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SCHUBERT VARIETIES AND REPRESENTATIONS OF DYNKIN QUIVERS

BY

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Abstract. We show that the types of singularities of Schubert varieties in the flag varieties Flag_n, $n \in \mathbb{N}$, are equivalent to the types of singularities of orbit closures for the representations of Dynkin quivers of type $\mathbb{A}$. Similarly, we prove that the types of singularities of Schubert varieties in products of Grassmannians $\mathrm{Grass}(n,a) \times \mathrm{Grass}(n,b)$, $a,b,n \in \mathbb{N}$, $a,b \leq n$, are equivalent to the types of singularities of orbit closures for the representations of Dynkin quivers of type $\mathbb{D}$. We also show that the orbit closures in representation varieties of Dynkin quivers of type $\mathbb{D}$ are normal and Cohen–Macaulay varieties.

1. Introduction. Throughout the paper k denotes a fixed algebraically closed field. All varieties considered are defined over k. Following Hesselink (see [8, (1.7)] and [1, (8.1)]) we call two pointed varieties $(\mathcal{X}, x_0)$ and $(\mathcal{Y}, y_0)$ *smoothly equivalent* if there are smooth morphisms $f : \mathcal{Z} \to \mathcal{X}$, $g : \mathcal{Z} \to \mathcal{Y}$ and a point $z_0 \in \mathcal{Z}$ with $f(z_0) = x_0$, $g(z_0) = y_0$. This is an equivalence relation and equivalence classes will be denoted by $\mathrm{Sing}(\mathcal{X}, x_0)$. If $\mathrm{Sing}(\mathcal{X}, x_0) = \mathrm{Sing}(\mathcal{Y}, y_0)$ then the variety $\mathcal{X}$ is regular (respectively normal) at x_0 if and only if the variety $\mathcal{Y}$ is regular (respectively normal) at y_0 (see [7, Section 17] for more information about smooth morphisms).

Let G be an algebraic group acting regularly on a variety X. We are interested in the types $\mathrm{Sing}(\overline{G \star x_1}, x_0)$, where x_0 and x_1 are points of X. The set of all such types will be denoted by $\mathrm{Sing}_G(X)$. Let B be a Borel subgroup of G. Any B-orbit closure in X will be called a *Schubert variety* in X. Recall that X is said to be *spherical* if it is a normal variety containing a dense open B-orbit (equivalently, a normal variety containing only a finite number of B-orbits). We will consider spherical varieties, where $G = \mathrm{Gl}_n$ is a general linear group. By B_n we denote a Borel subgroup of Gl_n.

The variety $\mathrm{Flag}_n = \mathrm{Gl}_n / B_n$ of full flags is spherical. It is known that all Schubert varieties in Gl_n / B_n have rational singularities (see for example [10]). We denote by Sing(Flag) the union $\bigcup_{n\in\mathbb{N}} \mathrm{Sing}_{B_n}(\mathrm{Flag}_n)$. Let P

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be a parabolic subgroup of Gl_n. We may assume that $B_n \subseteq P$. Using the canonical fibre bundle $\mathrm{Gl}_n/B_n \to \mathrm{Gl}_n/P$ with smooth fibre P/B_n, one can show that $\mathrm{Sing}_{B_n}(\mathrm{Gl}_n/P) \subseteq \mathrm{Sing}_{B_n}(\mathrm{Flag}_n)$.

Let $\mathrm{Grass}(n,a)$ denote the Grassmannian variety of the a-dimensional subspaces of k^n. The product $\mathrm{Grass}(n,a) \times \mathrm{Grass}(n,b)$ equipped with the diagonal action of Gl_n is also a spherical variety, for any nonnegative integers a and b with $a,b \leq n$. If $a+b \leq n$ we will denote by $\mathcal{O}(n,a,b)$ the maximal Gl_n-orbit of $\mathrm{Grass}(n,a) \times \mathrm{Grass}(n,b)$ consisting of the pairs (U,V) of subspaces of k^n such that $U \cap V = \{0\}$. Obviously, $\mathcal{O}(n,a,b)$ is also a spherical variety. We put

$$\mathrm{Sing}(\mathrm{Grass}^2) = \bigcup \{\mathrm{Sing}_{B_n}(\mathrm{Grass}(n,a) \times \mathrm{Grass}(n,b));\ a,b,n \in \mathbb{N},\ a,b \leq n\},$$
$$\mathrm{Sing}(\mathcal{O}) = \bigcup \{\mathrm{Sing}_{B_n}(\mathcal{O}(n,a,b));\ a,b,n \in \mathbb{N},\ a+b \leq n\}.$$

Let $Q = (Q_0, Q_1, s, e)$ be a finite quiver. Here Q_0 is the set of vertices, Q_1 is the set of arrows and $s,e : Q_1 \to Q_0$ are functions such that any arrow $\alpha \in Q_1$ has the starting vertex $s(\alpha)$ and the ending vertex $e(\alpha)$. We denote by $\mathrm{rep}(Q)$ the category of representations of the quiver Q. The objects of $\mathrm{rep}(Q)$ are the tuples $V = (V_i, f_\alpha)_{i \in Q_0, \alpha \in Q_1}$, where V_i are finite-dimensional vector spaces over k and $f_\alpha : V_{s(\alpha)} \to V_{e(\alpha)}$ are k-linear maps. A homomorphism between two representations $V = (V_i, f_\alpha)_{i \in Q_0, \alpha \in Q_1}$ and $W = (W_i, g_\alpha)_{i \in Q_0, \alpha \in Q_1}$ is a collection $(h_i)_{i \in Q_0}$ of linear maps $h_i : V_i \to W_i$ satisfying $h_{e(\alpha)} f_\alpha = g_\alpha h_{s(\alpha)}$ for any arrow $\alpha \in Q_1$. Furthermore, the sequence $\mathbf{dim}\, V = (\dim_k V_i)_{i \in Q_0}$ is called a *dimension vector* of V.

Let $\mathbf{d} = (d_i)_{i \in Q_0} \in \mathbb{N}^{Q_0}$ be a dimension vector. We define the affine space $\mathrm{rep}_Q(\mathbf{d})$ as the set of the tuples $V = (f_\alpha)_{\alpha \in Q_1}$, where f_α is a $d_{e(\alpha)} \times d_{s(\alpha)}$-matrix with coefficients in k for any $\alpha \in Q_1$. The product $\mathrm{Gl}(\mathbf{d}) = \prod_{i \in Q_0} \mathrm{Gl}_{d_i}$ of general linear groups acts on $\mathrm{rep}_Q(\mathbf{d})$ by conjugations

$$g \star V = (g_{e(\alpha)} f_\alpha g_{s(\alpha)}^{-1})_{\alpha \in Q_1}$$

for any $g = (g_i)_{i \in Q_0} \in \mathrm{Gl}(\mathbf{d})$ and $V = (f_\alpha)_{\alpha \in Q_1} \in \mathrm{rep}_Q(\mathbf{d})$. The orbits of this action correspond to the isomorphism classes of the representations of Q with dimension vector $\mathbf{d}$. We denote by $\mathrm{Sing}(\mathbb{A})$ and $\mathrm{Sing}(\mathbb{D})$ the set of types of singularities of $\mathrm{Gl}(\mathbf{d})$-orbit closures in $\mathrm{rep}_Q(\mathbf{d})$ for all dimension vectors $\mathbf{d} \in \mathbb{N}^{Q_0}$ and Dynkin quivers Q of type $\mathbb{A}_n$, $n \geq 1$, and $\mathbb{D}_n$, $n \geq 4$, respectively. If Q is a Dynkin quiver of type $\mathbb{A}_n$ and $\mathbf{d} \in \mathbb{N}^{Q_0}$, then the $\mathrm{Gl}(\mathbf{d})$-orbit closures in $\mathrm{rep}_Q(\mathbf{d})$ are varieties with rational singularities (see [9] and [3]). The idea of the proof is that

$$\mathrm{Sing}(\mathbb{A}) \subseteq \bigcup \{\mathrm{Sing}_{B_n}(\mathrm{Gl}_n/Q);\ Q \subseteq \mathrm{Gl}_n\text{-parabolic},\ n \in \mathbb{N}\}.$$

Observe that the latter set equals $\mathrm{Sing}(\mathrm{Flag})$. Our main result shows that the reverse inclusion also holds, and extends this to the case of Dynkin quivers of type $\mathbb{D}$.

THEOREM 1. $\mathrm{Sing}(\mathrm{Flag}) = \mathrm{Sing}(\mathbb{A})$ *and* $\mathrm{Sing}(\mathrm{Grass}^2) = \mathrm{Sing}(\mathcal{O}) = \mathrm{Sing}(\mathbb{D})$.

An interesting problem is to find if the orbit closures in representation varieties of Dynkin quivers of type $\mathbb{D}$ and $\mathbb{E}$ have rational singularities, or at least are normal or Cohen–Macaulay. We know that they are unibranch varieties, by [13, Corollary 3] and the connection between the representation varieties of a quiver Q and module varieties of the corresponding path algebra kQ, established in [4]. From [6, Theorem 2] it follows that the Schubert varieties in $\mathcal{O}(n,a,b)$ are normal and Cohen–Macaulay. Moreover, in the case of characteristic zero they have rational singularities (see [6, Remark 3]). As a consequence, we derive a new result on the geometry of orbit closures in representation varieties of Dynkin quivers of type $\mathbb{D}$.

COROLLARY 2. *Let Q be a Dynkin quiver of type $\mathbb{D}_n$ and $\mathbf{d} \in \mathbb{N}^{Q_0}$. Then the* $\mathrm{Gl}(\mathbf{d})$*-orbit closures in* $\mathrm{rep}_Q(\mathbf{d})$ *are normal and Cohen–Macaulay varieties. Furthermore, they have rational singularities if k is of characteristic zero.*

By Theorem 1, we also obtain the same result for the Schubert varieties in products of two Grassmannians.

COROLLARY 3. *The Schubert varieties in* $\mathrm{Grass}(n,a) \times \mathrm{Grass}(n,b)$, $a,b,n \in \mathbb{N}$, $a,b \le n$, *are normal and Cohen–Macaulay. In addition, they have rational singularities provided k is of characteristic zero.*

The next section contains a reduction of the proof of Theorem 1 to Proposition 11 about the existence of some special exact functors between categories of representations of quivers. Section 3 is devoted to the proof of Proposition 11. For basic background on the representation theory of algebras we refer to [2] and [11].

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2. Proof of Theorem 1. Let m and n be two positive integers and $\mathbb{M}_{n\times m}$ denote the affine space of $n \times m$-matrices with coefficients in k. The group $\mathrm{Gl}((n,m)) = \mathrm{Gl}_n \times \mathrm{Gl}_m$ acts on $\mathbb{M}_{n\times m}$ via $(g,h) \star f = gfh^{-1}$. We denote by $\mathcal{M}_{n\times m}$ the open subset in $\mathbb{M}_{n\times m}$ consisting of the matrices of rank m. Thus $\mathcal{M}_{n\times m}$ can be identified with the set of injective linear maps $k^m \to k^n$. The set $\mathcal{M}$ is empty provided $m > n$. It is well known that the canonical morphism

$$\mathcal{M}_{n\times m} \to \mathrm{Grass}(n,m), \qquad f \mapsto \mathrm{im}\, f,$$

is a Gl_n-equivariant principal Gl_m-bundle if $n \geq m$. Using this fact we shall construct more complicated principal bundles.

Let Q be a finite quiver and $\mathbf{d} \in \mathbb{N}^{Q_0}$. Then the set

$$\text{mono-rep}_Q(\mathbf{d}) = \prod_{\alpha\in Q_1} \mathcal{M}_{d_{e(\alpha)}\times d_{s(\alpha)}}$$

is a $\mathrm{Gl}(\mathbf{d})$-invariant open subset of $\mathrm{rep}_Q(\mathbf{d})$. Observe that $\text{mono-rep}_Q(\mathbf{d})$ is not empty if and only if $d_{e(\alpha)} \geq d_{s(\alpha)}$ for any arrow $\alpha \in Q_1$.

