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A NOTE ON L-SETS

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Abstract. Answering a question of Pisier, posed in [10], we construct an L-set which is not a finite union of translates of free sets.

Let G be a discrete group and $A(G) = l^2(G) * l^2(G)$ its Fourier algebra. In [2] Bożejko defines a set $E \subset G$ to be a *Leinert set* if any bounded function with support in E is a bounded multiplier of the function algebra $A(G)$ (with respect to pointwise multiplication).

In a natural way $A(G)$ is the pre-dual of the regular von Neumann algebra. Hence it is defined when an operator on $A(G)$ acts completely boundedly. Accordingly, a set $E \subset G$ is called a *strong* 2*-Leinert set* if, in addition to the above, pointwise multiplication by a bounded function with support in E is completely bounded on $A(G)$. (Originally Bożejko uses Herz–Schur multipliers, but see [3].) For the reader's convenience, among the various possible characterisations of strong 2-Leinert sets we take as definition:

Definition. A set $E \subset G$ is a *strong* 2*-Leinert set* if there exists $C > 0$ such that for all finitely supported functions $a : E \to B(H)$, where H is any Hilbert space,

$$\sum_{s\in E} \|a(s) \otimes \lambda(s)\|_{B(l^2(G,H))} \le C \max\Big\{\Big\|\Big(\sum_{s\in E} a(s)a(s)^*\Big)^{1/2}\Big\|_{B(H)}, \Big\|\Big(\sum_{s\in E} a(s)^*a(s)\Big)^{1/2}\Big\|_{B(H)}\Big\}.$$

Here $B(H)$ denotes the algebra of bounded operators on a Hilbert space H and $\lambda : G \to B(l^2(G))$ denotes the left regular representation: $\lambda(s)f(t) = f(s^{-1}t),\ f \in l^2(G)$.

Taking, in the above, $a : E \to \mathbb{C}$ to be scalar-valued, we see that on the linear span of $\{\lambda(s) : s \in E\}$, the norm operator and the Hilbert space norm $\|\cdot\|_{l^2(G)}$ are equivalent (see [9]). Those sets hence provide interesting operator spaces.

On the other hand these sets are closely connected to the question of unitarizability of uniformly bounded Hilbert space representations. It follows

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from [4] that any bounded function with support in a strong 2-Leinert set E is a coefficient of a uniformly bounded Hilbert space representation of the group. If E is infinite then in general this representation will not be unitarizable (i.e. not equivalent to a unitary representation). An example of this kind is given in [5]. Here sets satisfying Leinert's condition

$$(*)\qquad x_1^{-1}x_2\dots x_{2n-1}^{-1}x_{2n}\neq e$$
$$\forall n\in\mathbb{N} \text{ and } \forall x_1,\dots,x_{2n}\in E \text{ with } x_i\neq x_{i+1} \text{ for } i=1,\dots,2n-1$$

were used for this purpose. Those sets were the first examples of strong 2-Leinert sets, as discovered by Leinert [8], [9].

These sets were described by Akemann and Ostrand [1] to be left translates of sets which are algebraically free or unions of a free set and the group identity.

Pisier, in [10], considers lacunary sets in non-commutative groups. An *L-set*, in his language, is just a strong 2-Leinert set. Since elements in those sets satisfy only few relations he conjectures that any L-set is a finite union of left translates of free sets. We prepare to give a counterexample.

THEOREM. *Let (W,S) be a Coxeter system of large type, i.e. S generates the group W with the presentation*

$$s^2=e,\quad (ss')^{\mathrm{m(s,s')}}=e,\qquad s\neq s',\ s,s'\in S,$$

and $\mathrm{m(s,s')}>2$ *for all $s\neq s'$ $s,s'\in S$. Then S is an L-set.*

Proof. In fact, it is proved by Szwarc [11] that for each $w\in W$ the set $\mathrm{C}(w)=\{s\in S: l(ws)<l(w)\}$ contains at most two elements. Here l denotes the usual length with respect to the generating set S.

