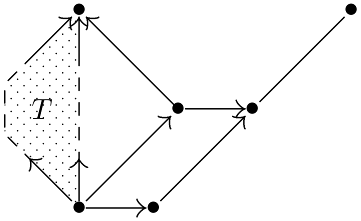
(3.1b)



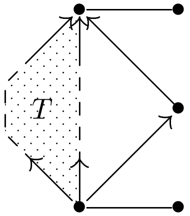
(3.1c)



(3.1d)



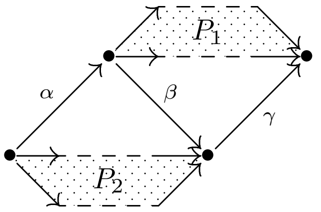
(3.1e)



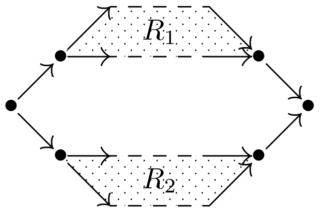
in which $T_1, T_2, Q_i, 1 \leq i \leq m$, and T are subposets of (2.9), where $\omega(T) = 2$; possibly $m = 1$, Q_1 or Q_m is a point, or both are points, or $m = 1$ and Q_1 is a point.

We also consider the incidence algebras of posets from the families

(3.2a)



(3.2b)



where each P_i and R_j is a subposet of (2.9) and $\omega(P_i) = \omega(R_j) = 2$.

Note that the incidence algebra $A(Q)$ of a poset Q of one of the forms (3.1), (3.2) is completely separating if and only if T_1, T_2, T, Q_i for $1 \leq i \leq m$, P_1, P_2, R_1, R_2, are all completely separating subposets of (2.9).

By an *admissible extension* (resp. *coextension*) of the incidence algebra of a poset Q we mean a one-point extension (resp. coextension) which contains neither a convex hypercritical algebra nor a convex crown, and such that the resulting algebra is an incidence algebra (for the larger poset). By a simple case by case investigation one can prove

LEMMA 3.3. *Let $A(Q)$ be the incidence algebra of a poset Q from the families* (3.1), (3.2). *If Q contains a convex subposet of a pg-critical algebra, then $A(Q)$ has no admissible extension* (*or coextension*).

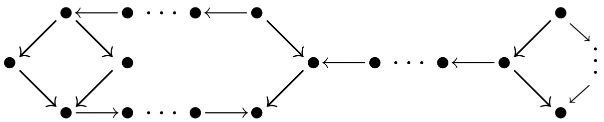
The main aim of this section is to prove the following fact.

PROPOSITION 3.4. *Let $A(Q)$ be the completely separating incidence algebra of a poset Q. Suppose that $A(Q)$ has no convex hypercritical subalgebra.*

If $A(Q)$ *contains a convex pg-critical algebra, then either* Q *or* Q^{op} *is a subposet of one of the posets from the families* (3.1), (3.2).

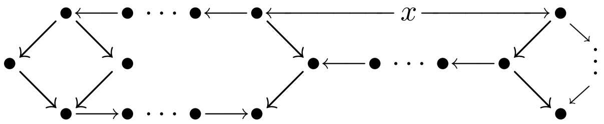
Proof. Consider extensions of the poset (2.4b). After a short inspection of all possible extensions one can see that the extension of any poset of the form (2.4b) by adding exactly one new arrow is not admissible. One can show that if our poset has the form

(3.5)



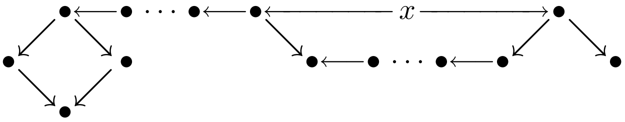
then there is no admissible extension. For example, the extension

(3.6)



contains a convex subposet

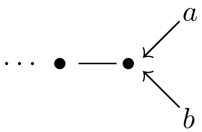
(3.7)



which is hypercritical of type $\overset{\approx}{\mathbb{D}}_n$. One can see that another extension with two new arrows contains (more than one) hypercritical poset. Similarly, if we add at least three new arrows (and one vertex), then we obtain a poset containing a hypercritical poset.