LEMMA 4. *Let Q be the equioriented Dynkin quiver of type $\mathbb{A}_n$*

$$1 \xrightarrow{\alpha_1} 2 \xrightarrow{\alpha_2} \ldots \xrightarrow{\alpha_{n-2}} (n-1) \xrightarrow{\alpha_{n-1}} n$$

and $\mathbf{d} = (d_1,\ldots,d_n) = (1,\ldots,n) \in \mathbb{N}^{Q_0}$. Then the map

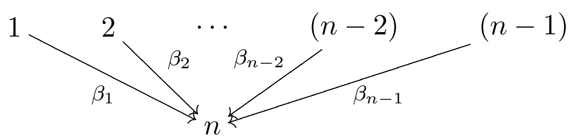
$$\pi : \text{mono-rep}_Q(\mathbf{d}) \to \mathrm{Flag}_n,$$

which sends a tuple $V = (f_{\alpha_i})_{1\leq i\leq n-1}$ to the flag

$$(0 \subset \mathrm{im}(f_{\alpha_{n-1}} f_{\alpha_{n-2}} \cdots f_{\alpha_2} f_{\alpha_1}) \subset \ldots \subset \mathrm{im}(f_{\alpha_{n-1}} f_{\alpha_{n-2}}) \subset \mathrm{im}(f_{\alpha_{n-1}})),$$

is a Gl_n-equivariant principal H-bundle, where $H = \prod_{i=1}^{n-1} \mathrm{Gl}_i$.

Proof. Let $\widetilde{Q}$ be the quiver



Since the set $\widetilde{Q}_0$ of vertices of $\widetilde{Q}$ equals Q_0, the variety $\text{mono-rep}_{\widetilde{Q}}(\mathbf{d})$ is defined. Consider the following commutative square:

$$\begin{array}{ccc} \text{mono-rep}_Q(\mathbf{d}) & \xrightarrow{\imath} & \text{mono-rep}_{\widetilde{Q}}(\mathbf{d}) \\ {\scriptstyle\pi}\downarrow & & \downarrow{\scriptstyle\varrho} \\ \mathrm{Flag}_n & \xrightarrow{\jmath} & \prod_{i=1}^{n-1} \mathrm{Grass}(n,i) \end{array}$$

where

$$\imath((f_{\alpha_i})_{1\leq i<n}) = (f_{\alpha_{n-1}} f_{\alpha_{n-2}} \cdots f_{\alpha_2} f_{\alpha_1}, \ldots, f_{\alpha_{n-1}} f_{\alpha_{n-2}}, f_{\alpha_{n-1}}),$$
$$\jmath(0 \subset V_1 \subset V_2 \subset \ldots \subset V_{n-1} \subset k^n) = (V_1, V_2, \ldots, V_{n-1}),$$
$$\varrho((f_{\beta_i})_{1\leq i<n}) = (\mathrm{im}\, f_{\beta_1}, \mathrm{im}\, f_{\beta_2}, \ldots, \mathrm{im}\, f_{\beta_{n-1}}).$$

The morphisms $\imath$ and $\jmath$ are closed immersions equivariant with respect to the action of $\mathrm{Gl}(\mathbf{d})$ and Gl_n, respectively. Moreover, the morphism ϱ is a product of the canonical morphisms

$$\mathcal{M}_{n\times i} \to \mathrm{Grass}(n,i), \qquad 1 \leq i < n.$$

Then ϱ is a Gl_n-equivariant principle H-bundle and, consequently, the same holds for π, since the above commutative square is cartesian. ■

We glue two copies of the quiver considered in the above lemma. More precisely, let $Q[n]$ denote the following Dynkin quiver of type $\mathbb{A}_{2n-1}$:

$$1 \xrightarrow{\alpha_1} 2 \xrightarrow{\alpha_2} \dots \xrightarrow{\alpha_{n-1}} n \xleftarrow{\beta_{n-1}} \dots \xleftarrow{\beta_2} (2n-2) \xleftarrow{\beta_1} (2n-1)$$

and $\mathbf{d}[n] = (1, 2, \dots, n-1, n, n-1, \dots, 2, 1) \in \mathbb{N}^{(Q[n])_0}$.

Proposition 5. $\mathrm{Sing}_{\mathrm{Gl}(\mathbf{d}[n])}(\text{mono-rep}_{Q[n]}(\mathbf{d}[n])) = \mathrm{Sing}_{B_n}(\mathrm{Flag}_n)$ *for any* $n \in \mathbb{N}$.

Proof. Consider the variety $\mathrm{Flag}_n \times \mathrm{Flag}_n$ equipped with the diagonal action of Gl_n. Since the Gl_n-equivariant projection

$$\mathrm{Flag}_n \times \mathrm{Flag}_n \to \mathrm{Flag}_n, \qquad (f, f') \mapsto f',$$

is a fibre bundle with the B_n-variety Flag_n as a typical fibre,

$$\mathrm{Sing}_{\mathrm{Gl}_n}(\mathrm{Flag}_n \times \mathrm{Flag}_n) = \mathrm{Sing}_{B_n}(\mathrm{Flag}_n).$$

We shall identify $\mathrm{Gl}(\mathbf{d}[n])$ with $\mathrm{Gl}_n \times H \times H$, where $H = \prod_{i=1}^{n-1} \mathrm{Gl}_i$. By Lemma 4, the Gl_n-equivariant morphism

$$\pi : \text{mono-rep}_{Q[n]}(\mathbf{d}[n]) \to \mathrm{Flag}_n \times \mathrm{Flag}_n,$$

which sends a tuple $V = (f_{\alpha_i}, f_{\beta_i})_{1 \le i \le n-1}$ to the pair of flags

$$(0 \subset \mathrm{im}(f_{\alpha_{n-1}} f_{\alpha_{n-2}} \cdots f_{\alpha_2} f_{\alpha_1}) \subset \dots \subset \mathrm{im}(f_{\alpha_{n-1}} f_{\alpha_{n-2}}) \subset \mathrm{im}(f_{\alpha_{n-1}}),$$
$$0 \subset \mathrm{im}(f_{\beta_{n-1}} f_{\beta_{n-2}} \cdots f_{\beta_2} f_{\beta_1}) \subset \dots \subset \mathrm{im}(f_{\beta_{n-1}} f_{\beta_{n-2}}) \subset \mathrm{im}(f_{\beta_{n-1}})),$$

is a principal $H \times H$-bundle. In particular, the inverse image π^{-1} induces a bijection, preserving closures and their types of singularities, between the set of Gl_n-orbits in $\mathrm{Flag}_n \times \mathrm{Flag}_n$ and the set of $\mathrm{Gl}(\mathbf{d}[n])$-orbits in the variety $\text{mono-rep}_{Q[n]}(\mathbf{d}[n])$. ■

As a direct consequence of the above proposition we get the inclusion $\mathrm{Sing}(\mathrm{Flag}) \subseteq \mathrm{Sing}(\mathbb{A})$. As was mentioned in Section 1, the reverse inclusion follows from [9] and [3]. However, we give a different proof, which will be easily generalized to the case of Dynkin quivers of type $\mathbb{D}$.

We shall use three operations on pairs $(Q, \mathbf{d})$, where Q is a quiver and $\mathbf{d} \in \mathbb{N}^{Q_0}$ a dimension vector. The first operation was introduced in [5, Section 5.2], and makes it possible to shrink an arrow $\alpha \in Q_1$ with $s(\alpha) \neq e(\alpha)$ and $d_{s(\alpha)} = d_{e(\alpha)}$. For example, if we perform the first operation on the quiver

$$1 \xrightarrow{\beta} 2' \underset{\gamma}{\overset{\sigma}{\rightleftarrows}} 2'' \xrightarrow{\delta} 3$$

with the dimension vector $(d_1, d_{2'}, d_{2''}, d_3) = (2, 6, 6, 9)$ and on the arrow σ, we get the quiver

$$1 \xrightarrow{\beta} 2 \xrightarrow{\delta} 3, \qquad \gamma : 2 \circlearrowleft$$

with the dimension vector $(d_1, d_2, d_3) = (2, 6, 9)$. Let $\widetilde{Q}$ denote the quiver obtained from Q by removing α and identifying the vertices $s(\alpha)$ and $e(\alpha)$. Furthermore, let $\widetilde{\mathbf{d}} \in \mathbb{N}^{\widetilde{Q}_0}$ be the dimension vector obtained from $\mathbf{d}$ by identifying the two equal coordinates $d_{s(\alpha)}$ and $d_{e(\alpha)}$.

LEMMA 6. *Assume that the pairs* $(Q, \mathbf{d})$ *and* $(\widetilde{Q}, \widetilde{\mathbf{d}})$ *are as above. Then* $\operatorname{Sing}_{\mathrm{Gl}(\mathbf{d})}(\text{mono-rep}_Q(\mathbf{d})) = \operatorname{Sing}_{\mathrm{Gl}(\widetilde{\mathbf{d}})}(\text{mono-rep}_{\widetilde{Q}}(\widetilde{\mathbf{d}}))$.

Proof. We have the obvious projection of varieties

$$\pi : \text{mono-rep}_Q(\mathbf{d}) \to \text{mono-rep}_{\widetilde{Q}}(\widetilde{\mathbf{d}}).$$

From the definition of the varieties $\text{mono-rep}_Q(\mathbf{d})$ we conclude that f_α is a bijection for any tuple $V = (f_\beta)_{\beta \in Q_1} \in \text{mono-rep}_Q(\mathbf{d})$. Hence the claim follows from [5, Section 5.2]. ■

The second operation replaces one arrow by two. Let Q be a quiver with a dimension vector $\mathbf{d} \in \mathbb{N}^{Q_0}$. Choose an arrow α in Q_1 such that $d_{e(\alpha)} - d_{s(\alpha)} \geq 2$, and an integer b such that $d_{s(\alpha)} < b < d_{e(\alpha)}$. We define a new quiver $\widetilde{Q}$ in the following way. The vertices of $\widetilde{Q}$ are all the vertices of Q and a new vertex x, while the arrows of $\widetilde{Q}$ are all the arrows of Q except α, and two new arrows $\alpha' : s(\alpha) \to x$ and $\alpha'' : x \to e(\alpha)$. Furthermore, we put $\widetilde{\mathbf{d}}$ to be the extension of the dimension vector $\mathbf{d}$ by a new coordinate $\widetilde{d}_x = b$.

LEMMA 7. *Assume that the pairs* $(Q, \mathbf{d})$ *and* $(\widetilde{Q}, \widetilde{\mathbf{d}})$ *are as above. Then any type of singularity in a closed irreducible* $\mathrm{Gl}(\mathbf{d})$*-invariant subset of the variety* $\text{mono-rep}_Q(\mathbf{d})$ *appears as a type of singularity of some closed irreducible* $\mathrm{Gl}(\widetilde{\mathbf{d}})$*-invariant subset of* $\text{mono-rep}_{\widetilde{Q}}(\widetilde{\mathbf{d}})$.

Proof. We have the obvious map

$$\pi' : \text{mono-rep}_{\widetilde{Q}}(\widetilde{\mathbf{d}}) \to \text{mono-rep}_Q(\mathbf{d}),$$

changing only two components $f_{\alpha'}$ and $f_{\alpha''}$ of a tuple $V = (f_\beta)_{\beta \in (\widetilde{Q})_1}$ into one component $f_\alpha = f_{\alpha''} f_{\alpha'}$ of $\pi'(V) = (f_\beta)_{\beta \in Q_1}$. According to the decomposition $\mathrm{Gl}(\widetilde{\mathbf{d}}) = \mathrm{Gl}(\mathbf{d}) \times \mathrm{Gl}_b$, the map π' is $\mathrm{Gl}(\mathbf{d})$-equivariant and Gl_b-invariant. Hence it suffices to show that π' is a bundle with irreducible and smooth fibre. Let $a = d_{s(\alpha)}$ and $c = d_{e(\alpha)}$. Thus $a < b < c$. Observe that the morphism π' is obtained by a base change from the morphism

$$\pi : \text{mono-rep}_{1 \xrightarrow{\alpha'} 2 \xrightarrow{\alpha''} 3}((a, b, c)) \to \text{mono-rep}_{1 \xrightarrow{\alpha} 3}((a, c)),$$

sending a pair $(f_{\alpha'}, f_{\alpha''})$ to its composition $f_{\alpha''}f_{\alpha'}$. Hence we may replace π' by π. If we assume that Gl_b acts trivially on $\text{mono-rep}_{1\xrightarrow{\gamma}3}((a,c))$, then π is $\mathrm{Gl}((a,b,c))$-equivariant. Since both varieties are $\mathrm{Gl}((a,b,c))$-orbits, the morphism π is a fibre bundle with smooth fibres. To show that the fibres are irreducible it suffices to prove that the isotropy group $\mathrm{Gl}_b \times H$ of an element $V \in \text{mono-rep}_{1\xrightarrow{\alpha}3}((a,c))$ is irreducible. The latter follows from the fact that H can be identified with an open subset of the k-algebra $\mathrm{End}_Q(V)$ of endomorphisms of the representation V. ∎

The last operation adds a new arrow. Let Q be a quiver with a dimension vector $\mathbf{d} \in \mathbb{N}^{Q_0}$. Choose a vertex $v \in Q_0$ with $d_v \geq 2$, and a positive integer a such that $a < d_v$. We define a new quiver $\widetilde{Q}$ obtained from Q by adding a new vertex x and a new arrow $\beta : x \to v$. Furthermore, let $\widetilde{\mathbf{d}}$ be the extension of the dimension vector $\mathbf{d}$ by a new coordinate $d_x = a$.

