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TRANSITIVITY OF PROXIMINALITY AND NORM ATTAINING FUNCTIONALS

BY

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Abstract. We study the question of when the set of norm attaining functionals on a Banach space is a linear space. We show that this property is preserved by factor reflexive proximinal subspaces in $\widetilde{R(1)}$ spaces and generally by taking quotients by proximinal subspaces. We show, for $\mathcal{K}(\ell_2)$ and c_0-direct sums of families of reflexive spaces, the transitivity of proximinality for factor reflexive subspaces. We also investigate the linear structure of the set of norm attaining functionals on hyperplanes of c_0 and show that, for some particular hyperplanes of c_0, linearity and orthogonal linearity coincide for the set of norm attaining functionals.

1. Introduction. We work only with real Banach spaces. For a Banach space X, we denote by B_X, S_X and $\mathrm{NA}(X)$ the closed unit ball of X, unit sphere of X and the set of all norm attaining functionals on X respectively. For a closed subspace Y of X we denote by Q_Y the canonical quotient map of X to X/Y. We are interested in Banach spaces for which $\mathrm{NA}(X)$ is a linear space. It is known that this is intimately related to the question of transitivity of proximinality ([4], [7]). We recall that Y is said to be a *proximinal subspace* of X if for every $x \in X$ there exists $y \in Y$ such that $\|x-y\| = d(x,Y)$, we then write $Y \overset{p}{\subseteq} X$.

In [9] W. Pollul raised the following question on *transitivity of proximinality.*

(A) *Which Banach spaces X have the following property*: *For any closed subspaces Y and Z of X with $Y \subseteq Z$, if* $\dim(X/Z) = \dim(Z/Y) = 1$ *and $Y \overset{p}{\subseteq} Z$, $Z \overset{p}{\subseteq} X$, then $Y \overset{p}{\subseteq} X$?*

In [7] V. Indumathi asked a more general question.

(B) *Which Banach spaces X have the following property*: *For any closed subspaces Y and Z of X with $Y \subseteq Z$, if* $\dim(X/Y) = n < \infty$ *and $Y \overset{p}{\subseteq} Z$, $Z \overset{p}{\subseteq} X$, then $Y \overset{p}{\subseteq} X$?*

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Following [7] we call a Banach space X with property described in (B) a $P(n)$ *space*, and we call X a P *space* if it is a $P(n)$ space for every $n \geq 2$. Examples of P spaces are c_0 and $\mathcal{K}(\ell_2)$ (the space of compact operators on ℓ_2). Also any finite-codimensional proximinal subspace of a P space is a P space ([8]).

A Banach space X is said to be an $R(1)$ *space* if every closed subspace Y of X of finite codimension with $Y^\perp \subseteq \mathrm{NA}(X)$ is proximinal in X. Examples of $R(1)$ spaces include c_0, all closed subspaces of c_0, reflexive spaces and $\mathcal{K}(\ell_2)$ (see [3] for c_0 and [4] for $\mathcal{K}(\ell_2)$).

To describe the connection between $R(1)$ and P spaces we need to recall the concept of orthogonal linearity from [7].

Let $f, g \in X^*$. Then f is said to be *strongly orthogonal* to g if the supremum of f on the unit ball of X is attained at some point of the unit ball of $\ker g$. A subset $F \subset X^*$ is said to be *orthogonally linear* if $f, g \in F$ and f strongly orthogonal to g implies that $\mathrm{span}\{f, g\} \subseteq F$. Recall that [7, Question 1] it is not known if there is a space X for which $\mathrm{NA}(X)$ is orthogonally linear but not linear. We answer this question in the case of hyperplanes of c_0.

It was proved in [7] that X is an $R(1)$ space and $\mathrm{NA}(X)$ is orthogonally linear if and only if X is a P space. Recently these properties were studied in [8] for direct sums of Banach spaces.

So far we have assumed that the subspaces are of finite codimension. We now consider subspaces with reflexive quotient, called factor reflexive spaces. Thus a closed subspace Y of a Banach space X is *factor reflexive* if X/Y is reflexive. Analogous to the above definitions, we call a Banach space X an $\widetilde{R(1)}$ *space* if for every factor reflexive subspace Y the condition $Y^\perp \subseteq \mathrm{NA}(X)$ implies that Y is proximinal in X. Since any reflexive quotient of c_0 is finite-dimensional, c_0 is an $R(1)$ as well as $\widetilde{R(1)}$ space.

We can now ask the following generalized version of questions (A) and (B).