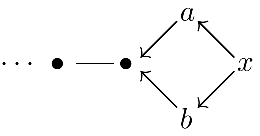
An inspection of all possible extensions shows that a poset of the form (2.4b) has an admissible extension, provided this extension is by one vertex and two or three new arrows, and the right part of (2.4b) is of the form

(3.8)



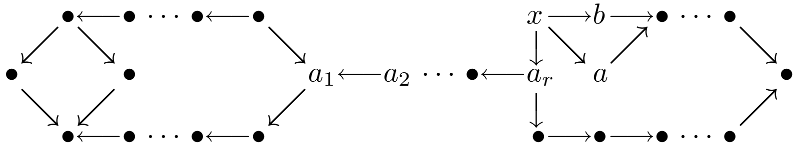
One can show that if after the admissible extension we add two arrows then the extensions are of the form

(3.9)

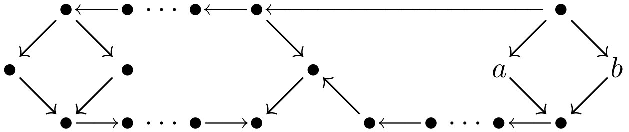


Assume now that after the extension we add three new arrows. We have the following admissible extensions:

(3.10a)



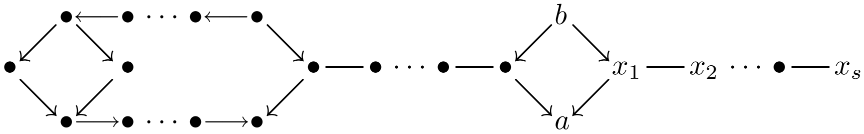
(3.10b)



One can prove that another extension with three new arrows is not admissible. Observe that a poset (3.10a) is of the form (3.1a), and (3.10b) is of the form (3.2a). Hence they have no admissible extension or coextension (Lemma 3.3).

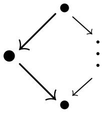
Consider now extensions of the poset of the form (3.9). For the poset

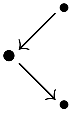
(3.11)



(it is possible that $s = 1$) we may investigate (as above) its admissible (co-) extensions. In this way one can show that any admissible iterated (co-) extension of (3.11) produces a subposet of a poset from the families (3.1), (3.2) or it contains a convex subposet of the form (2.4a) (which is a viper).

Assume now that a poset (2.4a) is not a viper, that is, it ends on at least one side with a subposet of one of the forms



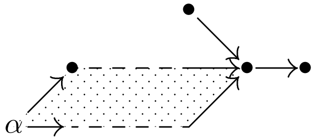


Then one can see that our poset has an admissible extension, if this extension is by one vertex and two or three new arrows and one side part of a poset (2.4a) is of the form (3.8). Then, as above, we may prove that any admissible iterated extension or coextension of our poset produces a poset which is a subposet of a poset from the families (3.1), (3.2) or a poset containing a convex subposet of the form (2.4a) (which is a viper). Hence the proof is reduced to the investigation of an iterated extension (or coextension) of a viper of non-polynomial growth.

Denote by r_i the number of incomparable pairs in the poset Q_i (in the notation of (2.12)). Observe that a viper is of non-polynomial growth if $r_1 + \ldots + r_n \geq 3$.