Lemma 8. *Assume that the pairs* $(Q, \mathbf{d})$ *and* $(\widetilde{Q}, \widetilde{\mathbf{d}})$ *are as above. Then any type of singularity in a closed irreducible* $\mathrm{Gl}(\mathbf{d})$*-invariant subset of the variety* $\text{mono-rep}_Q(\mathbf{d})$ *appears as a type of singularity of some closed irreducible* $\mathrm{Gl}(\widetilde{\mathbf{d}})$*-invariant subset of* $\text{mono-rep}_{\widetilde{Q}}(\widetilde{\mathbf{d}})$.

Proof. Let $b = d_v$. We have the obvious projection

$$\pi : \text{mono-rep}_{\widetilde{Q}}(\widetilde{\mathbf{d}}) \to \text{mono-rep}_Q(\mathbf{d}),$$

which is a trivial bundle with irreducible smooth fibre $\mathcal{M}_{b\times a}$. According to the decomposition $\mathrm{Gl}(\widetilde{\mathbf{d}}) = \mathrm{Gl}(\mathbf{d}) \times \mathrm{Gl}_a$, the map π is $\mathrm{Gl}(\mathbf{d})$-equivariant and Gl_a-invariant. Now the claim follows easily. ∎

Proposition 9. *Let* $m \in \mathbb{N}$, $\mathbf{d} = (d_1, \ldots, d_{2m-1}) \in \mathbb{N}^{(Q[m])_0}$ *and* $n = d_m$. *Then any type of singularity of the closure of a* $\mathrm{Gl}(\mathbf{d})$*-orbit in the variety* $\text{mono-rep}_{Q[m]}(\mathbf{d})$ *is a type of singularity of some* $\mathrm{Gl}(\mathbf{d}[n])$*-orbit closure in* $\text{mono-rep}_{Q[n]}(\mathbf{d}[n])$.

Proof. Observe that the set $\text{mono-rep}_{Q[m]}(\mathbf{d})$ is not empty if and only if

$$d_1 \leq \ldots \leq d_m \geq d_{m+1} \geq \ldots \geq d_{2m-1}.$$

We may assume that the above inequalities hold. There exists an iteration of the operations described above which leads from $(Q, \mathbf{d})$ to the pair $(Q[n], \mathbf{d}[n])$. In each intermediate step we get a Dynkin quiver $\widetilde{Q}$ of type $\mathbb{A}$. It is well known that the variety $\mathrm{rep}_{\widetilde{Q}}(\widetilde{\mathbf{d}})$ consists of finitely many $\mathrm{Gl}(\widetilde{\mathbf{d}})$-orbits. This implies that any irreducible closed $\mathrm{Gl}(\widetilde{\mathbf{d}})$-invariant subset of $\text{mono-rep}_{\widetilde{Q}}(\widetilde{\mathbf{d}})$ is the closure of some $\mathrm{Gl}(\widetilde{\mathbf{d}})$-orbit. Hence the claim is a consequence of Lemmas 6–8. ∎

We illustrate the operations used in the proof of Proposition 9 by an example. Consider the pair

$$(1' \xrightarrow{\gamma_1} 1'' \xrightarrow{\gamma_2} 4 \xleftarrow{\beta_3} 5 \xleftarrow{\beta_2} 6,\ (1,1,4,3,2)).$$

After the first operation performed on the arrow γ_1 we get

$$(1 \xrightarrow{\gamma_2} 4 \xleftarrow{\beta_3} 5 \xleftarrow{\beta_2} 6,\ (1,4,3,2)).$$

Now, if we apply the second operation to the arrow γ_2 with $b = 2$ we obtain

$$(1 \xrightarrow{\alpha_1} 2 \xrightarrow{\gamma_3} 4 \xleftarrow{\beta_3} 5 \xleftarrow{\beta_2} 6,\ (1,2,4,3,2)).$$

Applying once again the second operation, this time to the arrow γ_3 with $b = 3$, we get

$$(1 \xrightarrow{\alpha_1} 2 \xrightarrow{\alpha_2} 3 \xrightarrow{\alpha_3} 4 \xleftarrow{\beta_3} 5 \xleftarrow{\beta_2} 6,\ (1,2,3,4,3,2)).$$

Finally, after the third operation performed on the vertex 6 with $a = 1$, we obtain $(Q[4], \mathbf{d}[4])$.

Let Q and Q' be quivers and $\Phi : \mathrm{rep}(Q) \to \mathrm{rep}(Q')$ an exact functor. There exists a linear operator $\eta : \mathbb{Z}^{Q_0} \to \mathbb{Z}^{Q'_0}$ such that $\mathbf{dim}\,\Phi(U) = \eta(\mathbf{dim}\,U)$ for any $U \in \mathrm{rep}(Q)$. Furthermore, for each dimension vector $\mathbf{d} \in \mathbb{N}^{Q_0}$ there is a regular morphism

$$\Phi^{(\mathbf{d})} : \mathrm{rep}_Q(\mathbf{d}) \to \mathrm{rep}_{Q'}(\eta(\mathbf{d}))$$

corresponding to the functor Φ. We say that the functor Φ is *hom-controlled* if there is a bilinear form $\xi : \mathbb{Z}^{Q_0} \times \mathbb{Z}^{Q_0} \to \mathbb{Z}$ such that

$$\dim_k \mathrm{Hom}_{Q'}(\Phi(U), \Phi(V)) - \dim_k \mathrm{Hom}_Q(U, V) = \xi(\mathbf{dim}\,U, \mathbf{dim}\,V)$$

for any $U, V \in \mathrm{rep}(Q)$. The following result, which justifies the introduction of the above notion, follows from [12, Theorem 2] (see also [4] for the geometric relations between the varieties of representations of Q and Q', and the varieties of modules over the path algebras kQ and kQ', respectively).

PROPOSITION 10. *Let Q and Q' be quivers without oriented cycles and $\Phi : \mathrm{rep}(Q) \to \mathrm{rep}(Q')$ a hom-controlled functor. If $U, V \in \mathrm{rep}_Q(\mathbf{d})$ and $V \in \overline{\mathrm{Gl}(\mathbf{d}) \star U}$, then $\Phi^{(\mathbf{d})}(V)$ belongs to $\overline{\mathrm{Gl}(\eta(\mathbf{d})) \star \Phi^{(\mathbf{d})}(U)}$ and*

$$\mathrm{Sing}(\overline{\mathrm{Gl}(\eta(\mathbf{d})) \star \Phi^{(\mathbf{d})}(U)}, \Phi^{(\mathbf{d})}(V)) = \mathrm{Sing}(\overline{\mathrm{Gl}(\mathbf{d}) \star U}, V). \ \blacksquare$$

Recall that the *Auslander–Reiten quiver* Γ_Q of a Dynkin quiver Q is the translation quiver whose vertices are representatives of isomorphism classes of indecomposable representations of Q, there is an arrow $X \to Y$ in Γ_Q if and only if there is an irreducible map $X \to Y$ and the translation is induced by the Auslander–Reiten translate τ_Q. If $\mathcal{S}$ is a set of vertices of Γ_Q then we denote by $\operatorname{add} \mathcal{S}$ the smallest full subcategory of $\operatorname{rep}(Q)$ containing the vertices from $\mathcal{S}$ which is closed under direct sums and isomorphisms. We have the following method of constructing hom-controlled functors between the categories of representations of quivers. The proof of the proposition below is contained in Section 3.

PROPOSITION 11. *Let Q and Q' be Dynkin quivers and $F : \Gamma_Q \to \Gamma_{Q'}$ an injective morphism of translation quivers such that $F(\Gamma_Q)$ is a full subquiver of $\Gamma_{Q'}$. There exists a hom-controlled functor $\mathcal{F} : \operatorname{rep}(Q) \to \operatorname{rep}(Q')$ having the property $\mathcal{F}(M) \in \operatorname{add} \mathcal{S}$ for each $M \in \operatorname{rep}(Q)$, where $\mathcal{S}$ is the set of all vertices L of $\Gamma_{Q'}$ such that either L belongs to the image of F or there is an arrow $L \to F_X$ in $\Gamma_{Q'}$ for some nonprojective vertex X in Γ_Q.*

Let $\text{mono-rep}(Q[n])$ denote the full subcategory of $\operatorname{rep}(Q[n])$ consisting of representations $V = (V_i, f_\alpha)_{i\in(Q[n])_0,\, \alpha\in(Q[n])_1}$ such that all maps f_α are injective. We have the following observation.

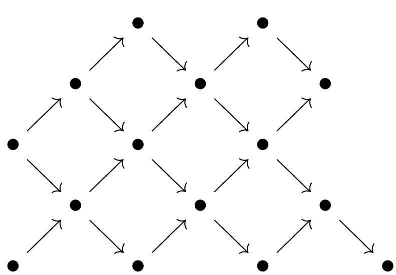
LEMMA 12. *Let Q be a Dynkin quiver of type $\mathbb{A}_n$, $n \in \mathbb{N}$. There exists an injective morphism $F : \Gamma_Q \to \Gamma_{Q[n]}$ of translation quivers such that $F(\Gamma_Q)$ is a full subquiver of $\Gamma_{Q[n]}$ and $F_X \in \text{mono-rep}(Q[n])$ for each $X \in \Gamma_Q$.*

Proof. The precise proof of the above lemma uses induction on n. Since it is an easy exercise in the representation theory of quivers of type $\mathbb{A}_n$, we will not present it here. However, in order to make things more accessible to nonexperts in representation theory we illustrate the situation by the following example.

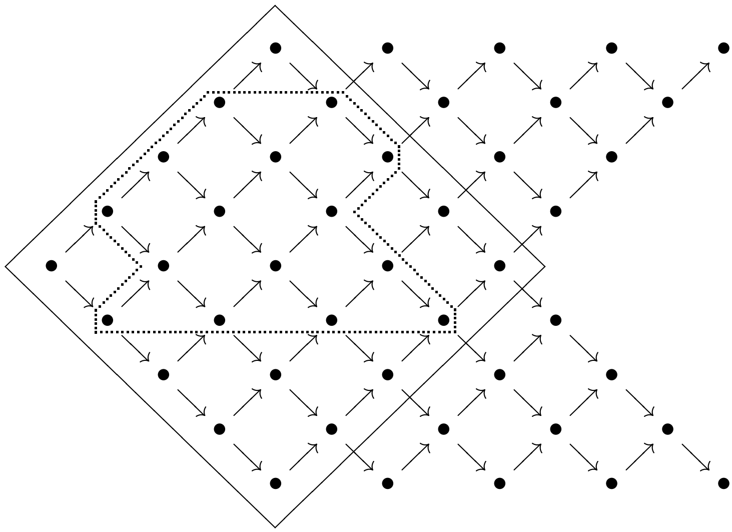
Let $n = 5$ and Q be the quiver

$$1 \to 2 \to 3 \leftarrow 4 \to 5.$$

Then Γ_Q has the form



and $\Gamma_{Q[n]}$ has the form



The vertices of $\Gamma_{Q[n]}$ which belong to mono-rep$(Q[n])$ are precisely the ones contained in the solid square. Furthermore, the dotted lines indicate the morphism F. ∎

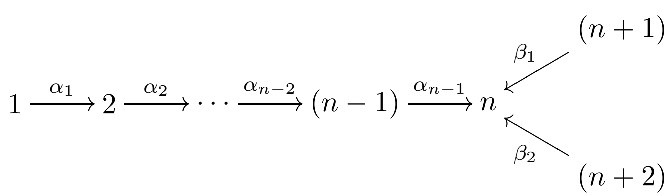
We derive the following consequence from the facts presented above.