(C) *Which Banach spaces X have the following property: For any factor reflexive closed subspaces Y and Z of X with $Y \subseteq Z$, if $Y \overset{p}{\subseteq} Z$ and $Z \overset{p}{\subseteq} X$ then $Y \overset{p}{\subseteq} X$?*

A Banach space with the property in (C) will be called a $\widetilde{P}$ *space*. Clearly any reflexive space and the space c_0 are examples of $\widetilde{P}$ spaces. Also any factor reflexive proximinal subspace of a $\widetilde{P}$ space is again a $\widetilde{P}$ space.

One of the aims of the present article is to contribute to the study of $\widetilde{R(1)}$ and $\widetilde{P}$ spaces. We now briefly describe the content of the article section-wise.

The second section contains investigations on the vector space structure of the norm attaining functionals on a Banach space X. In particular we study this for a factor reflexive proximinal subspace Y of a $\widetilde{P}$ space and for the quotient space X/Y of a P space X. We also give some stability results when X is an $\widetilde{R(1)}$ space and $\mathrm{NA}(X)$ is a vector space.

Motivated by Lemma 4.2 of [4] which identifies $\mathrm{NA}(\mathcal{K}(\ell_2))$ with the set of finite rank operators, in the third section we show that for any closed subspace of $\mathrm{NA}(\mathcal{K}(\ell_2))$ (by this we *always* mean that these subspaces are Banach spaces) the pre-annihilator is proximinal in $\mathcal{K}(\ell_2)$. We also show that $\mathcal{K}(\ell_2)$ and the c_0-direct sum of any family of reflexive spaces are $\widetilde{P}$ spaces.

In the fourth section we show that any separable $\widetilde{R(1)}$ space can be renormed with a Gateaux smooth norm retaining the proximinality properties. In particular we show that if X is a separable $\widetilde{R(1)}$ space then there exists an equivalent Gateaux smooth norm on X such that X with this new norm is still an $\widetilde{R(1)}$ space.

In the fifth section we study the vector space structure of norm attaining functionals in hyperplanes of c_0. We prove that orthogonal linearity and linearity are equivalent for hyperplanes in c_0, which gives a partial answer to Question 1 of [7].

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2. Linearity of $\mathrm{NA}(Y)$ for a closed subspace Y of a Banach space X. We start by recalling Garkavi's characterization for finite-codimensional proximinal subspaces which we use frequently.

LEMMA 2.1 (Garkavi [10]). *Let X be a normed linear space and Y be a closed subspace of finite codimension. Then Y is proximinal in X if and only if every closed subspace $Z \supseteq Y$ of X is proximinal in X.*

LEMMA 2.2. *Let Y be a proximinal subspace of X. Then Y is factor reflexive in X if and only if $Y^{\perp} \subseteq \mathrm{NA}(X)$.*

Proof. Suppose Y is factor reflexive in X. Equivalently, $(X/Y)^* \simeq Y^{\perp}$ is reflexive. Thus every $f \in Y^{\perp}$ is norm attaining on X/Y. Since Y is proximinal in X, $f \in \mathrm{NA}(X)$. Thus $Y^{\perp} \subseteq \mathrm{NA}(X)$. Conversely, suppose $Y^{\perp} \subseteq \mathrm{NA}(X)$. Then every element f in $Y^{\perp}$ attains its norm on X/Y. By a well known theorem of James we conclude that X/Y is reflexive. ∎

We now prove the following extension of Garkavi's characterization of finite-codimensional proximinal subspaces to the factor reflexive case.

Proposition 2.3. *Let X be a Banach space and let Y be a factor reflexive subspace. Then Y is proximinal in X if and only if every closed subspace $Z \supseteq Y$ of X is proximinal in X.*

Proof. Let Y be factor reflexive and assume that it is proximinal in X. Let Z be a closed subspace of X such that $Y \subseteq Z$. We need to show that Z is proximinal in X. Let S be the canonical map from X/Y to X/Z such that $S(Q_Y(x)) = Q_Z(x)$. Since Y is proximinal in X, $Q_Y(B_X) = B_{X/Y}$. Since X/Y is reflexive, $B_{X/Y}$ is weakly compact and dense in $Q_Y(B_X)$. So $S(B_{X/Y})$ is also weakly compact and dense in $Q_Z(B_X)$, hence we have $Q_Z(B_X) = S(Q_Y(B_X)) = S(B_{X/Y}) = B_{X/Z}$, which implies that Z is proximinal in X. The converse is trivial. ∎

Suppose Y is a proximinal subspace of X of finite codimension. Let $Q : X^* \to Y^* = X^*/Y^\perp$ denote the canonical quotient map. We note that $\{f \in X^* : f_{|Y}$ attains its norm on $Y\} = Q^{-1}(\mathrm{NA}(Y))$. This is the set $S(Y_1)$ in the notation of [7].