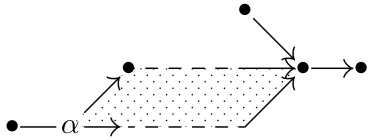
Suppose that Q is a viper with $n = 1$ and $r_1 \geq 3$. Then one can show that an admissible (co-) extension is a viper or a poset of the form

(3.12)



(which is a coextension of some viper with $n = 2$). The next admissible extension (or coextension) is a proper subposet of (3.1e) (without a vertex with one neighbour) or a poset of the form

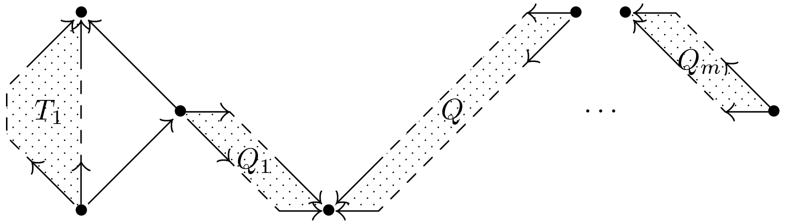
(3.13)



One can prove that any admissible extension (or coextension) of the first poset is of the form (3.1e). The poset (3.13) is a coextension of a viper with $n = 2$ or $n = 3$, depending on the orientation of the new arrow.

Consider a poset of the form

(3.14)



By a *left* (resp. *right*) *rolled viper* we mean a viper with the left (resp. right) part equal to (3.14) or its dual.

The posets from (3.1a) are *double-rolled vipers*, i.e. rolled on both ends.

By a *left* (resp. *right*) *armed viper* we mean a poset obtained from a viper by replacing the first (resp. last) Q_i not equal to the point by a poset equal to (3.12) (or by its opposite) with a gluing point $\alpha \in (3.12)$.

The posets (3.12) and (3.13) are armed vipers.

Observe that if a poset Q is of one of the forms: (i) a viper, (ii) a rolled or an armed viper, (iii) a double-rolled or double-armed viper, then Q is a subposet of (3.1a) (an armed viper is not a convex subposet of (3.1a)).

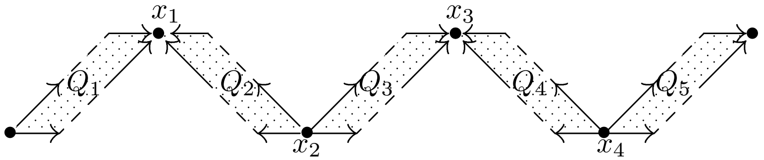
If in the notation of (3.14) we have $m = 1$, then the poset is an extension of a viper with $n = 2$ and $r_1 \neq 0 \neq r_2$ (also for Q_1 of width 2).

Observe that if there is an orientation in (3.1b) such that the vertex with one neighbour is a source, then (3.1b) is an extension of a viper with $n = 2$. Also (3.2b) is an extension of such a viper if $r_1 \neq 0 \neq r_2$. Suppose Q' is a poset of the shape (3.2b) for which only one of R_i is of width 2 (Q' is also an extension of some viper). One can see that after several admissible extensions and coextensions of the poset Q' we obtain a poset from the families (3.1), (3.2) which is either a rolled (or an armed) viper or a double-rolled or double-armed one.

Suppose Q is a viper with $n = 2, 3$, or 4 (here we mean that exactly 2, 3 or 4 subposets Q_i in the notation of (2.12) are not equal to the point). Then case by case inspection shows that iterated admissible extensions and coextensions lead to one of the following posets: a viper, a rolled (or an armed) viper, a double-armed or double-rolled viper, or a subposet of some poset from the families (3.1), (3.2).

For a viper Q with $n \geq 5$, one gets the same conclusion as above. We present more details of this proof for a viper Q with $n = 5$ of the form

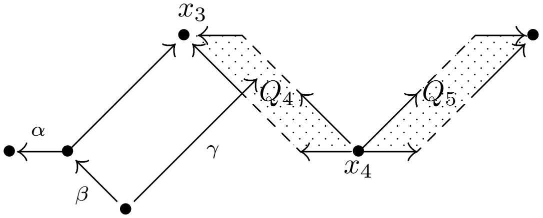
(3.15)



(Let us repeat that it is possible that Q_1 has two minimal points, or Q_5 has two maximal points.) Let Q' denote an extension of Q. Observe that if two new arrows end in $Q_2 \cup Q_3$ or in $Q_4 \cup Q_5$, then Q' contains a convex subposet $\widetilde{A}_{2,2}$. Similarly, if for an admissible extension of Q two new arrows end in Q_1, then they end at two minimal vertices of Q_1. The width of Q_i's (≤ 2) excludes three arrows going to one of Q_i.

