COLLOQUIUM MATHEMATICUM

VOL. 94 2002 NO. 2

HARDY'S THEOREM FOR THE HELGASON FOURIER TRANSFORM ON NONCOMPACT RANK ONE SYMMETRIC SPACES

BY

S. THANGAVELU (Bangalore)

Abstract. Let G be a semisimple Lie group with Iwasawa decomposition $G = KAN$. Let $X = G/K$ be the associated symmetric space and assume that X is of rank one. Let M be the centraliser of A in K and consider an orthonormal basis $\{Y_{\delta,j} : \delta \in \widehat{K}_0,\ 1 \le j \le d_\delta\}$ of $L^2(K/M)$ consisting of K-finite functions of type δ on K/M. For a function f on X let $\widetilde{f}(\lambda, b)$, $\lambda \in \mathbb{C}$, be the Helgason Fourier transform. Let h_t be the heat kernel associated to the Laplace–Beltrami operator and let $Q_\delta(i\lambda + \varrho)$ be the Kostant polynomials. We establish the following version of Hardy's theorem for the Helgason Fourier transform: Let f be a function on G/K which satisfies $|f(ka_r)| \le Ch_t(r)$. Further assume that for every δ and j the functions

$$F_{\delta,j}(\lambda) = Q_\delta(i\lambda + \varrho)^{-1} \int\limits_{K/M} \widetilde{f}(\lambda, b) Y_{\delta,j}(b)\, db$$

satisfy the estimates $|F_{\delta,j}(\lambda)| \le C_{\delta,j} e^{-t\lambda^2}$ for $\lambda \in \mathbb{R}$. Then f is a constant multiple of the heat kernel h_t.

1. Introduction. A classical theorem of Hardy on Fourier transform pairs says that if a nontrivial function f on $\mathbb{R}^n$ satisfies the estimates $|f(x)| \le Ce^{-a|x|^2}$ and $|\widehat{f}(\xi)| \le Ce^{-b|\xi|^2}$ for some constants $a, b \ge 0$ then $ab \le 1/4$, and if $ab = 1/4$ then f is essentially the Gaussian $e^{-a|x|^2}$. This can be viewed as a theorem on entire functions of order 2 on $\mathbb{C}^n$. In fact, if $F(\zeta)$ is an entire function of order 2 and type b on $\mathbb{C}^n$ which decays like $e^{-b|\xi|^2}$ when restricted to $\mathbb{R}^n$, then F is a constant multiple of the Gaussian $e^{-b|\xi|^2}$. The best possible result of this kind has been proved in [17].

Let us compare this with the classical Paley–Wiener theorem which characterises compactly supported smooth functions in terms of their Fourier transforms. This can be viewed as a theorem on entire functions of exponential type which have polynomial decay when restricted to $\mathbb{R}^n$. More

2000 *Mathematics Subject Classification*: Primary 22E30; Secondary 43A30.

Key words and phrases: semisimple Lie groups, symmetric spaces, Fourier transform, spherical harmonics, Jacobi functions, heat kernel.

precisely, if $F(\zeta)$ is entire and satisfies $|F(\zeta)| \leq C_N(1+|\zeta|)^{-N}e^{R|\mathrm{Im}\,\zeta|}$ for all N then $F(\xi) = \widehat{f}(\xi)$ for a smooth f supported in $|x| \leq R$. To motivate what we intend to do let us consider the following refinement of the Paley–Wiener theorem proved by Helgason [10].

We can view $\mathbb{R}^n$ as the homogeneous space $M(n)/O(n)$ where $M(n)$ is the group of all isometries of $\mathbb{R}^n$ and $O(n)$ is the orthogonal group. In this setup it is better to view the Fourier transform in polar coordinates. Writing $\xi = \lambda w$, $\lambda = |\xi|$ and $w \in S^{n-1}$, we have

$$\widehat{f}(\lambda, w) = \int_{\mathbb{R}^n} e^{-i\lambda x\cdot w} f(x)\,dx. \tag{1.1}$$

Note that $e^{-i\lambda x\cdot w}$ are all eigenfunctions of the Laplacian Δ which generates the class of all $M(n)$-invariant differential operators on $\mathbb{R}^n$. Helgason proved a Paley–Wiener theorem relating support and smoothness properties of f in terms of properties of $\widehat{f}(\lambda, w)$.

For each nonnegative integer m let $\mathcal{H}_m$ be the space of all spherical harmonics of degree m. If $\widehat{f}(\lambda, w)$ is an entire function of exponential type satisfying estimates uniformly in w and if for each $S_m \in \mathcal{H}_m$ the function

$$\lambda \mapsto \lambda^{-m} \int_{S^{n-1}} \widehat{f}(\lambda, w) S_m(w)\,dw \tag{1.2}$$

is even and holomorphic, then f is a compactly supported smooth function. In this article we are interested in a version of Hardy's theorem along these lines which serves as a motivation for a similar result on symmetric spaces.

Let G/K be a rank one symmetric space of noncompact type. The Helgason Fourier transform $\widetilde{f}(\lambda, b)$ of a function f on G/K is given by

$$\widetilde{f}(\lambda, b) = \int_{G/K} f(x) e^{(-i\lambda+\varrho)A(x,b)}\,dx \tag{1.3}$$

where $\lambda \in \mathbb{C}$ and $b \in K/M$ (see Section 3). Helgason [10] characterised compactly supported smooth functions on G/K in terms of holomorphic properties of the functions $\widetilde{f}(\lambda, b)$ and

$$\lambda \mapsto \int_{K/M} \widetilde{f}(\lambda, b) e^{(i\lambda+\varrho)A(x,b)}\,db. \tag{1.4}$$

There is a refinement of this theorem, due to Strichartz [24] and Bray [4], in terms of the spectral projections $f * \Phi_\lambda$ where Φ_λ are spherical functions on G/K.

For each irreducible unitary representation δ of K with a unique M-fixed vector, there are functions Y_δ on K/M which play the role of spherical harmonics. The holomorphic properties of the functions

$$\lambda \mapsto (Q_\delta(\varrho + i\lambda) Q_\delta(\varrho - i\lambda))^{-1} \int_K f * \Phi_\lambda(ka_r) Y_\delta(k)\, dk$$

are used in [4] to characterise compactly supported functions f on G/K. In this article we establish a version of Hardy's theorem in terms of exponential decay and growth of the functions

$$\lambda \mapsto Q_\delta(\varrho + i\lambda)^{-1} \int_{K/M} \widetilde{f}(\lambda, b) Y_\delta(b)\, db.$$

See Theorem 5.1 for the precise statement.