PROPOSITION 13. *Let Q be a Dynkin quiver of type $\mathbb{A}_n$, $n \in \mathbb{N}$. Then there is a linear operator $\eta : \mathbb{Z}^{Q_0} \to \mathbb{Z}^{(Q[n])_0}$ such that for each dimension vector $\mathbf{d} \in \mathbb{N}^{Q_0}$ any type of singularity of a* $\mathrm{Gl}(\mathbf{d})$*-orbit closure in* $\mathrm{rep}_Q(\mathbf{d})$ *is the type of singularity of the closure of a* $\mathrm{Gl}(\eta(\mathbf{d}))$*-orbit in* $\text{mono-rep}_{Q[n]}(\eta(\mathbf{d}))$.

Proof. It follows from the properties of the subcategory mono-rep$(Q[n])$ that if $L \to L'$ is an arrow in $\Gamma_{Q[n]}$ with $L' \in$ mono-rep$(Q[n])$ then also $L \in$ mono-rep$(Q[n])$. Thus Lemma 12 and Proposition 11 show that there exists a hom-controlled exact functor Φ : rep$(Q) \to$ mono-rep$(Q[n])$. Using Proposition 10 we get our claim. ∎

Proof of Theorem 1. The first equality $\mathrm{Sing}(\mathbb{A}) = \mathrm{Sing}(\mathrm{Flag})$ follows from Propositions 5, 9 and 13. Obviously, $\mathrm{Sing}(\mathcal{O}) \subseteq \mathrm{Sing}(\mathrm{Grass}^2)$. We shall explain how to change these propositions in order to show the inclusions $\mathrm{Sing}(\mathrm{Grass}^2) \subseteq \mathrm{Sing}(\mathbb{D}) \subseteq \mathrm{Sing}(\mathcal{O})$.

First, we redefine $Q[n]$ to be the following Dynkin quiver of type $\mathbb{D}_{n+2}$:



and put $\mathbf{d}[n,a,b] = (1,2,\dots,(n-1),n,a,b) \in \mathbb{N}^{(Q[n])_0}$ for any $n,a,b \in \mathbb{N}$ with $a,b \le n$. For any $\mathbf{d} \in \mathbb{N}^{(Q[n])_0}$, let $\mathcal{O}\text{-rep}_{Q[n]}(\mathbf{d})$ denote the open subset of mono-rep$(Q[n])$ consisting of the tuples $V = (f_\alpha)_{\alpha \in (Q[n])_1}$ such that $\operatorname{im} f_{\beta_1} \cap \operatorname{im} f_{\beta_2} = \{0\}$. Note that $\mathcal{O}\text{-rep}_{Q[n]}(\mathbf{d})$ is nonempty if and only if $a+b \le n$. We get a new version of Proposition 5.

PROPOSITION 14. *For any $n,a,b \in \mathbb{N}$ with $a,b \le n$ we hawe*

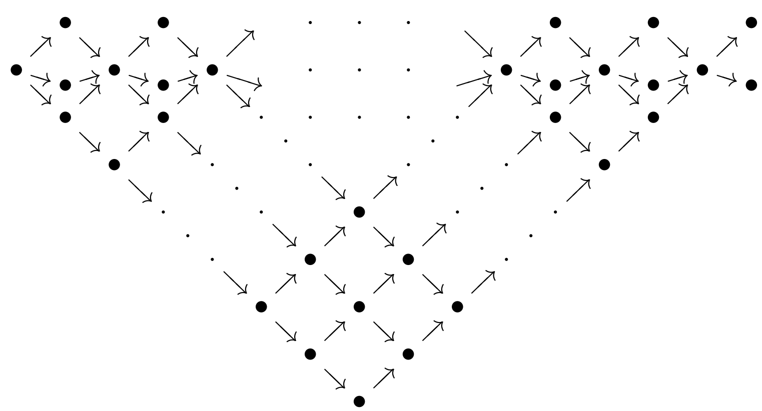
$$\operatorname{Sing}_{\mathrm{Gl}(\mathbf{d}[n,a,b])}(\text{mono-rep}_{Q[n]}(\mathbf{d}[n,a,b])) = \operatorname{Sing}_{B_n}(\operatorname{Grass}(n,a) \times \operatorname{Grass}(n,b)),$$
$$\operatorname{Sing}_{\mathrm{Gl}(\mathbf{d}[n,a,b])}(\mathcal{O}\text{-rep}_{Q[n]}(\mathbf{d}[n,a,b])) = \operatorname{Sing}_{B_n}(\mathcal{O}(n,a,b)).$$

In particular, $\operatorname{Sing}(\mathrm{Grass}^2) \subseteq \operatorname{Sing}(\mathbb{D})$. Applying the three operations to pairs $(Q[l],\mathbf{d})$, $l \in \mathbb{N}$, $\mathbf{d} \in \mathbb{N}^{(Q[l])_0}$, which do not change the two special arrows β_1 and β_2 in $Q[l]$, we get a result similar to Proposition 9.

PROPOSITION 15. *Let $m \in \mathbb{N}$, $\mathbf{d} = (d_1,\dots,d_{m+2}) \in \mathbb{N}^{(Q[m])_0}$, $n = d_m$, $a = d_{m+1}$ and $b = d_{m+2}$. Then any type of singularity of the closure of a $\mathrm{Gl}(\mathbf{d})$-orbit in the variety $\mathcal{O}\text{-rep}_{Q[m]}(\mathbf{d})$ is a type of singularity of some $\mathrm{Gl}(\mathbf{d}[n,a,b])$-orbit closure in $\mathcal{O}\text{-rep}_{Q[n]}(\mathbf{d}[n,a,b])$.*

To prove the above proposition we use Lemmas 6–8 replacing the varieties mono-rep$_Q(\mathbf{d})$ by $\mathcal{O}\text{-rep}_Q(\mathbf{d})$ for appropriate pairs $(Q,\mathbf{d})$.

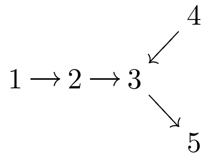
Let $\mathcal{O}\text{-rep}(Q[n])$ denote the full subcategory of mono-rep$(Q[n])$ consisting of the representations $V = (V_i, f_\alpha)_{i \in (Q[n])_0, \alpha \in (Q[n])_1}$ such that $\operatorname{im} f_{\beta_1} \cap \operatorname{im} f_{\beta_2} = \{0\}$. The full subquiver of $\Gamma_{Q[n]}$ consisting of the vertices belonging to $\mathcal{O}\text{-rep}(Q[n])$ is a translation quiver of the following shape:



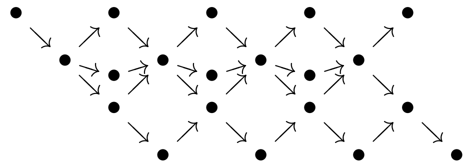
with $n+2$ τ-orbits, the longest three orbits consisting of n vertices. Let Q be a Dynkin quiver of type $\mathbb{D}_m$, $m \ge 4$. Then there is an injective morphism

$$F : \Gamma_Q \to \Gamma_{Q[n]}$$

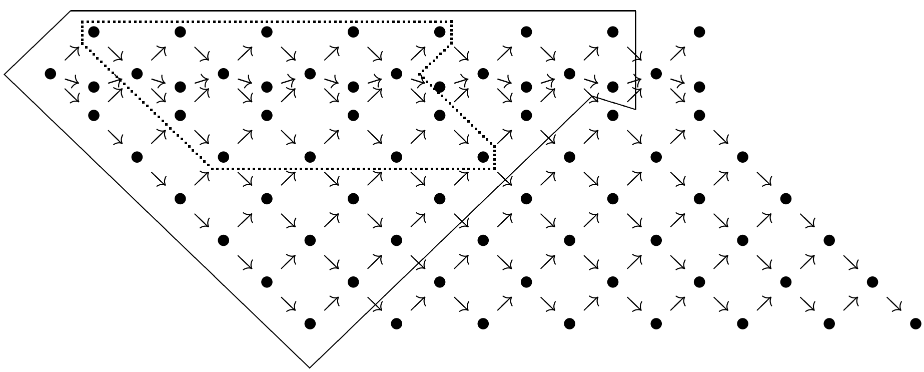
such that the vertices of $F(\Gamma_Q)$ correspond to indecomposable representations from $\mathcal{O}\text{-rep}(Q)$, where $n = 2m-3$. As an illustration of such a morphism we consider the Dynkin quiver Q of type $\mathbb{D}_5$



Then Γ_Q has the form



and $\Gamma_{Q[7]}$ has the form



The vertices of $\Gamma_{Q[7]}$ which belong to $\mathcal{O}$-rep$(Q[7])$ are precisely the ones contained in the solid polygon. Furthermore, the dotted lines indicate the morphism F. Applying Proposition 11 we get the following new version of Proposition 13.

Proposition 16. *Let Q be a Dynkin quiver of type $\mathbb{D}_m$, $m \geq 4$, and let $n = 2m-3$. Then there is a linear operator $\eta : \mathbb{Z}^{Q_0} \to \mathbb{Z}^{(Q[n])_0}$ such that for each dimension vector $\mathbf{d} \in \mathbb{N}^{Q_0}$ any type of singularity of a* $\mathrm{Gl}(\mathbf{d})$*-orbit closure in* $\mathrm{rep}_Q(\mathbf{d})$ *is the type of singularity of the closure of a* $\mathrm{Gl}(\eta(\mathbf{d}))$*-orbit in* $\mathcal{O}$-$\mathrm{rep}_{Q[n]}(\eta(\mathbf{d}))$.

Combining Propositions 14–16 we get $\mathrm{Sing}(\mathbb{D}) \subseteq \mathrm{Sing}(\mathcal{O})$. ∎

3. Proof of Proposition 11. Throughout this section, by an *algebra* we will mean a finite-dimensional k-algebra and by a *module* a finite-dimensional left module. We will denote the category of A-modules by $\mathrm{mod}\, A$. All categories considered will be k-categories and functors will be k-functors. If $\mathcal{A}$ is a category then we denote by $\mathrm{rad}_{\mathcal{A}}$ the Jacobson radical of $\mathcal{A}$. For an algebra A we abbreviate $\mathrm{rad}_{\mathrm{mod}\, A}$ by rad_A.

Let A be an algebra. We will denote by Γ_A the *Auslander–Reiten quiver* of A, i.e. the translation quiver whose vertices are representatives of isomorphism classes of indecomposable A-modules, the number of arrows between X and Y equals $\dim_k \operatorname{rad}_A(X,Y)/\operatorname{rad}_A^2(X,Y)$, and the translation is induced by the Auslander–Reiten translation τ_A.

Recall that if A and B are algebras then a functor $\mathcal{F} : \operatorname{mod} B \to \operatorname{mod} A$ is *exact* provided for each exact sequence

$$\theta : 0 \to M'' \xrightarrow{g} M \xrightarrow{f} M' \to 0 \tag{1}$$

in $\operatorname{mod} B$, the induced sequence

$$\mathcal{F}\theta : 0 \to \mathcal{F}M'' \xrightarrow{\mathcal{F}g} \mathcal{F}M \xrightarrow{\mathcal{F}f} \mathcal{F}M' \to 0$$

in $\operatorname{mod} A$ is also exact. Moreover, we call the functor $\mathcal{F}$ *hom-controlled* if $\mathcal{F}$ is exact and there exists a bilinear form $\xi : K_0(\operatorname{mod} B) \times K_0(\operatorname{mod} B) \to \mathbb{Z}$ such that

$$[\mathcal{F}M', \mathcal{F}M'']_A - [M', M'']_B = \xi(\mathbf{dim}\, M', \mathbf{dim}\, M'')$$

for any $M', M'' \in \operatorname{mod} B$. Here and later we use the notation $[U,V]_{\mathcal{A}} = \dim_k \operatorname{Hom}_{\mathcal{A}}(U,V)$ for two objects U and V of a category $\mathcal{A}$ and we abbreviate $[U,V]_{\operatorname{mod} A}$ by $[U,V]_A$ if U and V are modules over an algebra A. For a sequence (1) in $\operatorname{mod} B$ and $N \in \operatorname{mod} B$ we define

$$\delta_\theta(N) = [M' \oplus M'', N]_B - [M, N]_B$$

and dually

$$\delta^\theta(N) = [N, M' \oplus M'']_B - [N, M]_B.$$

From the definition of the Grothendieck group $K_0(\operatorname{mod} B)$ it follows that an exact functor $\mathcal{F} : \operatorname{mod} B \to \operatorname{mod} A$ is hom-controlled if and only if

$$\delta_{\mathcal{F}\theta}(\mathcal{F}N) = \delta_\theta(N) \quad \text{and} \quad \delta^{\mathcal{F}\theta}(\mathcal{F}N) = \delta^\theta(N)$$

for each short exact sequence θ in $\operatorname{mod} B$ and $N \in \operatorname{mod} B$.