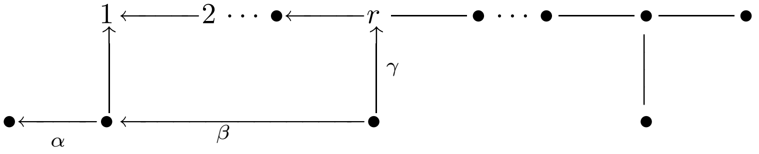
Assume that in Q' there are three new arrows, say α, β, γ, and α ends in Q_1, β in $Q_2 \cup Q_3$, and γ in $Q_4 \cup Q_5$. Then none of x_1, x_2 is the target of a new arrow. If Q_2, Q_3 is not equal to an arrow, then $Q' \setminus \{x_1, x_2\}$ contains the poset of a hypercritical algebra (remember that a viper is of non-polynomial growth, that is, at least one of Q_i's is of width 2). Assume that Q_2, Q_3 are each an arrow and suppose Q_5 is of width 2. Then Q' contains a convex subquiver of the form

(3.16)



and (3.16) contains a convex subposet of the form

(3.17)



which is the poset of a hypercritical algebra of type $\widetilde{\widetilde{\mathbb{D}}}_n$.

Similarly, if Q_1 is of width 2, then $Q' \setminus \{x_1\}$ contains a hypercritical poset. Hence, if the extension of Q is admissible and has three new arrows, then two of them end at minimal points of Q_1 and the extension is of the form (3.14) (with $m = 3$ or $m = 4$).

Any extension (of any poset) with 5 new arrows contains a hypercritical poset $\overset{\approx}{\mathbb{D}}_4$. If we have an extension with four new arrows, then from the above remarks we know that the targets of two of them are minimal points of Q_1, one target is in $Q_2 \cup Q_3$, and one in $Q_4 \cup Q_5$. Then none of x_1, x_2 is the target of a new arrow, and it is easy to show that $Q' \setminus \{x_1\}$ contains the poset of a hypercritical algebra (of type $\overset{\approx}{\mathbb{A}}_n$).

Similarly one can show that if an admissible extension Q' has two new arrows, then either the arrows end at two minimal points of Q_1 (and we obtain a viper with a longer Q_1), or Q_1 has only one minimal point, and one of the arrows ends at this point, the other in Q_2, and the extension is of the form (3.14).

If Q' is an admissible extension of Q with one new arrow, then Q' is either:

(a) a viper with
  (i) a longer Q_1, and the target of the new arrow is the minimal point of Q_1,
  (ii) $n = 6$, and the target of the new arrow is the (unique) maximal point of Q_5,
  (iii) a wider Q_1, and the target of the new arrow is the successor of the minimal point of Q_1,

or

(b) an armed viper, and the target of the new arrow is a (unique) predecessor of a (unique) maximal point of Q_5.

Hence, after an admissible extension we obtain either a new viper, an armed viper or a rolled viper. The same holds for admissible coextensions. Therefore, the iterated (co-) extended poset of our poset has no admissible (co-) extension if each side of the poset is rolled or armed, and it is a subposet of some poset of the form (3.1a) (if it is double-rolled, then it is one of (3.1a)).

In the above combinatorics, the first operation which we employed for the vipers was an admissible one-point extension. Dually, we can obtain the same conclusions for any n if we start with an admissible one-point coextension. ■

4. Proof of the tameness.

The aim of this section is to prove

PROPOSITION 4.1. *Suppose $A = A(Q)$ is the incidence algebra of a poset Q from the families* (3.1) *and* (3.2). *Then A is of tame representation type.*

In order to do this we need some lemmas.