We conclude this introduction with the following remarks and references on Hardy's theorem. In 1933, Hardy [8] proved his theorem for the Fourier transform on the real line. The most optimal result for the Euclidean Fourier transform was proved in [17] by Pfannschmidt. Analogues of Hardy's theorem for Fourier transforms on Lie groups have attracted considerable attention in recent years. It all started with the work of Sitaram and Sundari [21] who established a Hardy theorem for certain semisimple Lie groups. For other versions of this theorem for semisimple Lie groups see Cowling *et al.* [5] and Sengupta [20]. Analogues of Hardy's theorem for the Heisenberg group have been obtained in Sitaram *et al.* [22] and Thangavelu [25]–[27]. Step two nilpotent Lie groups were considered by Bagchi and Ray [3], Astengo *et al.* [2], and general nilpotent Lie groups by Kaniuth and Kumar [13]. General symmetric spaces of noncompact type were considered by Narayanan and Ray [16] and solvable extensions of H-type groups were treated in [2]. See also the works of Sarkar [18] and [19] for semisimple groups. For the latest works of the author on Hardy's theorem and related results we refer to [28] and [29].

The author wishes to thank Ms. Ashalata for her excellent job of typing the manuscript. We also thank the referee for his thorough reading of the manuscript and for making many valuable suggestions which have greatly improved the exposition of this paper.

2. The Euclidean case. Consider the Euclidean Fourier transform written in polar coordinates as

$$\widehat{f}(\lambda, w) = \int_{\mathbb{R}^n} e^{-i\lambda x \cdot w} f(x)\, dx \tag{2.1}$$

where $w \in S^{n-1}$ and $\lambda \geq 0$. Let $\mathcal{H}_m$ be the space of spherical harmonics of degree m on S^{n-1}. Let

$$p_t(x) = (4\pi t)^{-n/2} e^{-|x|^2/(4t)}$$

be the heat kernel associated to the Laplacian on $\mathbb{R}^n$. The following is our version of Hardy's theorem for the Euclidean Fourier transform written in the above form. It should be compared with the Paley–Wiener theorem proved in Helgason [9].

THEOREM 2.1. *Let f be a measurable function on $\mathbb{R}^n$ which satisfies the estimate $|f(x)| \leq Cp_s(x)$, $s > 0$. For each nonnegative integer m and $S_m \in \mathcal{H}_m$ assume that*

$$\Big|\lambda^{-m} \int_{S^{n-1}} \widehat{f}(\lambda, w) S_m(w)\, dw\Big| \leq C_m e^{-t\lambda^2}$$

for all $\lambda > 0$ for some constants $C_m \geq 0$ and $t > 0$. Then

(i) *$f = 0$ when $s < t$;*

(ii) *f is a constant multiple of p_t when $s = t$;*

(iii) *there are infinitely many linearly independent functions satisfying the above two conditions when $s > t$.*

Proof. Consider the integral

$$\int_{S^{n-1}} \widehat{f}(\lambda, w) S_m(w)\, dw = \int_{\mathbb{R}^n} \Big(\int_{S^{n-1}} e^{i\lambda x \cdot w} S_m(w)\, dw\Big) f(x)\, dx.$$

Writing $x = rx',\ x' \in S^{n-1}$, and using the identity (see Helgason [9])

$$\int_{S^{n-1}} e^{i\lambda r x' \cdot w} S_m(w)\, dw = C_{n,m} \frac{J_{n/2+m-1}(\lambda r)}{(\lambda r)^{n/2-1}} S_m(x') \tag{2.2}$$

we get

$$\int_{S^{n-1}} \widehat{f}(\lambda, w) S_m(w)\, dw = C_{n,m} \int_0^\infty f_m(r) \frac{J_{n/2+m-1}(\lambda r)}{(\lambda r)^{n/2+m-1}} (\lambda r)^m r^{n-1}\, dr,$$

where $f_m(r)$ is defined by

$$f_m(r) = \int_{S^{n-1}} f(rx') S_m(x')\, dx' \tag{2.3}$$

and J_α stands for the Bessel function of order α.

From the above equation it follows that the function

$$F_m(\lambda) = \lambda^{-m} \int_{S^{n-1}} \widehat{f}(\lambda, w) S_m(w)\, dw$$

is an even function of $\lambda \in \mathbb{R}$ and satisfies the estimate

$$|F_m(\lambda)| \leq C_m e^{-t\lambda^2}$$

for $\lambda \in \mathbb{R}$. Using the formula for the Fourier transform of a radial function on $\mathbb{R}^n$ we infer that

$$F_m(\lambda) = C \int_{\mathbb{R}^{n+2m}} f_m(|x|)|x|^{-m} e^{-i\lambda x\cdot w}\, dx \tag{2.4}$$

for any $w \in S^{n+2m-1}$. Under the hypothesis on f, we see that $|f_m(|x|)| \leq C_m p_s(x)$ and hence F_m can be extended to the complex plane as an entire function of λ. It is easy to see that for $\lambda \in \mathbb{C}$ we have

$$|F_m(\lambda)| \leq C_m(1+|\lambda|)^{n+m-1} e^{s|\operatorname{Im}\lambda|^2},$$

which follows from the estimate $|p_s(x)| \leq Ce^{-|x|^2/(4s)}$.

We now appeal to the following complex-analytic lemma, a proof of which can be found in [16], [18].

LEMMA 2.2. *Let f be an entire function of one complex variable which satisfies the following estimates for some $a > 0$:*

(i) $|f(z)| \leq C(1+|z|)^m e^{a|\operatorname{Im} z|^2}$ *for* $z \in \mathbb{C}$;

(ii) $|f(x)| \leq C(1+|x|)^m e^{-ax^2}$ *for* $x \in \mathbb{R}$.

Then $f(z) = P(z)e^{-az^2}$ where P is a polynomial of degree $\leq m$.

Applying this lemma to $F_m(\lambda)$ we conclude that $F_m(\lambda) = C_m e^{-t\lambda^2}$ in the case when $s \leq t$. Since $F_m(\lambda)$ is the Fourier transform of $f_m(|x|)|x|^{-m}$ on $\mathbb{R}^{n+2m}$ we see that

$$f_m(|x|) = C_m|x|^m p_t(x).$$

If $s < t$, this is not compatible with the estimate $|f_m(|x|)| \leq Cp_s(x)$ unless of course $C_m = 0$ for all m. Thus we get $f = 0$ when $s < t$. Again when $s = t$ we get $f_m = 0$ for all $m > 0$ and therefore $f(x) = f_0(|x|) = C_0 p_t(x)$.