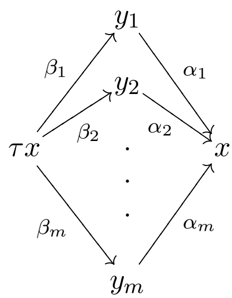
Recall that an algebra B is *representation directed* provided the Auslander–Reiten quiver Γ_B of B is a finite directed quiver. The aim of this section is to prove the following.

Proposition 17. *Let A be an algebra, B a representation directed algebra and $F : \Gamma_B \to \Gamma_A$ an injective morphism of translation quivers such that $F(\Gamma_B)$ is a full subquiver of Γ_A. Then there exists a hom-controlled exact functor $\mathcal{F} : \operatorname{mod} B \to \operatorname{mod} A$ having the property $\mathcal{F}(M) \in \operatorname{add} \mathcal{S}$ for each $M \in \operatorname{mod} B$, where $\mathcal{S}$ is the set of all vertices L of Γ_A such that either L belongs to the image of F or there is an arrow $L \to F_X$ in Γ_A for some nonprojective vertex X in Γ_B.*

As a consequence of the above we obtain Proposition 11. Indeed, for a Dynkin quiver Q the category $\operatorname{rep}(Q)$ is equivalent to the category $\operatorname{mod} kQ$, where kQ is the path algebra of Q. Using this equivalence we may identify Γ_Q with Γ_{kQ} and $\mathbb{Z}^{Q_0}$ with $K_0(kQ)$. Finally, Γ_Q is a finite directed quiver provided Q is a Dynkin quiver.

Given a translation quiver Γ we will denote by Γ_0 the set of vertices of Γ, by Γ_1 the set of arrows of Γ and by Γ_0' the set of nonprojective vertices of Γ. We will also denote by $k\Gamma$ the *mesh category* of Γ, i.e. the path category of Γ modulo the mesh ideal. Recall (see for example [11]) that the *path category* of Γ is a Krull–Schmidt category whose indecomposable objects are the vertices of Γ, for two vertices x and y of Γ the homomorphism space is the linear space with basis formed by all paths from x to y, and the composition of maps is induced by the composition of paths. The *mesh ideal* is the ideal in the path category of Γ generated by all maps of the form $\sum_{i=1}^{m} \alpha_i \beta_i$, where

(2)



is a mesh in Γ.

Let Γ be a translation quiver and $\mathcal{A}$ a category. A *representation* $G : \Gamma \to \mathcal{A}$ of Γ in $\mathcal{A}$ is a system $(G_x, G_\alpha)_{x\in\Gamma_0,\,\alpha\in\Gamma_1}$ of objects G_x, $x \in \Gamma_0$, of $\mathcal{A}$, and maps $G_\alpha : G_x \to G_y$, $\alpha : x \to y$, in $\mathcal{A}$. If the representation G satisfies the mesh relations, that is, $\sum_{i=1}^{m} G_{\alpha_i} G_{\beta_i} = 0$ for each mesh in Γ of the form (2), then we have an induced functor $k\Gamma \to \mathcal{A}$ defined in the obvious way.

Let A be an algebra and $G : \Gamma \to \operatorname{mod} A$ a representation of a translation quiver Γ. We call G *exact* if for each mesh θ in Γ of the form (2) the induced sequence in $\operatorname{mod} A$

$$G\theta : 0 \to G_{\tau x} \xrightarrow{(G_{\beta_i})^{\mathrm{tr}}} \bigoplus_{i=1}^{m} G_{y_i} \xrightarrow{(G_{\alpha_i})} G_x \to 0$$

is exact. We call the representation G *hom-controlled* if G is exact and

$$\delta_{G\theta}(G_z) = \delta_\theta(z) \quad \text{and} \quad \delta^{G\theta}(G_z) = \delta^\theta(z)$$

for each mesh θ and vertex z in Γ, where for a mesh θ of the form (2) we put

$$\delta_\theta(z) = [x,z]_{k\Gamma} + [\tau x, z]_{k\Gamma} - \sum_{i=1}^{m} [y_i, z]_{k\Gamma},$$
$$\delta^\theta(z) = [z,x]_{k\Gamma} + [z, \tau x]_{k\Gamma} - \sum_{i=1}^{m} [z, y_i]_{k\Gamma}.$$

Note that each exact representation $G : \Gamma \to \operatorname{mod} A$ satisfies the mesh relations, thus G induces a functor $k\Gamma \to \operatorname{mod} A$. Recall that an algebra A is called *standard* provided the categories $k\Gamma_A$ and $\operatorname{mod} A$ are equivalent. Hence, if A is a standard algebra and $G : \Gamma_A \to \mathcal{A}$ is an exact representation then we have an induced functor $\operatorname{mod} A \to \mathcal{A}$. The following observation is fundamental for the proof of Proposition 17.

LEMMA 18. *Let B be a representation finite standard algebra and A an algebra. If $G : \Gamma_B \to \operatorname{mod} A$ is a hom-controlled representation then the induced functor $\mathcal{F} : \operatorname{mod} B \to \operatorname{mod} A$ is hom-controlled as well.*

Proof. For a short exact sequence θ in $\operatorname{mod} B$ let $\Delta_\theta := \sum_{X \in (\Gamma_B)_0} \delta_\theta(X)$. We prove inductively on Δ_θ that the sequence $\mathcal{F}\theta$ is exact and $\delta_{\mathcal{F}\theta}(\mathcal{F}X) = \delta_\theta(X)$ and $\delta^{\mathcal{F}\theta}(\mathcal{F}X) = \delta^\theta(X)$ for each $X \in (\Gamma_B)_0$. Since every B-module is a direct sum of indecomposable ones and the vertices of Γ_B constitute a full set of representatives of isomorphism classes of indecomposable B-modules, this will finish the proof.

Note that $\Delta_\theta = 0$ if and only if θ splits. Thus we may assume $\Delta_\theta > 0$ and for any short exact sequence θ' in $\operatorname{mod} B$ with $\Delta_{\theta'} < \Delta_\theta$ the sequence $\mathcal{F}\theta'$ is exact and we have $\delta_{\mathcal{F}\theta'}(\mathcal{F}X) = \delta_{\theta'}(X)$ and $\delta^{\mathcal{F}\theta'}(\mathcal{F}X) = \delta^{\theta'}(X)$ for each $X \in (\Gamma_B)_0$. Let θ be of the form

$$0 \to M'' \xrightarrow{g} M \xrightarrow{f} M' \to 0.$$

Since $\Delta_\theta > 0$ the sequence θ does not split. In particular, there exists an indecomposable direct summand M_1' of M' such that the map $p_1 f$ does not split, where $p_1 : M' \to M_1'$ is the appropriate projection. Let $M_2' = \ker p_1$ and $p_2 : M \to M_2'$ be the projection along M_1', that is, $\binom{p_1}{p_2} : M' \to M_1' \oplus M_2' = M'$ is the identity map. Since $p_1 f$ is an epimorphism which does not split, M_1' cannot be projective. Let

$$\theta_1 : 0 \to \tau_B M_1' \xrightarrow{v} N \xrightarrow{u} M_1' \to 0$$

be an Auslander–Reiten sequence. Using the fact that $p_1 f$ is not a split epimorphism we get a map $h : M \to N$ such that $p_1 f = uh$. Denote by $h'' : M'' \to \tau_B M_1'$ the map induced by h, that is, $vh'' = hg$. Because

$$\begin{array}{ccccccccc}
0 & \longrightarrow & M'' & \xrightarrow{g} & M & \xrightarrow{f} & M' & \longrightarrow & 0 \\
 & & \big\downarrow{\scriptstyle h''} & & \big\downarrow{\scriptstyle \left(\begin{smallmatrix} h \\ p_2 f \end{smallmatrix}\right)} & & \big\| & & \\
0 & \longrightarrow & \tau_B M_1' & \xrightarrow{\left(\begin{smallmatrix} v \\ 0 \end{smallmatrix}\right)} & N \oplus M_2' & \xrightarrow{\left(\begin{smallmatrix} u & 0 \\ 0 & \mathrm{Id}_{M_2''} \end{smallmatrix}\right)} & M_1' \oplus M_2' & \longrightarrow & 0
\end{array}$$

is a commutative diagram with exact rows, the sequence

$$\theta_2 : 0 \to M'' \xrightarrow{\left(\begin{smallmatrix} g \\ -h'' \end{smallmatrix}\right)} M \oplus \tau_B M_1' \xrightarrow{\left(\begin{smallmatrix} h & v \\ p_2 f & 0 \end{smallmatrix}\right)} N \oplus M_2' \to 0$$

is exact. By easy calculations we have $\delta_\theta(X) = \delta_{\theta_1}(X) + \delta_{\theta_2}(X)$ and $\delta^\theta(X) = \delta^{\theta_1}(X) + \delta^{\theta_2}(X)$ for each $X \in (\Gamma_B)_0$. In particular, we have $\Delta_\theta = \Delta_{\theta_1} + \Delta_{\theta_2}$. Since Auslander–Reiten sequences do not split we have $\Delta_{\theta_1} > 0$. Consequently, $\Delta_{\theta_2} < \Delta_\theta$, and by the inductive hypothesis the sequence $\mathcal{F}\theta_2$ is exact and we have $\delta_{\mathcal{F}\theta_2}(\mathcal{F}X) = \delta_{\theta_2}(X)$ and $\delta^{\mathcal{F}\theta_2}(\mathcal{F}X) = \delta^{\theta_2}(X)$ for each $X \in (\Gamma_B)_0$. Moreover, from the assumptions on G it follows that $\mathcal{F}\theta_1$ is exact and we have $\delta_{\mathcal{F}\theta_1}(\mathcal{F}X) = \delta_{\theta_1}(X)$ and $\delta^{\mathcal{F}\theta_1}(\mathcal{F}X) = \delta^{\theta_1}(X)$ for each $X \in (\Gamma_B)_0$. Since $\mathcal{F}\theta_1$ and $\mathcal{F}\theta_2$ are exact, the commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathcal{F}M'' & \xrightarrow{\mathcal{F}g} & \mathcal{F}M & \xrightarrow{\mathcal{F}f} & \mathcal{F}M' & \longrightarrow & 0 \\
 & & \big\downarrow{\scriptstyle \mathcal{F}h''} & & \big\downarrow{\scriptstyle \left(\begin{smallmatrix} \mathcal{F}h \\ \mathcal{F}(p_2 f) \end{smallmatrix}\right)} & & \big\| & & \\
0 & \longrightarrow & \mathcal{F}(\tau_B M_1') & \xrightarrow{\left(\begin{smallmatrix} \mathcal{F}v \\ 0 \end{smallmatrix}\right)} & \mathcal{F}N \oplus \mathcal{F}M_2' & \xrightarrow{\left(\begin{smallmatrix} \mathcal{F}u & 0 \\ 0 & \mathrm{Id}_{\mathcal{F}M_2''} \end{smallmatrix}\right)} & \mathcal{F}M_1' \oplus \mathcal{F}M_2' & \longrightarrow & 0
\end{array}$$

shows that $\mathcal{F}\theta$ is exact. Finally, by obvious calculations we get

$$\delta_{\mathcal{F}\theta}(\mathcal{F}X) = \delta_{\mathcal{F}\theta_1}(\mathcal{F}X) + \delta_{\mathcal{F}\theta_2}(\mathcal{F}X) = \delta_{\theta_1}(X) + \delta_{\theta_2}(X) = \delta_\theta(X)$$

and

$$\delta^{\mathcal{F}\theta}(\mathcal{F}X) = \delta^{\mathcal{F}\theta_1}(\mathcal{F}X) + \delta^{\mathcal{F}\theta_2}(\mathcal{F}X) = \delta^{\theta_1}(X) + \delta^{\theta_2}(X) = \delta^\theta(X)$$

for any $X \in (\Gamma_B)_0$. ■

From now on we use the notation of Proposition 17. Since B is a representation directed algebra, we know that B is standard (see for example [11, Lemma 2.3.3]). In particular, we may identify the arrows of Γ_B with the corresponding maps in $\operatorname{mod} B$.