Consider the bound quiver S of the form

(4.2)
$$\begin{tikzcd}
1 \arrow[r, "a"] \arrow[d, "c"'] & 2 \arrow[d, "b"] \\
3 \arrow[r, "d"'] & 4
\end{tikzcd}$$

with the ideal $I_S = (ba - dc)$, and the quiver T of the form

(4.3)
$$\begin{tikzcd}
f \arrow[r, "\alpha"] & g \arrow[loop above, "\beta"] \arrow[r, "\gamma"] & h
\end{tikzcd}$$

with the ideal $I_T = (\beta^2 - \beta, \gamma\alpha)$.

Assume that S is a full convex subquiver of some quiver Q in which vertex 2 (resp. 3) is the target of only one arrow a (resp. c) and the source of only one arrow b (resp. d). Assume that the quiver Q is bounded by the ideal J in KQ such that if $\sum_r k_\sigma \omega_r$ is a relation in J such that for some r we have $\omega_r = \omega'_r a \omega''_r$ (resp. $\omega_r = \omega'_r b \omega''_r$), then $\omega'_r = \omega'''_r b$ (resp. $\omega''_r = a\omega'''_r$), and assume a similar property for the arrows c, d (if one of them appears in a relation, then it appears in the composition dc in this relation). We consider the quiver Q' which is obtained from Q by replacing S by the subquiver T (of type (4.3), with vertices 1, 4 glued to A, C, respectively). Now we define the ideal J' in KQ'. If w is a relation in I other than $dc - ba$, then we obtain the relation w' by replacing any composition dc, ba (if it appears in some path in w) by the composition $\gamma\beta\alpha$. The generators of J' are the relations w' together with $\beta^2 - \beta$, $\gamma\alpha$.

Then we have the following

LEMMA 4.4. (a) *The algebras KS/I_S and KT/I_T are isomorphic.*

(b) *There is a degeneration* (see [G]) *of the algebra KT/I_T isomorphic to the special biserial algebra KT/I, where $I = (\beta^2, \gamma\alpha)$.*

(c) *The algebras KQ/J and KQ'/J' are isomorphic.*

Proof. (a) Let $\overline{\overline{\overline{F}}} : KS \to KT$ be a K-algebra homomorphism mapping the generators $e_1, e_2, e_3, e_4, a, b, c, d$ of KS to $e_f, e_g, e_g - \beta, \beta\alpha, -\gamma\beta, \alpha - \beta\alpha, \gamma - \gamma\beta$ (in this order). Observe that $\overline{\overline{\overline{F}}}$ is a surjection. Hence the induced homomorphism $\overline{F} : KS \to KT/I_T$ is a surjection and

$$\overline{F}(dc - ba) = (\gamma - \gamma\beta)(\alpha - \beta\alpha) + \gamma\beta^2\alpha = \gamma\alpha - 2\gamma\beta\alpha + 2\gamma\beta^2\alpha = 0$$

in KT/I_T, since $\beta^2 = \beta$ and $\gamma\alpha = 0$. Therefore the induced homomorphism $F : KS/I_S \to KT/I_T$ is an isomorphism.

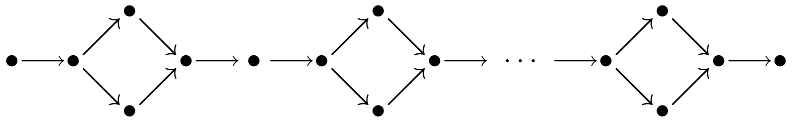
(b) If we set $A = KT/(\gamma\alpha)$ and (following [G] and [CB]) $\varrho_\lambda = \beta^2 - \lambda\beta$, $A_\lambda = A/(\varrho_\lambda)$, then one can check that $A_1 = A_\lambda$ for $\lambda \neq 0$ (by taking $\beta \mapsto \lambda\beta$), $A_0 = KS/I_S$ (hence A_0 is a degeneration of $A_1 = KT/I_T$) and one can see that A_0 is special biserial (see [CB] for the definition).