Let P be a solid harmonic of degree $k \geq 1$. When $s > t$ choose $\delta > 0$ such that $s > (1+\delta)t$ and consider

$$h_{k,\delta}(x) = P(x)p_{(1+\delta)^{-1}s}(x). \tag{2.5}$$

Then it is clear that

$$|h_{k,\delta}(x)| \leq Cp_s(x).$$

We also have $\widehat{h}_{k,\delta}(\lambda, w) = CP(\lambda w)e^{-s\lambda^2/(1+\delta)}$. It follows that for any $S \in \mathcal{H}_m$,

$$\left|\lambda^{-m} \int_{S^{n-1}} \widehat{h}_{k,\delta}(\lambda, w)S(w)\, dw\right| \leq Ce^{-t\lambda^2}$$

since $s > (1+\delta)t$. This completes the proof of Theorem 2.1.

We conclude this section with the following observation. Let $L_k^\alpha(t)$ be the Laguerre polynomials of type $\alpha > -1$ and consider the conditions

$$\Big|\lambda^{-m}\int_{S^{n-1}}\widehat{f}(\lambda,w)S_{mj}(w)\,dw\Big| \le C_{mj}|L_k^{n/2+m-1}(\lambda^2)|e^{-\lambda^2/2} \tag{2.6}$$

where $\{S_{mj} : j = 1,\dots,d_m\}$ is an orthonormal basis for $\mathcal{H}_m$. Suppose f satisfies the estimate

$$|f(x)| \le C(1+|x|^2)^N e^{-|x|^2/2}$$

where N is a nonnegative integer. Then by appealing to the general form of the complex-analytic Lemma 2.2 we can conclude that

$$\lambda^{-m}\int_{S^{n-1}}\widehat{f}(\lambda,w)S_{mj}(w)\,dw = P_{mj}(\lambda)e^{-\lambda^2/2}$$

where $P_{mj}(\lambda)$ is a polynomial satisfying

$$|P_{mj}(\lambda)| \le C_{mj}|L_k^{n/2+m-1}(\lambda^2)|.$$

As all the zeros of the Laguerre polynomial $L_k^\alpha(t)$ are real we conclude that

$$P_{mj}(\lambda) = C_{mj}L_k^{n/2+m-1}(\lambda^2).$$

If we let $f_{mj}(|x|) = \int_{S^{n-1}} f(|x|w)S_{mj}(w)\,dw$ then we conclude that the Fourier transform of $f_{mj}(|x|)|x|^{-m}$ considered as a function on $\mathbb{R}^{n+2m}$ is given by the Laguerre function. Thus we have

$$\int_{\mathbb{R}^{n+2m}} f_{mj}(|x|)|x|^{-m}e^{-i\lambda x\cdot w}\,dx = C_{mj}L_k^{n/2+m-1}(\lambda^2)e^{-\lambda^2/2}.$$

Since Laguerre functions $L_k^\alpha(t^2)e^{-t^2/2}$ are eigenfunctions of the Hankel transform we get

$$f_{mj}(|x|) = C_{mj}|x|^m L_k^{n/2+m-1}(|x|^2)e^{-|x|^2/2}.$$

The estimate on $f(x)$ implies that $f_{mj}(|x|) = 0$ for $m > 2(N-k)$. In conclusion

$$f(x) = \sum_{m=0}^{2(N-k)}\Big(\sum_{j=1}^{d_m} C_{mj}S_{mj}(x')|x|^m\Big)L_k^{n/2+m-1}(|x|^2)e^{-|x|^2/2}.$$

Thus we have

THEOREM 2.3. *Suppose f satisfies $|f(x)| \le C(1+|x|^2)^{N+k}e^{-|x|^2/2}$ for some nonnegative integers N and k, and for each $S_m \in \mathcal{H}_m$,*

$$\Big|\lambda^{-m}\int_{S^{n-1}}\widehat{f}(\lambda,w)S_m(w)\,dw\Big| \le C_m|L_k^{n/2+m-1}(\lambda^2)|e^{-\lambda^2/2}.$$

Then

$$f(x) = \Big(\sum_{m=0}^{2N} P_m(x) L_k^{n/2+m-1}(|x|^2)\Big) e^{-|x|^2/2}$$

where P_m are homogeneous harmonic polynomials of degree m.

3. Preliminaries on symmetric spaces. In this section we collect relevant material from the theory of symmetric spaces. General references for this section are the monographs [9] and [10] of Helgason.

Let $X = G/K$ be a noncompact, rank one symmetric space. The semisimple Lie group G is assumed to be connected with finite centre. Let $G = NAK$ be the Iwasawa decomposition with N nilpotent, K maximal compact and A one-dimensional. Every $g \in G$ has the unique decomposition $g = n(g) \exp A(g) k(g)$ where $A(g)$ belongs to the Lie algebra of A. Let M be the centraliser of A in K. Then the function $A(gK, kM) = A(k^{-1}g)$ is right K-invariant in g and right M-invariant in K. We use the symbols x and b to denote elements of X and K/M respectively.

In the rank one case there are two roots, denoted by γ and 2γ, and we define $\varrho = \frac{1}{2}(m_\gamma + 2m_{2\gamma})$ where m_γ and $m_{2\gamma}$ are the multiplicities of γ and 2γ. Then for each $\lambda \in \mathbb{C}$, the function $x \mapsto e^{(i\lambda+\varrho)A(x,b)}$ is a joint eigenfunction of all invariant differential operators on X. Using these functions we define the *Helgason Fourier transform* of a function by

$$\widetilde{f}(\lambda, b) = \int_X f(x) e^{(-i\lambda+\varrho)A(x,b)}\, dx \tag{3.1}$$

where dx is the measure induced from the Haar measure dg on G via

$$\int_G f(gK)\, dg = \int_X f(x)\, dx.$$

For the Helgason Fourier transform we have inversion and Plancherel theorems. For instance, the inversion formula for compactly supported smooth f says that

$$f(x) = C \int_{-\infty}^{\infty} \int_{K/M} \widetilde{f}(\lambda, b) e^{(i\lambda+\varrho)A(x,b)} |c(\lambda)|^{-2}\, d\lambda\, db.$$

Here $d\lambda$ is the usual Lebesgue measure on $\mathbb{R}$, db is the normalised measure on K/M and $c(\lambda)$ is the Harish-Chandra c-function.