Let $p_Z : P_Z \to Z$ be a projective cover of an indecomposable B-module Z. Then for any B-module M, the map p_Z induces the inclusion $(p_Z, M)_B : (Z, M)_B \to (P_Z, M)_B$ of k-vector spaces, where $(-, M)_B = \operatorname{Hom}_B(-, M)$. We will denote $\operatorname{Coker}(p_Z, M)_B$ by $\langle Z, M\rangle$. If $f : M \to N$ is a B-homomorphism then f induces a map $\langle Z, M\rangle \to \langle Z, N\rangle$ which will be denoted by $\langle Z, f\rangle$. It easily follows that in this way we obtain a covariant functor $\langle Z, -\rangle : \operatorname{mod} B \to \operatorname{mod} k$. Other properties of $\langle Z, -\rangle$ are the following.

Lemma 19. *Let Z be an indecomposable B-module and*

$$0 \to M'' \xrightarrow{g} M \xrightarrow{f} M' \to 0 \tag{3}$$

an exact sequence in $\operatorname{mod} B$. *Then* $\langle Z, g\rangle$ *is a monomorphism*, $\langle Z, f\rangle$ *is an epimorphism and* $\langle Z, f\rangle\langle Z, g\rangle = 0$. *Moreover*, *if the sequence* (3) *is an Auslander–Reiten sequence then* $\operatorname{im}\langle Z, g\rangle = \ker\langle Z, f\rangle$ *if* $Z \not\simeq M'$ *and* $\ker\langle Z, f\rangle/\operatorname{im}\langle Z, g\rangle \simeq k$ *if* $Z \simeq M'$.

Proof. We have the following commutative diagram with exact rows and vertical maps being monomorphisms:

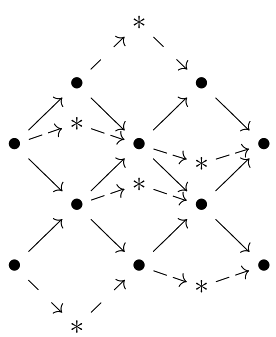
$$\begin{array}{ccccccccc} 0 & \longrightarrow & (Z,M'')_B & \xrightarrow{(Z,g)_B} & (Z,M) & \xrightarrow{(Z,f)_B} & (Z,M')_B & & \\ & & \downarrow{\scriptstyle (p_Z,M'')_B} & & \downarrow{\scriptstyle (p_Z,M)_B} & & \downarrow{\scriptstyle (p_Z,M')_B} & & \\ 0 & \longrightarrow & (P_Z,M'')_B & \xrightarrow{(P_Z,g)_B} & (P_Z,M) & \xrightarrow{(P_Z,f)_B} & (P_Z,M')_B & \longrightarrow & 0 \end{array}$$

This implies the first part of the lemma. Assume now that (3) is an Auslander–Reiten sequence. Then $\operatorname{im}(Z, f)_B = \operatorname{rad}_B(Z, M')$ and

$$\dim_k \operatorname{Hom}_B(Z, M')/\operatorname{rad}_B(Z, M') = \begin{cases} 1, & Z \simeq M', \\ 0, & Z \not\simeq M'. \end{cases} \blacksquare$$

We define a translation quiver Γ in the following way. The vertices of Γ are $X \in (\Gamma_B)_0$, and W_X for $X \in (\Gamma_B)'_0$, the arrows are $\alpha \in (\Gamma_B)_1$ and $\psi_X : \tau_B X \to W_X$, $\phi_X : W_X \to X$ for $X \in (\Gamma_B)'_0$, and the translation in Γ is just τ_B. Recall that $(\Gamma_B)'_0$ denotes the set of nonprojective vertices of Γ_B. Note that all new vertices of Γ are projective-injective ones. As Γ_B is a finite directed quiver, the same holds for Γ. Consequently, there exists a positive integer n such that $\operatorname{rad}^n_{k\Gamma} = 0$, since for any vertices W' and W'' of Γ, $\operatorname{rad}^m_{k\Gamma}(W', W'')$ is a k-linear subspace of $\operatorname{Hom}_{k\Gamma}(W', W'')$ spanned by all paths from W' to W'' of length at least m.

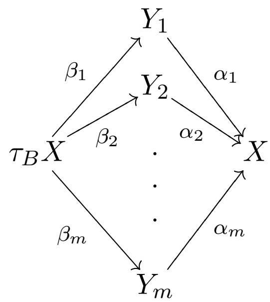
We illustrate the above construction by the following example. Let B be the path algebra of the quiver $\bullet \to \bullet \leftarrow \bullet \to \bullet$. Then Γ has the form



where we denoted by asterisks the new vertices and by dashed arrows the new arrows.

If

(4)



is a mesh in Γ_B, then we put $V_X = \bigoplus_{i=1}^m Y_i \oplus W_X$ and set

$$\nu_X = \begin{pmatrix} \beta_1 \\ \vdots \\ \beta_m \\ \psi_X \end{pmatrix} : \tau_B X \to Y_1 \oplus \ldots \oplus Y_m \oplus W_X,$$

$$\mu_X = (\alpha_1, \ldots, \alpha_m, \phi_X) : Y_1 \oplus \ldots \oplus Y_m \oplus W_X \to X.$$

We want to define a representation $G : \Gamma_B \to k\Gamma$ having some desired properties to be stated later. First we define G_X for the vertices X of Γ_B. We put

$$G_X = X \oplus U_X, \quad \text{where} \quad U_X = \bigoplus_{Z \in (\Gamma_B)'_0} W_Z \otimes_k \langle Z, X \rangle.$$

If $\alpha : X \to Y$ is an arrow in Γ_B, then we will denote by $U_\alpha : U_X \to U_Y$ the map induced by the maps $W_Z \otimes_k \langle Z, \alpha \rangle : W_Z \otimes_k \langle Z, X \rangle \to W_Z \otimes_k \langle Z, Y \rangle$, $Z \in (\Gamma_B)'_0$. The crucial properties of the above assignment are collected in the following lemma.

Lemma 20. *Consider a mesh in* Γ_B *of the form* (4). *Then there exists an isomorphism* $h_X : \bigoplus_{i=1}^m U_{Y_i} \to W_X \oplus U_{\tau_B X} \oplus U_X$ *with the following properties*:

(1) *The composition* $U_{\tau_B X} \xrightarrow{(U_{\beta_i})^{\mathrm{tr}}} \bigoplus_{i=1}^m U_{Y_i} \xrightarrow{h_X} W_X \oplus U_{\tau_B X} \oplus U_X$ *equals* $\begin{pmatrix} 0 \\ \mathrm{Id}_{U_{\tau_B X}} \\ 0 \end{pmatrix}$.

(2) *The composition* $W_X \oplus U_{\tau_B X} \oplus U_X \xrightarrow{h_X^{-1}} \bigoplus_{i=1}^m U_{Y_i} \xrightarrow{(U_{\alpha_i})} U_X$ *equals* $(0, 0, \mathrm{Id}_{U_X})$.

Proof. This is a direct consequence of the properties of the functor $\langle Z, - \rangle$ presented in Lemma 19 and the well known fact that monomorphisms and epimorphisms in $\mathrm{mod}\, k$ split. ∎

For simplicity we will identify $\bigoplus U_{Y_i}$ and $W_X \oplus U_{\tau_B X} \oplus U_X$ via h_X. We denote by γ_{X,Y_i} and $\sigma_{Y_i,X}$ the maps $W_X \to U_{Y_i}$ and $U_{Y_i} \to W_X$ induced by the equality $\bigoplus_{i=1}^m U_{Y_i} = W_X \oplus U_{\tau_B X} \oplus U_X$.

We want to define representations $G^{(n)} : \Gamma_B \to k\Gamma$, $n \geq 2$, such that $G^{(n)}_X = G_X$, $X \in (\Gamma_B)_0$, and for each mesh in Γ_B of the form (4) we have

$$\text{(I) } (G^{(n)}_{\beta_i})^{\mathrm{tr}} = \begin{pmatrix} \nu_X & \Psi_{12} \\ 0 & \Psi_{22} \\ \Psi_{31} & \Psi_{32} \end{pmatrix} : \tau_B X \oplus U_{\tau_B X} \to V_X \oplus U_{\tau_B X} \oplus U_X,$$

$$(G^{(n)}_{\alpha_i}) = \begin{pmatrix} \mu_X & \Phi_{12} & 0 \\ \Phi_{21} & \Phi_{22} & \Phi_{23} \end{pmatrix} : V_X \oplus U_{\tau_B X} \oplus U_X \to X \oplus U_X,$$

(II) $\Psi_{31}, \Psi_{12}, \Psi_{32}, \Phi_{12}, \Phi_{21}, \Phi_{22} \in \mathrm{rad}_{k\Gamma}$, Ψ_{22} and Φ_{23} are isomorphisms,

(III) $\Phi_{21}\nu_X + \Phi_{23}\Psi_{31}, \mu_X\Psi_{12} + \Phi_{12}\Psi_{22}, \Phi_{21}\Psi_{12} + \Phi_{22}\Psi_{22} + \Phi_{23}\Psi_{32} \in \mathrm{rad}^n_{k\Gamma}$.

Obviously we only need to define $G^{(n)}_\alpha$ for arrows α of Γ_B.

We define $G^{(2)}_\alpha$ for $\alpha : X \to Y$ by

$$G^{(2)}_\alpha = \begin{pmatrix} \alpha & \phi_Y \sigma_{X,Y} \\ \gamma_{\tau_B^{-1}X,Y}\psi_{\tau_B^{-1}X} & U_\alpha \end{pmatrix} : X \oplus U_X \to Y \oplus U_Y.$$

Note that the map $\phi_Y\sigma_{X,Y}$ is defined only for Y nonprojective. If this is not the case then we replace it by the zero map. The same remark applies to $\gamma_{\tau_B^{-1}X,Y}\psi_{\tau_B^{-1}X}$, which is defined only for noninjective X. For a mesh in Γ_B of the form (4) we have $(G^{(2)}_{\beta_i})^{\mathrm{tr}}(\tau_B X) \subseteq V_X$ and the induced map equals ν_X. Moreover, the induced map $U_{\tau_B X} \to W_X \oplus U_{\tau_B X} \oplus U_X$ is

$$(U_{\beta_i})^{\mathrm{tr}} = \begin{pmatrix} 0 \\ \mathrm{Id}_{U_{\tau_B X}} \\ 0 \end{pmatrix}$$

by Lemma 20 and the induced map $U_{\tau_B X} \to Y_i$ belongs to $\mathrm{rad}_{k\Gamma}$. Thus

$$(G^{(2)}_{\beta_i})^{\mathrm{tr}} = \begin{pmatrix} \nu_X & \Phi_{12} \\ 0 & \mathrm{Id}_{U_{\tau_B X}} \\ 0 & 0 \end{pmatrix} : \tau_B X \oplus U_{\tau_B X} \to V_X \oplus U_{\tau_B X} \oplus U_X$$

with $\Phi_{12} \in \mathrm{rad}_{k\Gamma}$. Similarly one shows that

$$(G^{(2)}_{\alpha_i}) = \begin{pmatrix} \mu_X & 0 & 0 \\ \Phi_{21} & 0 & \mathrm{Id}_{U_X} \end{pmatrix} : V_X \oplus U_{\tau_B X} \oplus U_X \to X \oplus U_X,$$

where $\Phi_{21} \in \mathrm{rad}_{k\Gamma}$. The above remarks imply that $G^{(2)}$ satisfies conditions (I) and (II). Direct calculations show that condition (III) also holds.