Statement (c) is a consequence of (a) and of the constructions of the quiver Q' and the ideal J'. ■

LEMMA 4.5. *The incidence algebras of the posets of the forms* (3.1a), (3.2a), (3.2b) *are of tame representation type.*

Proof. Without loss of generality, invoking Lemmas 2.1 and 2.2, we assume that each of the posets T_i, Q_i, P_i, R_i is of the form

(4.6)



We denote the unique longest path in such a subquiver simply by T_i, Q_i, P_i, R_i. Now, for the quiver of the shape (3.1a) (resp. for the quiver (3.2a)) bounded by all commutativity relations of the squares of T_i, Q_i, P_i, R_i, we denote by B the corresponding bound quiver algebra. Hence, we have $A(Q) \cong B/(\beta\alpha - T_1, \delta\gamma - T_2)$ for the quiver Q of the shape (3.1a) (resp. $A(Q) \cong B/(\gamma\beta - P_1, \beta\alpha - P_2)$ for the quiver Q of the shape (3.2a)).

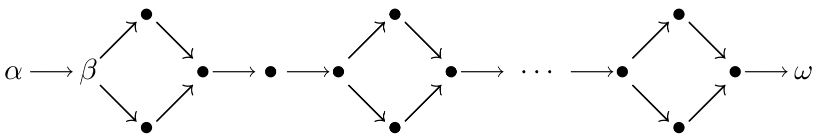
We define the functions $f_1, f_2 : K \to B$ by $f_1(\lambda) = \beta\alpha - \lambda T_1$, $f_2(\lambda) = \delta\gamma - \lambda T_2$ (resp. $f_1(\lambda) = \gamma\beta - \lambda P_1$, $f_2(\lambda) = \beta\alpha - \lambda P_2$) in the notations from Theorem B in [CB]. By [CB, Theorem B], the tameness of our incidence algebras $A(Q) \cong A_1$ reduces to the tameness of the bound quiver algebras A_0 with the quivers (3.1a), (3.2a) bounded by the relations: all commutativity relations in squares from T_i, Q_i, P_i, R_i and $\beta\alpha = 0$, $\delta\gamma = 0$ for (3.1a) ($\beta\alpha = 0$, $\gamma\beta = 0$ for (3.2a)).

Now, in our quiver Q, we replace each square in T_i, Q_i, P_i, R_i by the subquiver T of the form (4.3) and obtain a new isomorphic algebra (Lemma 4.4(c)) having a special biserial degeneration (by iterating the modification from Lemma 4.4(b)). Special biserial algebras are of tame representation type [WW] (see also [DS2]). By [CB] our incidence algebras are of tame representation type. ■

LEMMA 4.7. *The incidence algebras of posets of the forms* (3.1b), (3.1c), (3.1d), (3.1e) *are of tame representation type.*

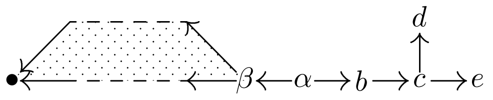
Proof. Suppose $A(Q)$ is the incidence algebra of a poset Q of the form (3.1b). Without loss of a generality (Lemma 2.1), we may assume that the subposet T (3.1b) is of the form

(4.8)



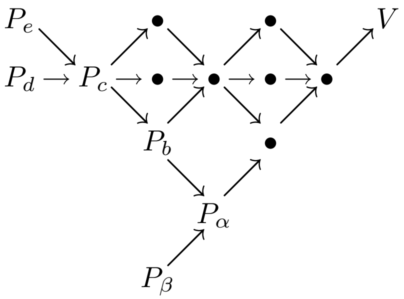
We choose the orientation of the arrow not lying on the path from the large commutativity relation and we denote the vertices of $Q' = Q\setminus\{\omega\}$ as follows:

(4.9)