In the spectral Paley–Wiener theorem proved in [4] a key role is played by certain irreducible unitary representations of K with M-fixed vectors. Let $\widehat{K}_0 \subset \widehat{K}$ stand for the set of all irreducible unitary representations of K with M-fixed vectors. Let V_δ, $\delta \in \widehat{K}_0$, be the finite-dimensional vector space on which δ is realised. Then it is known (see Kostant [15]) that V_δ contains a unique normalised M-fixed vector. Let $\{v_1,\dots,v_{d_\delta}\}$ be an orthonormal basis for V_δ with v_1 as the M-fixed vector. Define the functions

$$Y_{\delta,j}(kM)=(v_j,\delta(k)v_1)$$

on K/M for $\delta\in\widehat{K}_0$ and $1\le j\le d_\delta$. We have the following result (see Helgason [10]).

PROPOSITION 3.1. *The system* $\{Y_{\delta,j}:1\le j\le d_\delta,\ \delta\in\widehat{K}_0\}$ *is an orthonormal basis for* $L^2(K/M)$.

If we make use of the identification of K/M with the unit sphere in the Lie algebra corresponding to AN, we can get an explicit realisation of $\widehat{K}_0$. With this identification $L^2(K/M)$ has the spherical harmonic decomposition and so the functions $Y_{\delta,j}$ can be identified with spherical harmonics. The spherical harmonic decomposition leads to a parametrisation of $\widehat{K}_0$ by a pair (p,q) of integers. This was first proved by Kostant [15]; see also the works of Johnson [11] and Johnson and Wallach [12]. In the rank one case p and q are integers, $p\ge 0$ and $p\pm q$ is always even and nonnegative (see Bray [4]).

For each $\delta\in\widehat{K}_0$ and $\lambda\in\mathbb{C}$ we can define the functions

$$\Phi_{\lambda,\delta}(x)=\int_K e^{(i\lambda+\varrho)A(x,kM)}Y_{\delta,1}(kM)\,dk. \tag{3.2}$$

These are called *spherical functions of type* δ. Note that $\Phi_{\lambda,\delta}$ are K-biinvariant and they are eigenfunctions of the Laplace–Beltrami operator $\mathcal{L}$ with eigenvalue $-(\lambda^2+\varrho^2)$. When δ is the unit representation, $\Phi_{\lambda,\delta}$ is denoted by Φ_λ and is simply called the *spherical function*. This is given by

$$\Phi_\lambda(x)=\int_K e^{(i\lambda+\varrho)A(x,kM)}\,dk. \tag{3.3}$$

The spherical functions are expressible in terms of Jacobi functions (see Helgason [10]). In fact, let $\alpha=\frac12(m_\gamma+m_{2\gamma}-1)$ and $\beta=\frac12(m_{2\gamma}-1)$. Then for each $\delta\in\widehat{K}_0$ there is a pair (p,q) of integers such that

$$\Phi_{\lambda,\delta}(x)=Q_\delta(i\lambda+\varrho)(\alpha+1)_p^{-1}(\operatorname{sh}r)^p(\operatorname{ch}r)^q\varphi_\lambda^{(\alpha+p,\beta+q)}(r), \tag{3.4}$$

where $\varphi_\lambda^{(\alpha+p,\beta+q)}$ are the Jacobi functions with parameters $(\alpha+p,\beta+q)$, $(z)_m=\Gamma(z+m)/\Gamma(z)$ and Q_δ are the polynomials

$$Q_\delta(i\lambda+\varrho)=\left(\frac12(\alpha+\beta+1+i\lambda)\right)_{(p+q)/2}\left(\frac12(\alpha-\beta+1+i\lambda)\right)_{(p-q)/2} \tag{3.5}$$

(called the *Kostant polynomials*). In the above formula $r=\log a$ if $x=gK$ and $g=kak'$ is the polar decomposition of g. By abuse of notation we will denote this correspondence by writing $x=ka_r$.

We conclude this section by recalling the following formula which is crucial for us. For each $\delta \in \widehat{K}_0$ we have

$$\int\limits_K e^{(i\lambda+\varrho)A(x,k'M)} Y_{\delta,j}(k'M)\,dk' = Y_{\delta,j}(kM)\Phi_{\lambda,\delta}(a_r) \tag{3.6}$$

if $x = ka_r$. A proof can be found in Helgason [10].

4. Results from Jacobi analysis. In this section we will collect some information about Jacobi functions which are needed in the proof of Hardy's theorem for symmetric spaces. General reference for this section is Koornwinder [14]. See also Anker *et al.* [1].

When f is a K-invariant function on X the Helgason Fourier transform $\widetilde{f}(\lambda, b)$ is independent of b and is given by

$$\widetilde{f}(\lambda) = \int\limits_X f(x)\Phi_\lambda(x)\,dx. \tag{4.1}$$

Writing this in the polar form we get

$$\widetilde{f}(\lambda) = \int\limits_0^\infty f(a_r)\varphi_\lambda(r)\Delta(r)\,dr, \tag{4.2}$$

where $\Delta(r) = \Delta_{\alpha,\beta}(r) = (2\operatorname{sh} r)^{2\alpha+1}(2\operatorname{ch} r)^{2\beta+1}$ and $\varphi_\lambda(r) = \varphi_\lambda^{(\alpha,\beta)}(r)$ is the Jacobi function of type (α, β). Thus results about the spherical Fourier transforms of K-biinvariant functions on G follow from the general theory of Jacobi transform.

The Jacobi functions $\varphi_\lambda^{(\alpha,\beta)}(r)$ are defined by hypergeometric functions for all $\alpha, \beta, \lambda \in \mathbb{C}$, α not a negative integer. These functions are eigenfunctions of the Jacobi operator

$$\mathcal{L}_{\alpha,\beta} = \frac{d^2}{dr^2} + ((2\alpha+1)\coth r + (2\beta+1) + \operatorname{th} r)\,\frac{d}{dr}$$

with eigenvalues $-(\lambda^2 + \varrho^2)$ where $\varrho = \alpha + \beta + 1$. The Jacobi transform of a suitable function f on $\mathbb{R}^+$ is given by

$$\widetilde{f}(\lambda) = \int\limits_0^\infty f(r)\varphi_\lambda^{(\alpha,\beta)}(r)\Delta_{\alpha,\beta}(r)\,dr. \tag{4.3}$$

For this transform we have inversion, Plancherel and Paley–Wiener theorems. For instance we have

THEOREM 4.1. *Let α, β be real, $\alpha > -1$ and $|\beta| \le \alpha+1$. For $f \in C_0^\infty(\mathbb{R})$ which is even we have*

$$f(r) = \frac{1}{2\pi}\int\limits_0^\infty \widetilde{f}(\lambda)\varphi_\lambda^{(\alpha,\beta)}(r)|c_{\alpha,\beta}(\lambda)|^{-2}\,d\lambda$$

where $c_{\alpha,\beta}(\lambda)$ *is the Harish-Chandra c-function*

$$c_{\alpha,\beta}(\lambda) = \frac{2^{\varrho - i\lambda}\Gamma(\alpha+1)\Gamma(i\lambda)}{\Gamma\big(\frac{1}{2}(i\lambda+\varrho)\big)\Gamma\big(\frac{1}{2}(i\lambda+\alpha-\beta+1)\big)}.$$