Assume now that $n \geq 2$ and we have defined the representation $G^{(n)}$ satisfying conditions (I)–(III). Since B is representation directed we may number the nonprojective vertices $X_1, \ldots, X_l$ of Γ_B in such a way that if X_s precedes X_t in Γ_B then $s \leq t$. We want to define representations $G^{(n+1,s)} : \Gamma_B \to k\Gamma$, $s = 0, \ldots, l$, such that $G^{(n+1,s)}_X = G_X$ for any vertex X of Γ_B, and for each mesh in Γ_B of the form (4) with $X = X_t$ conditions (I) and (II) hold together with the following new version of condition (III):

(III′) $\Phi_{21}\nu_X + \Phi_{23}\Psi_{31}, \mu_X\Psi_{12} + \Phi_{12}\Psi_{22}, \Phi_{21}\Psi_{12} + \Phi_{22}\Psi_{22} + \Phi_{23}\Psi_{32} \in \mathrm{rad}^n_{k\Gamma}$ if $t > s$, and $\Phi_{21}\nu_X + \Phi_{23}\Psi_{31}, \mu_X\Psi_{12} + \Phi_{12}\Psi_{22}, \Phi_{21}\Psi_{12} + \Phi_{22}\Psi_{22} + \Phi_{23}\Psi_{32} \in \mathrm{rad}^{n+1}_{k\Gamma}$ if $t \le s$.

Obviously we may put $G^{(n+1,0)} = G^{(n)}$.

Let $s \in \{1, \ldots, l\}$ and assume we have defined $G^{(n+1,s-1)}$ with the desired properties. Let $X = X_s$ and consider the mesh in Γ_B of the form (4). We have

$$(G^{(n+1,s)}_{\beta_i})^{\mathrm{tr}} = \begin{pmatrix} \nu_X & \Psi_{12} \\ 0 & \Psi_{22} \\ \Psi_{31} & \Psi_{32} \end{pmatrix} : \tau_B X \oplus U_{\tau_B X} \to V_X \oplus U_{\tau_B X} \oplus U_X,$$

$$(G^{(n+1,s)}_{\alpha_i}) = \begin{pmatrix} \mu_X & \Phi_{12} & 0 \\ \Phi_{21} & \Phi_{22} & \Phi_{23} \end{pmatrix} : V_X \oplus U_{\tau_B X} \oplus U_X \to X \oplus U_X,$$

with $f_{21} = \Phi_{21}\nu_X + \Phi_{23}\Psi_{31} \in \mathrm{rad}^n_{k\Gamma}$, $f_{12} = \mu_X\Psi_{12} + \Phi_{12}\Psi_{22} \in \mathrm{rad}^n_{k\Gamma}$ and $f_{22} = \Phi_{21}\Psi_{12} + \Phi_{22}\Psi_{22} + \Phi_{23}\Psi_{32} \in \mathrm{rad}^n_{k\Gamma}$. We define $G^{(n+1,s)}$ by

$$G^{(n+1,s)}_{\alpha} = G^{(n+1,s-1)}_{\alpha}, \qquad \alpha \ne \alpha_i, \beta_i,$$

$$(G^{(n+1,s)}_{\beta_i})^{\mathrm{tr}} = \begin{pmatrix} \nu_X & \Psi_{12} \\ 0 & \Psi_{22} \\ \Psi_{31} - \Phi_{23}^{-1} f_{21} & \Psi_{32} \end{pmatrix} : \tau_B X \oplus U_{\tau_B X} \to V_X \oplus U_{\tau_B X} \oplus U_X,$$

$$(G^{(n+1,s)}_{\alpha_i}) = \begin{pmatrix} \mu_X & \Phi_{12} - f_{12}\Psi_{22}^{-1} & 0 \\ \Phi_{21} & \Phi_{22} - f_{22}\Psi_{22}^{-1} & \Phi_{23} \end{pmatrix} : V_X \oplus U_{\tau_B X} \oplus U_X \to X \oplus U_X.$$

Note that

$$G^{(n+1,s)}_{\beta_i} = G^{(n+1,s-1)}_{\beta_i} + \begin{pmatrix} 0 & 0 \\ g_{21} & 0 \end{pmatrix} : \tau_B X \oplus U_{\tau_B X} \to Y_i \oplus U_{Y_i},$$

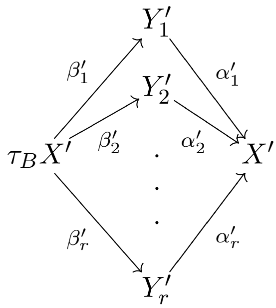
where $g_{21} \in \mathrm{rad}^n_{k\Gamma}$. This follows since Y_i is a direct summand of V_X, U_{Y_i} is a direct summand of $V_X \oplus U_{\tau_B X} \oplus U_X$ and $f_{21} \in \mathrm{rad}^n_{k\Gamma}$. Similarly,

$$G^{(n+1,s)}_{\alpha_i} = G^{(n+1,s-1)}_{\alpha_i} + \begin{pmatrix} 0 & g_{12} \\ 0 & g_{22} \end{pmatrix} : Y_i \oplus U_{Y_i} \to X \oplus U_X,$$

where $g_{12}, g_{22} \in \mathrm{rad}^n_{k\Gamma}$. This is again a consequence of the facts that Y_i is a direct summand of V_X, U_{Y_i} is a direct summand of $W_X \oplus U_{\tau_B X} \oplus U_X$ and $f_{12}, f_{22} \in \mathrm{rad}^n_{k\Gamma}$. These observations play a fundamental role in the proof of the following lemma.

LEMMA 21. *The representation* $G^{(n+1,s)}$ *satisfies conditions* (I), (II) *and* (III).

Proof. Condition (II) holds by the inductive hypothesis, since $G^{(n+1,s)}_{\alpha} - G^{(n+1,s-1)}_{\alpha} \in \mathrm{rad}^n_{k\Gamma}$ for any arrow α in Γ_B. In order to verify the remaining conditions assume that



is a mesh in Γ_B with $X' = X_t$. Note that if $X' \neq X, Y_i, \tau_B^- Y_i$ for any i then $(G_{\beta_p'}^{(n+1,s)})^{\mathrm{tr}} = (G_{\beta_p'}^{(n+1,s-1)})^{\mathrm{tr}}$ and $(G_{\alpha_p'}^{(n+1,s)}) = (G_{\alpha_p'}^{(n+1,s-1)})$. Hence (I) and (III$'$) are satisfied by the inductive hypothesis, since $t \neq s$, and consequently $t \leq s$ if and only if $t \leq s-1$.

Assume now $X' = Y_i$ for some i. In particular, $t < s$. Since B is representation directed it follows that $\beta_p' \neq \alpha_j, \beta_j$ for any p and j. Thus $(G_{\beta_p'}^{(n+1,s)})^{\mathrm{tr}} = (G_{\beta_p'}^{(n+1,s-1)})^{\mathrm{tr}}$ and the first part of (I) is satisfied by the inductive hypothesis. We also have $\alpha_p' \neq \alpha_j$ for any p and j. On the other hand, we have $\alpha_q' = \beta_i$ for some q and $\alpha_p' \neq \beta_j$ for $p \neq q$ or $j \neq i$ (note that there are no multiple arrows in Γ). It follows from the above remarks about the connection between $G_{\beta_j}^{(n+1,s)}$ and $G_{\beta_j}^{(n+1,s-1)}$ that

$$(G_{\alpha_p'}^{(n+1,s)}) = (G_{\alpha_p'}^{(n+1,s-1)}) + \begin{pmatrix} 0 & 0 & 0 \\ \Phi_{21}'' & 0 & 0 \end{pmatrix} : V_{X'} \oplus U_{\tau_B X'} \oplus U_{X'} \to X' \oplus U_{X'}$$

for some $\Phi_{21}'' \in \mathrm{rad}_{k\Gamma}^n$, since $\tau_B X$ is a direct summand of $V_{X'}$ and $U_{Y_j} = U_{X'}$. This and the inductive hypothesis imply the second part of (I). Finally, by direct calculations and the inductive hypothesis we obtain

$$(G_{\alpha_p'}^{(n+1,s)})(G_{\beta_p'}^{(n+1,s)})^{\mathrm{tr}} = (G_{\alpha_p'}^{(n+1,s-1)})(G_{\beta_p'}^{(n+1,s-1)})^{\mathrm{tr}} + \begin{pmatrix} 0 & 0 \\ \Phi_{21}''\nu_{X'} & \Phi_{21}''\Psi_{12}' \end{pmatrix} : \tau_B X' \oplus U_{\tau_B X'} \to X' \oplus U_{X'}$$

for some $\Psi_{12}' \in \mathrm{rad}_{k\Gamma}$. Using again the inductive hypothesis we get (III$'$), since $\Phi_{21}'' \in \mathrm{rad}_{k\Gamma}^n$ and $\nu_X, \Psi_{12}' \in \mathrm{rad}_{k\Gamma}$.

If $X' = X$, and thus $t = s$, then (I) and (III$'$) follow immediately from the definition of $G^{(n+1,s)}$.

Finally suppose $X' = \tau_B^- Y_j$, thus $\tau_B X' = Y_j$, for some j. In particular, $t > s$. Similarly to the case $X' = Y_j$ we obtain $(G_{\alpha_i'}^{(n+1,s)}) = (G_{\alpha_i'}^{(n+1,s-1)})$ and

$$(G_{\beta_i'}^{(n+1,s)})^{\mathrm{tr}} = (G_{\beta_i'}^{(n+1,s)})^{\mathrm{tr}} + \begin{pmatrix} 0 & \Psi_{12}'' \\ 0 & \Psi_{22}'' \\ 0 & \Psi_{32}'' \end{pmatrix} : \tau_B X' \oplus U_{\tau_B X'} \to V_{X'} \oplus U_{\tau_B X'} \oplus U_X'$$

for some $\Psi_{12}'', \Psi_{22}'', \Psi_{32}'' \in \mathrm{rad}_{k\Gamma}^n$. These remarks together with the inductive hypothesis imply that (I) is satisfied. Condition (III′) follows by direct calculations from the above observations and the inductive hypothesis. ∎

We put $G^{(n+1)} = G^{(n+1,l)}$. As an obvious consequence of the previous lemma, the representation $G^{(n+1)}$ satisfies (I)–(III). This finishes the inductive construction of the representations $G^{(n)}$, $n \geq 2$.

Recall that there exists a positive integer n such that $\operatorname{rad}^n_{k\Gamma} = 0$. We fix it and define $G = G^{(n)}$. We summarize properties of G in the following.

Corollary 22. *For each mesh in Γ_B of the form* (4) *we have*

$$(G_{\beta_i})^{\mathrm{tr}} = \begin{pmatrix} \nu_X & \Psi_{12} \\ 0 & \Psi_{22} \\ \Psi_{31} & \Psi_{32} \end{pmatrix} : \tau_B X \oplus U_{\tau_B X} \to V_X \oplus U_{\tau_B X} \oplus U_X,$$

$$(G_{\alpha_i}) = \begin{pmatrix} \mu_X & \Phi_{12} & 0 \\ \Phi_{21} & \Phi_{22} & \Phi_{23} \end{pmatrix} : V_X \oplus U_{\tau_B X} \oplus U_X \to X \oplus U_X,$$

Ψ_{22} and Φ_{23} are isomorphisms, *and* $(G_{\alpha_i})(G_{\beta_i})^{\mathrm{tr}} = 0$.

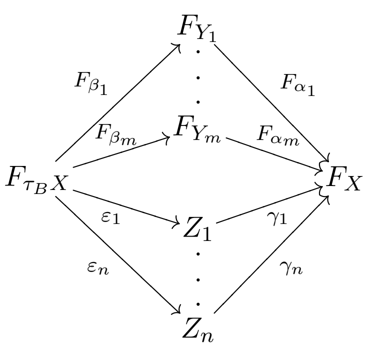
Lemma 23. *There exists a representation $H : \Gamma \to \operatorname{mod} A$ such that $H_X = F_X$ for $X \in (\Gamma_B)_0$ and*

$$0 \to H_{\tau_B X} \xrightarrow{\begin{pmatrix} H_{\beta_1} \\ \vdots \\ H_{\beta_m} \\ H_{\psi_X} \end{pmatrix}} \bigoplus_{i=1}^{m} H_{Y_i} \oplus H_{W_X} \xrightarrow{(H_{\alpha_1} \ \ldots \ H_{\alpha_m} \ H_{\phi_X})} H_X \to 0 \tag{5}$$

is an Auslander–Reiten sequence in $\operatorname{mod} A$ *for a mesh in Γ_B of the form* (4).