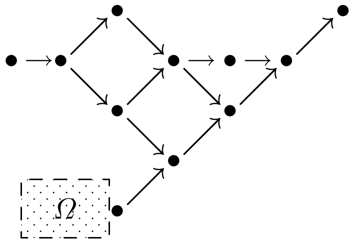
The algebra $A(Q)$ is a one-point coextension of $A(Q')$ by a thin module V with $\operatorname{Supp} V = Q' \setminus \{e\}$. By computing the Auslander–Reiten transformation $\tau = D\operatorname{Tr}$ (see [ARS]) one can check that $\operatorname{Supp}(\tau V) = b \to c \to e$, $\operatorname{Supp}(\tau^2 V) = c \to d$ and $\tau^3 V = P_e$ (a simple projective module), and the corresponding part of the Auslander–Reiten quiver of $A(Q')$ is

(4.10)



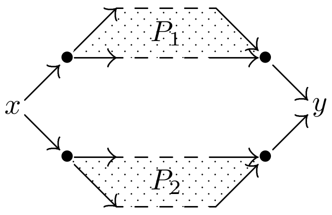
The corresponding vector space category (see [GR], [R1], [S2]) has the form

(4.11)



where Ω is the category $\operatorname{Hom}(A(Q')\text{-mod}, P_\beta)$ (see [S1], [S2] or [R1]). Observe that if, for an $A(Q')$-module X, we have $\operatorname{Hom}(X, P_\beta) \neq 0$, then $\operatorname{Supp} X \subseteq T \setminus \{\alpha, \omega\}$. Denote the poset $T \setminus \{\alpha, \omega\}$ by R. The tameness of the poset C of type (3.2b)

(3.2b)



(with $R_1 = R_2 = T$) implies the tameness of $C \setminus \{x\}$, which is a one-point coextension of $R_1 \cup R_2$ by a thin decomposable module V (with $\operatorname{Supp} V = R_1 \cup R_2$). Denote the direct summands of V by V_1, V_2. The corresponding vector space category is a disjoint union of two vector space categories $\operatorname{Hom}(A(R_1)\text{-mod}, V_1)$, $\operatorname{Hom}(A(R_2)\text{-mod}, V_2)$, for which $\operatorname{Hom}(X_1, X_2) = 0$ and $\operatorname{Hom}(X_2, X_1) = 0$ for $X_i \in \operatorname{ob}\operatorname{Hom}(A(R_i)\text{-mod}, V_i)$, $i = 1, 2$. Each of these categories is equal to Ω, and their union is of tame representation type. Hence Ω is a poset such that any of its finite subposets is a subposet of (2.9) and hence the vector space category (4.11) is of tame representation type. Therefore the poset (3.1b) is representation-tame. Similarly one can prove the tameness of each of the posets in (3.1c)–(3.1e). ■

5. The proof of the main result. For algebras of polynomial growth the Theorem is proved in [Sk3]. Thus it remains to prove it for incidence algebras of non-polynomial growth.

The implication (i)⇒(ii) is known [P1] and (ii)⇒(iii) is a direct consequence of the fact that the Tits form of any concealed algebra of wild type is not weakly non-negative [K].

Assume condition (iii) and that $A(Q)$ is of non-polynomial growth. Then, by Proposition 3.4, either Q or Q^{op} is of one of the forms (3.1), (3.2) and it follows from Proposition 4.1 that $A(Q)$ is of tame representation type. ■

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REFERENCES

[ARS] M. Auslander, I. Reiten and S. O. Smalø, *Representation Theory of Artin Algebras*, Cambridge Stud. Adv. Math. 36, Cambridge Univ. Press, 1995.

[BLS] R. Bautista, F. Larrión and L. Salmerón, *On simply connected algebras*, J. London Math. Soc. (2) 27 (1983), 212–220.

[Bo1] K. Bongartz, *Algebras and quadratic forms*, ibid. 28 (1983), 461–469.