We need asymptotic properties of the Jacobi functions. If $\operatorname{Im}\lambda < 0$, then

$$\varphi_\lambda^{(\alpha,\beta)}(r) = c_{\alpha,\beta}(\lambda) e^{(i\lambda-\varrho)r}(1+O(1)) \tag{4.4}$$

as $r \to \infty$. A more precise expansion of $\varphi_\lambda^{(\alpha,\beta)}$ in terms of Bessel functions can be found in Stanton and Tomas [23]. We will make use of the estimate

$$|\varphi_\lambda^{(\alpha,\beta)}(r)| \le C(1+r)e^{r(|\operatorname{Im}\lambda|-\varrho)} \tag{4.5}$$

valid for all $r \ge 0$ and $\lambda \in \mathbb{C}$. A proof of this estimate can be found in Flensted-Jensen [7].

The Jacobi transform and the Euclidean Fourier transform are related via the Abel transform. This transform is defined as the composition of two Weyl fractional integral operators. For $\operatorname{Re}\mu > 0$, $\tau > 0$ define

$$W_\mu^\tau f(r) = \frac{1}{\Gamma(\mu)} \int_r^\infty f(s)(\operatorname{ch}\tau s - \operatorname{ch}\tau r)^{\mu-1}\, d(\operatorname{ch}\tau s). \tag{4.6}$$

The *Abel transform* Af of a function f is then given by

$$Af(r) = 2^{3\alpha+1/2}\pi^{-1/2}\Gamma(\alpha+1) W^1_{\alpha-\beta} W^2_{\beta+1/2} f(r). \tag{4.7}$$

The Jacobi transform and the Abel transform are related by

$$\widetilde{f}(\lambda) = \int_{-\infty}^{\infty} e^{i\lambda r} Af(r)\, dr. \tag{4.8}$$

Therefore, if we can invert A then an inversion formula for the Jacobi transform can be obtained.

In order to invert the Abel transform we can make use of the fact that $\{W_\mu^\tau : \mu \in \mathbb{C}\}$ is a one-parameter group of transformations with

$$W_{-1}^\tau f(r) = -\frac{d}{d(\operatorname{ch}\tau r)} f(r).$$

Thus A^{-1} is given by

$$A^{-1} f(r) = \pi^{1/2} 2^{-3\alpha-1/2}\Gamma(\alpha+1)^{-1} W^2_{-\beta-1/2} W^1_{\beta-\alpha} f(r).$$

More explicitly, letting D_τ stand for the differential operator $-d/d(\operatorname{ch}\tau r)$, we have

$$A^{-1} f(r) = c(\alpha,\beta) D_2^{\beta+1/2} D_1^{\alpha-\beta} f(r) \tag{4.9}$$

when $\beta+1/2$ and $\alpha-\beta$ are integers. If $\alpha-\beta$ is an integer and $2\beta+1$ is an odd integer, then

$$A^{-1}f(r) = c'(\alpha,\beta)\int_r^\infty D_2^{\beta+1}D_1^{\alpha-\beta}f(s)\,\frac{d(\operatorname{ch} s)}{\sqrt{\operatorname{ch}2s-\operatorname{ch}2r}}. \tag{4.10}$$

We obtain f by applying A^{-1} to the Euclidean inverse Fourier transform of $\widetilde{f}(\lambda)$.

Let $h_t(r) = h_t^{(\alpha,\beta)}(r)$, $t>0$, $r\ge 0$, be the heat kernel associated with the operator $\mathcal{L}_{\alpha,\beta}$. Since $\varphi_\lambda^{(\alpha,\beta)}$ are eigenfunctions of the operator $\mathcal{L}_{\alpha,\beta}$, the heat kernel is defined by the condition

$$\int_0^\infty h_t(r)\varphi_\lambda^{(\alpha,\beta)}(r)\Delta_{\alpha,\beta}(r)\,dr = e^{-(\lambda^2+\varrho^2)t}. \tag{4.11}$$

By the inversion formula,

$$h_t(r) = \frac{1}{2\pi}\int_0^\infty e^{-(\lambda^2+\varrho^2)t}\varphi_\lambda^{(\alpha,\beta)}(r)|c_{\alpha,\beta}(\lambda)|^{-2}\,d\lambda. \tag{4.12}$$

In terms of the Abel transform

$$Ah_t(r) = (4\pi t)^{-1/2}e^{-\varrho^2 t}e^{-r^2/(4t)}. \tag{4.13}$$

We require the following sharp estimate on the heat kernel proved in Anker *et al.* [1].

THEOREM 4.2. *Let $\alpha\ge\beta$ be integers*, $2\beta+1\ge 0$. *Let h_t*, $t>0$, *be the heat kernel* (4.12) *associated to the operator $\mathcal{L}_{\alpha,\beta}$. Then there are constants C_1 and C_2 such that*

$$C_1 t^{-3/2}e^{-\varrho^2 t}H_t(r)\le h_t(r)\le C_2 t^{-3/2}e^{-\varrho^2 t}H_t(r)$$

where $H_t(r) = H_t^{(\alpha,\beta)}(r)$ is given by

$$H_t(r) = (1+r)(1+(1+r)/t)^{\alpha-1/2}e^{-\varrho r}e^{-r^2/(4t)},\qquad \varrho=\alpha+\beta+1.$$

In [1] the authors estimated the heat kernel associated to the Laplace–Beltrami operator on an NA group. There the parameters are given by $\alpha = (m+k-1)/2$ and $\beta = (k-1)/2$ where m is an even integer. The same proof applies to our kernels $h_t^{(\alpha,\beta)}$ under the conditions on α and β stated in the theorem. This covers all rank one symmetric spaces except the real hyperbolic case in which $\alpha = (n-2)/2$ and $\beta=-1/2$. For this case heat kernel estimates of the above type are already known (see for example Davies and Mandouvalos [6]).

5. Hardy's theorem. We are now ready to state and prove our version of Hardy's theorem for the Helgason Fourier transform on rank one symmetric spaces. Let h_t be the heat kernel associated with the Laplace–Beltrami operator on G/K.