If one defines

$$\varGamma_1=\{(u,v)\in W\times W: vu\in S,\ l(u)=l(v)-1\},$$
$$\varGamma_2=\{(u,v)\in W\times W: vu\in S,\ l(u)=l(v)+1\}$$

then for any $v\in W$ one has $\operatorname{card}\{u\in W:(u,v)\in\varGamma_1\}\le 2$ and for any $u\in W$, similarly, $\operatorname{card}\{v\in W:(u,v)\in\varGamma_2\}\le 2$. Because for all elements v in a Coxeter group one has $l(vs)\in\{l(v)+1,l(v)-1\}$ for all $s\in S$, one obtains $R:=\{(u,v):vu\in S\}=\varGamma_1\cup\varGamma_2$. Now, let $a:S\to B(H)$ be finitely supported. Then

$$\sum_{s\in E}\|a(s)\otimes\lambda(s)\|_{B(l^2(W,H))}$$
$$=\sup\Big\{\sum_{s,t\in W\times W}\langle a(s)h(s^{-1}t),g(t)\rangle_H:\|h\|_{l^2(W,H)}=\|g\|_{l^2(W,H)}=1\Big\}$$
$$=\sup\Big\{\sum_{(s,t)\in R}\langle a(ts)h(s),g(t)\rangle_H:\|h\|_{l^2(W,H)}=\|g\|_{l^2(W,H)}=1\Big\}.$$

Hence we take $h, g \in l^2(W, H)$ of norm one and estimate

$$\begin{aligned}
&\Big|\sum_{(s,t)\in\Gamma_1} \langle a(ts)h(s), g(t)\rangle_H\Big| \\
&\quad \le \Big(\sum_{(s,t)\in\Gamma_1} \|a(ts)h(s)\|_H^2\Big)^{1/2}\Big(\sum_{(s,t)\in\Gamma_1} \|g(t)\|_H^2\Big)^{1/2} \\
&\quad \le \Big(\sum_{(s,t)\in W\times W} \langle a(ts)^*a(ts)h(s), h(s)\rangle_H\Big)^{1/2}\Big(2\sum_{t\in W} \|g(t)\|_H^2\Big)^{1/2} \\
&\quad \le \sqrt{2}\Big(\sum_{s\in W}\Big\langle \sum_{t\in W} a(ts)^*a(ts)h(s), h(s)\Big\rangle_H\Big)^{1/2} \\
&\quad \le \sqrt{2}\Big(\sum_{s\in W}\Big\|\Big(\sum_{t\in W} a(ts)^*a(ts)\Big)^{1/2}h(s)\Big\|_H^2\Big)^{1/2} \\
&\quad \le \sqrt{2}\Big\|\Big(\sum_{t\in W} a(t)^*a(t)\Big)^{1/2}\Big\|_{B(H)}\Big(\sum_{s\in W} \|h(s)\|_H^2\Big)^{1/2}.
\end{aligned}$$

The summation over Γ_2 can be dealt with similarly, yielding

$$\begin{aligned}
&\sum_{s\in E} \|a(s)\otimes\lambda(s)\|_{B(l^2(W,H))} \\
&\quad \le \sqrt{2}\Big\|\Big(\sum_{s\in W} a(s)a(s)^*\Big)^{1/2}\Big\|_{B(H)} + \sqrt{2}\Big\|\Big(\sum_{t\in W} a(t)^*a(t)\Big)^{1/2}\Big\|_{B(H)}.
\end{aligned}$$

From this we obtain the requirement of the definition with the constant $C = 2\sqrt{2}$. ■

COROLLARY. *There exist L-sets which are not finite unions of left translates of free sets.*

We just take a countable set of generators S of a Coxeter system of large type, with $\mathrm{m}(s, s') = 3$ for all $s \ne s'$, $s, s' \in S$. Then S cannot be a finite union of translates of free sets, since none of its two-element subsets satisfies Leinert's condition $(*)$. This would be a contradiction to the relations of the presentation.

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