[Bo2] —, *Critical simply connected algebras*, Manuscripta Math. 46 (1984), 117–136.

[CB] W. W. Crawley-Bovey, *Tameness of biserial algebras*, Arch. Math. (Basel) 65 (1995), 399–407.

[DS1] P. Dowbor and A. Skowroński, *On the representation type of locally bounded categories*, Tsukuba J. Math. 10 (1986), 63–72.

[DS2] —, —, *Galois coverings of representation-infinite algebras*, Comment. Math. Helv. 62 (1987), 311—337.

[D] P. Dräxler, *Completely separating algebras*, J. Algebra 165 (1994), 550–565.

[DN] P. Dräxler and R. Nörenberg (eds.), *CREP Manual*, Part II, Preprint 97-009 (1997), Univ. Bielefeld.

[Dr] Yu. A. Drozd, *Tame and wild matrix poblems*, in: Representation Theory II, Lecture Notes in Math. 832, Springer, 1980, 242–258.

[GR] P. Gabriel and A. Roiter, *Representations of Finite-Dimensional Algebras*, Algebra VIII, Encyclopedia Math. Sci. 73, Springer, 1992.

[G] C. Geiss, *On degenerations of tame and wild algebras*, Arch. Math. (Basel) 64 (1995), 11–16.

[HR] D. Happel and C. M. Ringel, *Tilted algebras*, Trans. Amer. Math. Soc. 274 (1982), 399–443.

[HV] D. Happel and D. Vossieck, *Minimal algebras of infinite representation type with preprojective component*, Manuscripta Math. 42 (1983), 221–243.

[K] O. Kerner, *Tilting wild algebras*, J. London Math. Soc. 39 (1989), 29–47.

[L] M. Lersch, *Minimal wilde Algebren*, Diplomarbeit, Düsseldorf, 1987.

[Lo] M. Loupias, *Représentations indécomposables des ensembles ordonnés finis*, thèse, Univ. François Rabelais de Tours, 1975.

[NS] R. Nörenberg and A. Skowroński, *Tame minimal non-polynomial growth simply connected algebras*, Colloq. Math. 73 (1997), 301–330.

[P1] J. A. de la Peña, *Algebras with hypercritical Tits form*, in: Topics in Algebra, Banach Center Publ. 26 (Part 1), PWN, Warszawa, 1990, 353–369.

[P2] —, *The families of two-parametric tame algebras with sincere directing modules*, in: Representations of Algebras, CMS Conf. Proc. 14, Amer. Math. Soc., 1993, 361–392.

[R1] C. M. Ringel, *Tame algebras (On algorithms for solving vector space problems, II)*, Lecture Notes in Math. 831, Springer, 1980, 137–287.

[R2] —, *Tame Algebras and Integral Quadratic Forms*, Lecture Notes in Math. 1099, Springer, 1984.

[S1] D. Simson, *Vector space categories, right peak rings and their socle projective modules*, J. Algebra 92 (1985), 532–571.

[S2] —, *Linear Representations of Partially Ordered Sets and Vector Space Categories*, Algebra Logic Appl. 4, Gordon and Breach, 1992.

[Sk1] A. Skowroński, *Group algebras of polynomial growth*, Manuscripta Math. 59 (1987), 499–516.

[Sk2] —, *Simply connected algebras and Hochschild cohomologies*, in: Representations of Algebras, CMS Conf. Proc. 14, Amer. Math. Soc., 1993, 421–447.

[Sk3] —, *Simply connected algebras of polynomial growth*, Compositio Math. 109 (1997), 99–133.

[U] L. Unger, *The connected algebras of the minimal wild hereditary algebras*, Bayreuth. Math. Schr. 31 (1990), 145–154.

[WW] B. Wald and J. Waschbüsch, *Tame biserial algebras*, J. Algebra 95 (1985), 480–500.

[Wi] J. Wittman, *Verkleidete zahme und minimal wilde Algebren*, Diplomarbeit, Bayreuth, 1990.

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