THEOREM 5.1. *Let f be a measurable function on G/K which satisfies the following two conditions for some $s, t > 0$:*

(i) $|f(ka_r)| \le Ch_s(r)$ *for all* $ka_r \in G/K$;

(ii) *for each* $\delta \in \widehat{K}_0$ *and* $1 \le j \le d_\delta$ *the function*

$$F_{\delta,j}(\lambda) = Q_\delta(i\lambda + \varrho)^{-1} \int\limits_{K/M} \widetilde{f}(\lambda, kM) Y_{\delta,j}(kM)\, dk$$

satisfies the estimate $|F_{\delta,j}(\lambda)| \le C_{\delta,j} e^{-t\lambda^2}$ *for all* $\lambda \in \mathbb{R}$.

Then

(a) $f = 0$ *whenever* $s < t$;

(b) $f(x) = ch_t(x)$ *when* $s = t$;

(c) *there are infinitely many linearly independent functions satisfying* (i) *and* (ii) *when* $s > t$.

Proof. Let $\widetilde{F}_{\delta,j}(\lambda) = \int_{K/M} \widetilde{f}(k, b) Y_{\delta,j}(b)\, db$. Recalling the definition of $\widetilde{f}(\lambda, b)$ we have

$$\widetilde{F}_{\delta,j}(\lambda) = \int\limits_{G/K} \int\limits_{K/M} f(x) e^{(-i\lambda+\varrho)A(x,b)} Y_{\delta,j}(b)\, db\, dx.$$

Writing $x = ka_r$ and using the formula (3.6) we have

$$\widetilde{F}_{\delta,j}(\lambda) = \int\limits_{G/K} f(x) Y_{\delta,j}(kM) \Phi_{\lambda,\delta}(a_r)\, dx.$$

Integrating in polar coordinates we get the formula

$$\widetilde{F}_{\delta,j}(\lambda) = \int\limits_0^\infty f_{\delta,j}(r) \Phi_{\lambda,\delta}(a_r) \Delta_{\alpha,\beta}(r)\, dr, \tag{5.1}$$

where $\alpha,\ \beta$ are the parameters associated to the group G and

$$f_{\delta,j}(r) = \int\limits_K f(ka_r) Y_{\delta,j}(kM)\, dk. \tag{5.2}$$

Recall that for each δ there are integers p and q such that

$$\Phi_{\lambda,\delta}(a_r) = Q_\delta(i\lambda + \varrho)(\alpha+1)_p^{-1} (\operatorname{sh} r)^p (\operatorname{ch} r)^q \varphi_\lambda^{(\alpha+p,\beta+q)}(r).$$

In view of this we have

$$F_{\delta,j}(\lambda) = \frac{4^{p+q}}{(\alpha+1)_p} \int\limits_0^\infty \widetilde{f}_{\delta,j}(r) \varphi_\lambda^{(\alpha+p,\beta+q)}(r) \Delta_{\alpha+p,\beta+q}(r)\, dr \tag{5.3}$$

where we have written

$$\widetilde{f}_{\delta,j}(r) = f_{\delta,j}(r)(\operatorname{sh} r)^{-p}(\operatorname{ch} r)^{-q}. \tag{5.4}$$

Now the condition $|f(ka_r)| \le Ch_s(r)$ leads to the estimate

$$|f_{\delta,j}(r)| \le C_1(\delta,j)(1+r)(1+(1+r)/s)^{\alpha-1/2}e^{-\varrho r}e^{-r^2/(4s)}. \tag{5.5}$$

If we use this estimate in the integral defining $F_{\delta,j}(\lambda)$ then in view of the estimate (4.5) for the Jacobi functions we get

$$|F_{\delta,j}(\lambda)| \le C_2(\delta,j)\int_0^\infty (1+r)^2(1+(1+r)/s)^{\alpha-1/2}e^{-r^2/(4s)+r|\operatorname{Im}\lambda|}\,dr.$$

From this it is clear that $F_{\delta,j}(\lambda)$ extends to an entire function of order 2 which satisfies

$$|F_{\delta,j}(\lambda)| \le C_3(\delta,j)(1+|\lambda|)^{\alpha+3/2}e^{s|\operatorname{Im}\lambda|^2} \tag{5.6}$$

for all $\lambda \in \mathbb{C}$.

With this estimate and hypothesis (ii) on $F_{\delta,j}(\lambda)$ we can appeal to the complex-analytic lemma to conclude that if $s \le t$, then

$$F_{\delta,j}(\lambda) = C_4(\delta,j)e^{-t\lambda^2}.$$

But $F_{\delta,j}(\lambda)$ is the Jacobi transform of type $(\alpha+p,\beta+q)$ of the function $\widetilde{f}_{\delta,j}(r)$ and so we get, by the inversion formula for the Jacobi transform,

$$\widetilde{f}_{\delta,j}(r) = C_5(\delta,j)\int_0^\infty e^{-t\lambda^2}\varphi_\lambda^{(\alpha+p,\beta+q)}(r)|c_{\alpha+p,\beta+q}(\lambda)|^{-2}\,d\lambda. \tag{5.7}$$

If h_t^δ is the heat kernel associated to $\mathcal{L}_{\alpha+p,\beta+q}$ then we have proved

$$f_{\delta,j}(r) = C_6(\delta,j)e^{(\varrho+p+q)^2t}(\operatorname{sh} r)^p(\operatorname{ch} r)^q h_t^\delta(r). \tag{5.8}$$

Since $f_{\delta,j}$ satisfies the estimate (5.5) we conclude that

$$(\operatorname{sh} r)^p(\operatorname{ch} r)^q h_t^\delta(r) \le C_7(\delta,j)(1+(1+r)/s)^{\alpha-1/2}(1+r)e^{-\varrho r}e^{-r^2/(4s)}.$$

In view of the estimates given in Theorem 4.2 this is not possible for $s < t$ unless $C_7(\delta,j) = 0$. As this is true for all j and δ, we conclude that $f = 0$.

When $s = t$, again by Theorem 4.2 the above estimate is possible only when $p = q = 0$. Therefore, $f_{\delta,j} = 0$ for all δ except the unit representation. Hence f has to be a constant multiple of h_t.

In the case of the group $G = \mathrm{SU}(1,n)$, the integer q parametrising $\delta \in \widehat{K}_0$ can be negative. Since $\beta = 0$ in this case we have

$$\Phi_{\lambda,\delta}(r) = Q_\delta(i\lambda+\varrho)(\alpha+1)_p^{-1}(\operatorname{sh} r)^p(\operatorname{ch} r)^q\varphi_\lambda^{(n-1+p,q)}(r).$$

If q is negative we can use the relation

$$\varphi_\lambda^{(\alpha,\beta)}(r) = (2\operatorname{ch} r)^{-2\beta}\varphi_\lambda^{(\alpha,-\beta)}(r)$$

to get the formula

$$\Phi_{\lambda,\delta}(r) = 2^{-2q} Q_\delta(i\lambda + \varrho)(\alpha+1)_p^{-1}(\operatorname{sh} r)^p(\operatorname{ch} r)^{-q}\varphi_\lambda^{(n-1+p,-q)}(r)$$

and therefore, there is no problem in appealing to Theorem 4.2 for estimating the kernel h_t^δ.