Proof. We put $H_X = F_X$ for each vertex X of Γ_B. Let X be a nonprojective vertex in Γ_B. Since $F : \Gamma_B \to \Gamma_A$ is an injective morphism of translation quivers, we have $F_{\tau_B X} = \tau_A F_X$ and the mesh in Γ_A ending at F_X has the form

(6)



for some $n \geq 0$, provided the mesh in Γ_B ending at X has the form (4). We define $H_{W_X} = \bigoplus_{1 \leq i \leq n} Z_i$. Thus H is defined on the vertices of Γ and we need to define H_α for arrows α in Γ. The construction of the homomorphisms H_α, $\alpha \in \Gamma_1$, is similar to the one presented in the proof of [11, Lemma 2.3.3] and it involves the notions and properties of irreducible, sink and source homomorphisms, which can be found in [11].

We first define H_α for all arrows $\alpha : Y \to X$ in Γ with X a projective vertex of Γ_B. In this case α belongs to Γ_B and we choose as H_α an arbitrary irreducible homomorphism $H_Y \to H_X$. By induction on the number of predecessors of a nonprojective vertex X in Γ_B, we define the irreducible homomorphisms $H_{\alpha_i} : F_{Y_i} \to F_X$, $1 \le i \le m$, and homomorphisms $H_{\psi_X} : F_{\tau_B X} \to \bigoplus Z_i$ and $H_{\phi_X} : \bigoplus Z_i \to F_X$, where the mesh in Γ_A ending at F_X has the form (6). Recall that the quiver Γ_B is finite and has no oriented cycles, thus by the inductive hypothesis we may assume that the irreducible homomorphisms H_{β_i}, $1 \le i \le m$, are already defined. Since the Auslander–Reiten quiver of a representation-finite algebra has no multiple arrows and F is an injective map, the modules F_{Y_i}, $1 \le i \le m$, are pairwise nonisomorphic. Hence we may choose irreducible homomorphism $g_{\varepsilon_i} : F_{\tau_B X} \to Z_i$, $1 \le i \le n$, such that the homomorphism

$$g = (H_{\beta_1}, \dots, H_{\beta_m}, g_{\varepsilon_1}, \dots, g_{\varepsilon_n})^{\mathrm{tr}} : F_{\tau_B X} \to \bigoplus_{1\le i\le m} F_{Y_i} \oplus \bigoplus_{1\le j\le n} Z_j$$

is a source map. The cokernel of g can be identified with F_X. We put $H_{\psi_X} = (g_{\varepsilon_i})^{\mathrm{tr}}$. Let $(H_{\alpha_1}, \dots, H_{\alpha_m}, H_{\phi_X})$ be a cokernel homomorphism of the source map $g = (H_{\beta_1}, \dots, H_{\beta_m}, H_\psi)^{\mathrm{tr}}$. Then (5) is an Auslander–Reiten sequence in $\operatorname{mod} A$. In particular, the components H_{α_i}, $1 \le i \le n$, of the cokernel homomorphism are irreducible. This finishes the inductive step of the construction of H. ∎

Let $H : \Gamma \to \operatorname{mod} A$ be a representation as described in the above lemma. Note that H satisfies the mesh relations, thus we have the induced functor $\mathcal{H} : k\Gamma \to \operatorname{mod} A$. Consequently, we also have the representation $\mathcal{H} \circ G : \Gamma_B \to \operatorname{mod} A$ defined in the natural way.

Lemma 24. *The representation $\mathcal{H} \circ G$ is hom-controlled.*

Proof. We first show that the representation $\mathcal{H} \circ G$ is exact. Let θ be a mesh in Γ_B of the form (4). Note that $(\mathcal{H} \circ G)_{\tau_B X} = H_{\tau_B X} \oplus \mathcal{H}(U_{\tau_B X})$, $(\mathcal{H} \circ G)_X = H_X \oplus \mathcal{H}(U_X)$ and $\bigoplus_{i=1}^m (\mathcal{H} \circ G)_{Y_i} = \mathcal{H}(V_X) \oplus \mathcal{H}(U_{\tau_B X}) \oplus \mathcal{H}(U_X)$, where $\mathcal{H}(V_X) = \bigoplus_{i=1}^m H_{Y_i} \oplus H_{W_X}$. Since $\mathcal{H}$ sends isomorphisms to isomorphisms we also deduce from Corollary 22 and Lemma 23 that

$$((\mathcal{H} \circ G)_{\beta_i})^{\mathrm{tr}} = \begin{pmatrix} \Psi_{11} & \Psi_{12} \\ 0 & \Psi_{22} \\ \Psi_{31} & \Psi_{32} \end{pmatrix} :$$

$$H_{\tau_B X} \oplus \mathcal{H}(U_{\tau_B X}) \to \mathcal{H}(V_X) \oplus \mathcal{H}(U_{\tau_B X}) \oplus \mathcal{H}(U_X),$$

$$((\mathcal{H}\circ G)_{\alpha_i}) = \begin{pmatrix} \Phi_{11} & \Phi_{12} & 0 \\ \Phi_{21} & \Phi_{22} & \Phi_{23} \end{pmatrix} : \\ \mathcal{H}(V_X)\oplus\mathcal{H}(U_{\tau_B X})\oplus\mathcal{H}(U_X)\to H_X\oplus\mathcal{H}(U_X),$$

where

$$\Psi_{11} = \begin{pmatrix} H_{\beta_1} \\ \vdots \\ H_{\beta_m} \\ H_{\psi_X} \end{pmatrix}, \qquad \Phi_{11} = (H_{\alpha_1},\dots,H_{\alpha_m},H_{\phi_X}),$$

Ψ_{22} and Φ_{23} are isomorphisms, and $((\mathcal{H}\circ G)_{\alpha_i})((\mathcal{H}\circ G)_{\beta_i})^{\mathrm{tr}} = 0$. Since $\dim_k \mathcal{H}(V) = \dim_k H_{\tau_B X} + \dim_k H_X$, in order to show that the sequence $(\mathcal{H}\circ G)\theta$ is exact it is enough to show that $((\mathcal{H}\circ G)_{\beta_i})^{\mathrm{tr}}$ is a monomorphism and $((\mathcal{H}\circ G)_{\alpha_i})$ is an epimorphism. The former follows from the fact that Ψ_{11} and Ψ_{22} are injective maps and the latter follows from the surjectivity of Φ_{11} and Φ_{23}. This finishes the proof that $\mathcal{H}\circ G$ is exact.

We now show that the representation $\mathcal{H}\circ G$ is hom-controlled. Let Z be a vertex of Γ_B. By basic properties of Auslander–Reiten sequences it follows that $\delta_\theta(Z) = \delta_{\tau_B X,Z}$, where $\delta_{x,y}$ is the Kronecker delta. On the other hand, we have

$$\begin{aligned}\delta_{(\mathcal{H}\circ G)\theta}((\mathcal{H}\circ G)_Z) &= [H_X\oplus\mathcal{H}(U_X)\oplus H_{\tau_B X}\oplus\mathcal{H}(U_{\tau_B X}), H_Z\oplus\mathcal{H}(U_Z)]_A \\ &\quad - [\mathcal{H}(V_X)\oplus\mathcal{H}(U_X)\oplus\mathcal{H}(U_{\tau_B X}), H_Z\oplus\mathcal{H}(U_Z)]_A \\ &= [H_X\oplus H_{\tau_B X}, H_Z\oplus\mathcal{H}(U_Z)]_A \\ &\quad - [\mathcal{H}(V_X), H_Z\oplus\mathcal{H}(U_Z)]_A.\end{aligned}$$

Since

$$0\to H_{\tau_B X}\to\mathcal{H}(V_X)\to H_X\to 0$$

is an Auslander–Reiten sequence in $\operatorname{mod} A$, we infer using again properties of the Auslander–Reiten sequences that $\delta_{(\mathcal{H}\circ G)\theta}((\mathcal{H}\circ G)_Z)$ is the multiplicity of $H_{\tau_B X}$ as a direct summand of $H_Z\oplus\mathcal{H}(U_Z)$. Recall that U_Z is a direct sum of objects of the form $W_{X'}$ and consequently $\mathcal{H}(U_Z)$ is a direct sum of modules of the from $H_{W_{X'}}$. Because the image of F is a full translation subquiver of Γ_B and for each indecomposable direct summand L of $H_{W_{X'}}$ we have an arrow $L\to F_{X'}$ in Γ_A which does not belong to the image of F, it follows that $H_{\tau_B X} = F_{\tau_B X}$ cannot be a direct summand of $\mathcal{H}(U_Z)$. Thus $\delta_{(\mathcal{H}\circ G)\theta}((\mathcal{H}\circ G)_Z) = \delta_{H_{\tau_B X},H_Z} = \delta_{F_{\tau_B X},F_Z} = \delta_{\tau_B X,Z}$, the last equality following from the fact that F is injective. Similarly, we show $\delta^\theta(Z) = \delta_{X,Z} = \delta^{(\mathcal{H}\circ G)\theta}((\mathcal{H}\circ G)_Z)$. ∎

Proof of Proposition 17. Since $\mathcal{H}\circ G : \Gamma_B\to\operatorname{mod} A$ is a hom-controlled representation and directed algebras are representation finite and standard, we deduce using Lemma 18 that the induced functor $\mathcal{F} : \operatorname{mod} B\to\operatorname{mod} A$ is hom-controlled. Moreover, it follows from the definition of the induced functor that $\mathcal{F}(M)$ is a direct sum of modules of the form $(\mathcal{H} \circ G)_X$, $X \in (\Gamma_B)_0$. We also have $(\mathcal{H} \circ G)_X = F_X \oplus \mathcal{H}(U_X)$ and $\mathcal{H}(U_X)$ is a direct sum of modules of the form $H_{W_{X'}}$, $X' \in (\Gamma_B)'_0$. Finally, for each direct summand L of $H_{W_{X'}}$ we have an arrow $L \to F_{X'}$ in Γ_A, hence $\mathcal{F}(M) \in \operatorname{add} \mathcal{S}$. ∎

REFERENCES

[1] S. Abeasis, A. Del Fra and H. Kraft, *The geometry of representations of* A_m, Math. Ann. 256 (1981), 401–418.

[2] M. Auslander, I. Reiten and S. O. Smalø, *Representation Theory of Artin Algebras*, Cambridge Stud. Adv. Math. 36, Cambridge Univ. Press, 1995.

[3] G. Bobiński and G. Zwara, *Normality of orbit closures for Dynkin quivers of type* $\mathbb{A}_n$, Manuscripta Math. 105 (2001), 103–109.

[4] K. Bongartz, *A geometric version of the Morita equivalence*, J. Algebra 139 (1991), 159–171.

[5] —, *Minimal singularities for representations of Dynkin quivers*, Comment. Math. Helv. 69 (1994), 575–611.

[6] M. Brion, *Multiplicity-free subvarieties of flag varieties*, preprint.

[7] A. Grothendieck (avec la collaboration de J. A. Dieudonné), *Éléments de géométrie algébrique IV. Étude locale des schémas et des morphismes de schémas IV*, Inst. Hautes Études Sci. Publ. Math. 32 (1967).

[8] W. Hesselink, *Singularities in the nilpotent scheme of a classical group*, Trans. Amer. Math. Soc. 222 (1976), 1–32.

[9] V. Lakshmibai and P. Magyar, *Degeneracy schemes, quiver schemes, and Schubert varieties*, Internat. Math. Res. Notices 12 (1998), 627–640.

[10] A. Ramanathan, *Schubert varieties are arithmetically Cohen–Macaulay*, Invent. Math. 80 (1985), 283–294.

[11] C. M. Ringel, *Tame Algebras and Integral Quadratic Forms*, Lecture Notes in Math. 1099, Springer, 1984.

[12] G. Zwara, *Smooth morphisms of module schemes*, Proc. London Math. Soc. (3) 84 (2002), 539–558.

[13] —, *Unibranch orbit closures in module varieties*, Ann. Sci. École Norm. Sup., in press.

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