Given $\delta \in \widehat{K}_0$ which is not the unit representation consider

$$f(ka_r) = Y_{\delta,1}(k)h_t^\delta(r)(\operatorname{sh} r)^p(\operatorname{ch} r)^q. \tag{5.9}$$

Then for any δ' not equivalent to δ, $F_{\delta',j}(\lambda) = 0$ and $F_{\delta,j}(\lambda) = 0$ for any $j > 1$. Since $F_{\delta,1}(\lambda) = Ce^{-t\lambda^2}$ condition (ii) of the theorem is satisfied for these functions. As in the Euclidean case, given $s > t$ choose $\varepsilon > 0$ such that $s > (1+\varepsilon)t$ and let

$$f_{p,q}(ka_r) = Y_{\delta 1}(k)h^\delta_{s/1+\varepsilon}(r)(\operatorname{sh} r)^p(\operatorname{ch} r)^q. \tag{5.10}$$

Then we see that the estimate

$$|f_{p,q}(ka_r)| \le Ch_s(r)$$

holds and as $s > (1+\varepsilon)t$ the second condition of the theorem is also satisfied for these functions. This proves (c) of the conclusion.

COROLLARY 5.2. *In the above theorem replace condition* (i) *by the estimate*

$$|f(ka_r)| \le C(1+r)^N h_t(r)$$

for some nonnegative integer N. Then $f(ka_r)$ is a finite linear combination of terms of the form $Y_{\delta,j}(k)(\operatorname{sh} r)^p(\operatorname{ch} r)^q h_t^\delta(r)$.

6. Some remarks. We would like to conclude with the following remarks concerning Theorems 2.1 and 5.1. First of all, we can prove a "spectral version" of Hardy's theorem for the Euclidean Fourier transform. Let us write down the inversion formula on $\mathbb{R}^n$ in the form

$$f(x) = C_n \int_0^\infty f * \varphi_\lambda(x)\lambda^{n-1}\,d\lambda \tag{6.1}$$

where $\varphi_\lambda(x) = 2^{n/2-1}\Gamma(n/2)J_{n/2-1}(\lambda|x|)(\lambda|x|)^{-n/2+1}$. The following version of Hardy's theorem for the spectral projections $f * \varphi_\lambda$ is an easy consequence of Theorem 2.1.

THEOREM 6.1. *Let f satisfy for some $t > 0$ the estimates $|f(x)| \le Cp_t(x)$ and*

$$\int_{\mathbb{R}^n} f * \varphi_\lambda(x)\bar{f}(x)\,dx \le Ce^{-t\lambda^2}, \qquad \lambda \in \mathbb{R}^+.$$

Then f is a constant multiple of the heat kernel p_t.

To see this, let $\{S_{mj} : 1 \le j \le d_m\}$ be an orthonormal basis for $\mathcal{H}_m$. Then using the formula (2.2) we have the addition theorem for the Bessel functions:

$$\varphi_\lambda(x-y) = \sum_{m=0}^{\infty}\sum_{j=1}^{d_m} C_{mj} S_{mj}(x') S_{mj}(y') \,\frac{J_{n/2+m-1}(\lambda|x|)}{(\lambda|x|)^{n/2-1}} \cdot \frac{J_{n/2+m-1}(\lambda|y|)}{(\lambda|y|)^{n/2-1}}$$

where $x', y' \in S^{n-1}$. From this it follows that

$$\int_{\mathbb{R}^n} f * \varphi_\lambda(x) \bar{f}(x)\,dx = \sum_{m=0}^{\infty}\sum_{j=1}^{d_m} |C_{mj}|^2 \left| \int_0^\infty f_{mj}(r)\,\frac{J_{n/2+m-1}(\lambda r)}{(\lambda r)^{n/2-1}}\, r^{n-1}\,dr\right|^2.$$

Therefore, we see that hypothesis (ii) of Theorem 2.1 is satisfied. Hence we obtain the result.

A similar version of Theorem 5.1 is also available. Consider the spectral projections $f * \Phi_\lambda$ defined by

$$f * \Phi_\lambda(x) = \int_G f(y)\Phi_\lambda(y^{-1}x)\,dy \tag{6.2}$$

where f and Φ_λ are considered as right K-invariant functions on the group G. A simple calculation shows that

$$\int_{G/K} f * \Phi_\lambda(x)\bar{f}(x)\,dx = \int_{K/M} |\widetilde{f}(\lambda, b)|^2\,db. \tag{6.3}$$

This follows from the fact that

$$f * \Phi_\lambda(x) = \int_{K/M} e^{(i\lambda+\varrho)A(x,b)}\widetilde{f}(\lambda, b)\,db.$$

Therefore, the condition

$$\Big(\int_{K/M} |\widetilde{f}(\lambda, b)|^2\,db\Big)^{1/2} \le Ce^{-t\lambda^2}$$

will guarantee that condition (ii) of Theorem 5.1 is true. Hence we have

THEOREM 6.2. *Let f satisfy for some $t > 0$ the estimates $|f(ka_r)| \le Ch_t(r)$ and*

$$\int_{G/K} f * \Phi_\lambda(x)\bar{f}(x)\,dx \le Ce^{-t\lambda^2}$$

for all $\lambda \in \mathbb{R}$. Then f is a constant multiple of the heat kernel h_t.

In view of the above remarks, this theorem is a restatement of Theorem 3.2 in [16] for the rank one case. Finally, we indicate how to get a version of Hardy's theorem for the group Fourier transform on the semisimple Lie group G.

For each $\lambda \in \mathbb{R}$ there is an irreducible unitary representation π_λ of G realised on $L^2(K/M)$ which is given explicitly by

$$\pi_\lambda(g)f(k) = e^{(i\lambda+\varrho)A(g,k)} f(\kappa(g^{-1}k))$$

where $\kappa(g)$ is the k-part of the KAN decomposition of g. These are called the *spherical principal series representations.* Define the group Fourier transform of a function f on G by

$$\widehat{f}(\lambda) = \int_G f(g)\pi_\lambda(g)\, dg.$$

For right K-invariant functions, the Plancherel measure is supported on the spherical principal series. Using the orthonormal basis $\{Y_{\delta j} : \delta \in \widehat{K}_0,\ 1 \le j \le d_\delta\}$ we calculate that

$$\|\widehat{f}(\lambda)\|^2_{\mathrm{HS}} = \int_{K/M} |\widetilde{f}(\lambda, b)|^2\, db = \int_{K/M} |\widehat{f}(\lambda)Y_0(b)|^2\, db \tag{6.4}$$

where Y_0 is the constant function corresponding to the unit representation.

Therefore, the condition on $f * \Phi_\lambda$ in the above theorem can be replaced by $\|\widehat{f}(\lambda)\|_{\mathrm{HS}} \le Ce^{-t\lambda^2}$, which gives Theorem 3.1 in [16]. If we use Theorem 5.1 we get the following refinement.

THEOREM 6.3. *Let f be a right K-invariant function on the Lie group G which satisfies the estimate $|f(ka_rk')| \le Ch_t(r)$ for some $t > 0$. Further assume that*

$$|Q_\delta(i\lambda + \varrho)^{-1}(\widehat{f}(\lambda)Y_0, Y_{\delta,j})| \le C_{\delta,j}e^{-t\lambda^2}$$

for every $\delta \in \widehat{K}_0$, $1 \le j \le d_\delta$ and $\lambda \in \mathbb{R}$. Then f is a constant multiple of h_t.

The relation between the Helgason Fourier transform and the group Fourier transform of a right K-invariant function is given by $\widetilde{f}(\lambda, b) = \widehat{f}(\lambda)Y_0(b)$. Hence

$$\int_{K/M} \widetilde{f}(\lambda, b)Y_{\delta,j}(b)\, db = (\widehat{f}(\lambda)Y_0, Y_{\delta,j})$$

and so Theorem 6.3 follows from Theorem 5.1.

REFERENCES

[1] J.-P. Anker, E. Damek and C. Yacoub, *Spherical analysis on harmonic AN groups*, Ann. Scoula Norm. Sup. Pisa 23 (1996), 643–679.

[2] F. Astengo, M. Cowling, B. Di Blasio and M. Sundari, *Hardy's uncertainty principle on some Lie groups*, J. London Math. Soc. (2) 62 (2000), 461–472.

[3] S. C. Bagchi and S. Ray, *Uncertainty principles like Hardy's theorem on some Lie groups*, J. Austral. Math. Soc. Ser. A 65 (1998), 289–302.

[4] W. Bray, *Generalized spectral projections on symmetric spaces of non-compact type*: *Paley–Wiener theorems*, J. Funct. Anal. 135 (1996), 206–232.
[5] M. Cowling, A. Sitaram and M. Sundari, *Hardy's uncertainty principles on semisimple Lie groups*, Pacific J. Math. 192 (2000), 293–296.
[6] E. B. Davies and N. Mandouvalos, *Heat kernel bounds on hyperbolic spaces and Kleinian groups*, Proc. London Math. Soc. 57 (1988), 182–208.
[7] M. Flensted-Jensen, *Paley–Wiener type theorems for a differential operator connected with symmetric spaces*, Ark. Mat. 10 (1972), 143–162.
[8] G. H. Hardy, *A theorem concerning Fourier transforms*, J. London Math. Soc. 8 (1933), 227–231.
[9] S. Helgason, *Groups and Geometric Analysis*, Academic Press, 1984.
[10] —, *Geometric Analysis on Symmetric Spaces*, Math. Surveys Monogr. 39, Amer. Math. Soc., 1994.
[11] K. D. Johnson, *Composition series and intertwining operators for the spherical principal series II*, Trans. Amer. Math. Soc. 215 (1976), 269–283.
[12] K. D. Johnson and N. Wallach, *Composition series and intertwining operators for the spherical principal series I*, ibid. 229 (1977), 137–173.
[13] E. Kaniuth and A. Kumar, *Hardy's theorem for simply connected nilpotent Lie groups*, Math. Proc. Cambridge Philos. Soc. 131 (2001), 487–494.
[14] T. H. Koornwinder, *Jacobi functions and analysis on noncompact semisimple Lie groups*, in: Special Functions: Group Theoretical Aspects and Applications, R. Askey, T. H. Koornwinder and W. Schempp (eds.), Reidel, Dordrecht, 1984, 1–85.
[15] B. Kostant, *On the existence and irreducibility of certain series of representations*, Bull. Amer. Math. Soc. 75 (1969), 627–642.
[16] E. K. Narayanan and S. Ray, *The heat kernel and Hardy's theorem on symmetric spaces of noncompact type*, Proc. Indian Acad. Sci. 112 (2002), 321–330.
[17] C. Pfannschmidt, *A generalization of the theorem of Hardy*: *A most general version of the uncertainty principle for Fourier integrals*, Math. Nachr. 182 (1996), 317–327.
[18] R. Sarkar, *Revisiting Hardy's theorems on semisimple Lie groups*, Colloq. Math., to appear.
[19] —, *Revisiting Hardy's theorem on* $\mathrm{SL}(2,\mathbb{R})$, Proc. Indian Acad. Sci., to appear.
[20] J. Sengupta, *An analogue of Hardy's theorem for semisimple Lie groups*, Proc. Amer. Math. Soc. 128 (2000), 2493–2499.
[21] A. Sitaram and M. Sundari, *An analogue of Hardy's theorem for very rapidly decreasing functions on semisimple groups*, Pacific J. Math. 177 (1997), 187–200.
[22] A. Sitaram, M. Sundari and S. Thangavelu, *Uncertainty principles on certain Lie groups*, Proc. Indian Acad. Sci. Math. Sci. 105 (1995), no. 2, 135–151.
[23] R. Stanton and P. Tomas, *Expansions for spherical functions on non-compact symmetric spaces*, Acta Math. 140 (1978), 251–271.
[24] R. Strichartz, *Harmonic analysis as spectral theory of Laplacians*, J. Funct. Anal. 87 (1989), 51–148.
[25] S. Thangavelu, *Harmonic Analysis on the Heisenberg Group*, Progr. Math. 159, Birkhäuser, Boston, 1998.
[26] —, *An analogue of Hardy's theorem for the Heisenberg group*, Colloq. Math. 87 (2001), 137–145.
[27] —, *Revisiting Hardy's theorem for the Heisenberg group*, Math. Z. 242 (2002), 761–779.
[28] —, *Uniqueness theorems for the Fourier transform on symmetric spaces of noncompact type*, preprint, 2001.

[29] S. Thangavelu, *On theorems of Hardy, Gelfand–Shilov and Beurling for semisimple groups*, preprint, 2002.

Stat-Math Division
Indian Statistical Institute
8th Mile, Mysore Road
Bangalore-560 059, India
E-mail: veluma@isibang.ac.in

Received 10 September 2001;
revised 29 May 2002 (4116)