It was proved in [7] that X is a P space if and only if it is an $R(1)$ space and $\mathrm{NA}(X)$ is orthogonally linear. We also recall the following result from [7].

(i) If X is a $P(2)$ space, then $\mathrm{NA}(X)$ is orthogonally linear (Prop. 5).
(ii) For any normed linear space X, $\mathrm{NA}(X)$ is orthogonally linear if and only if $Q^{-1}(\mathrm{NA}(Y)) \subseteq \mathrm{NA}(X)$ for every proximinal hyperplane Y in X (Prop. 10).

In the following results we establish by direct and simple arguments the relationship between the set of norm attaining functionals in X, Y and X/Y for some special subspaces $Y \subset X$.

Proposition 2.4. *Let X be a $\widetilde{P}$ space. Let Y be a proximinal, factor reflexive subspace of X. Then $Q^{-1}(\mathrm{NA}(Y)) \subseteq \mathrm{NA}(X)$.*

Proof. Let $g \in \mathrm{NA}(Y)$ and let $f \in X^*$ be such that $Q(f) = f_{|Y} = g$. Consider $Z = \ker g$ in Y. We have $Z \overset{p}{\subseteq} Y$ since $g \in \mathrm{NA}(Y)$. Then $Z \overset{p}{\subseteq} X$ since $Z \overset{p}{\subseteq} Y \overset{p}{\subseteq} X$ and X is a $\widetilde{P}$ space. This implies that $Z^\perp \subset \mathrm{NA}(X)$ by Lemma 2.2. Since $Q(f) = g = f_{|Y}$ and $f_{|Y}(z) = 0$ for all $z \in Z$, we have $f \in Z^\perp$ and thus $f \in \mathrm{NA}(X)$. Thus we have $Q^{-1}(\mathrm{NA}(Y)) \subseteq \mathrm{NA}(X)$. ∎

We next show that if in addition one assumes that X is a $\widetilde{P}$ space and $\mathrm{NA}(X)$ is linear then the same conclusion holds for any proximinal factor reflexive subspace. If we assume linearity of $\mathrm{NA}(X)$, we will have equality of $Q^{-1}(\mathrm{NA}(Y))$ and $\mathrm{NA}(X)$ for any factor reflexive proximinal subspace Y.

PROPOSITION 2.5. *Let X be an $\widetilde{R(1)}$ space such that* $\mathrm{NA}(X)$ *is a vector space. Let Y be a factor reflexive proximinal subspace of X. Then* $Q^{-1}(\mathrm{NA}(Y)) = \mathrm{NA}(X)$ *and* $Q(\mathrm{NA}(X)) = \mathrm{NA}(Y)$. *In particular* $\mathrm{NA}(Y)$ *is a vector space.*

Proof. Let $Y^\perp \subseteq \mathrm{NA}(X)$. Let $f \in Q^{-1}(\mathrm{NA}(Y))$. Let f_0 be a norm preserving extension of $f_{|Y}$. Clearly $f_0 \in \mathrm{NA}(X)$ and $f - f_0 \in Y^\perp \subset \mathrm{NA}(X)$. Since $\mathrm{NA}(X)$ is a vector space, $f \in \mathrm{NA}(X)$.

Now suppose $f \in \mathrm{NA}(X)$. Let Z be the closed subspace such that $Z^\perp = \operatorname{span}\{f, Y^\perp\}$. By our hypothesis we have $Z \subset Y$ and Z is proximinal in X and in particular in Y. Thus $f_{|Y}$ attains its norm on Y so that $f \in Q^{-1}(\mathrm{NA}(Y))$. We then have $Q(\mathrm{NA}(X)) = Q[Q^{-1}(\mathrm{NA}(Y))] \subseteq \mathrm{NA}(Y)$. On the other hand, $Q(\mathrm{NA}(X)) \supseteq \mathrm{NA}(Y)$ by the Hahn–Banach theorem. Hence $Q(\mathrm{NA}(X)) = \mathrm{NA}(Y)$ and $\mathrm{NA}(Y)$ is a vector space. ∎

LEMMA 2.6. *Let X be an $\widetilde{R(1)}$ space such that* $\mathrm{NA}(X)$ *is a vector space. Let Y be a factor reflexive subspace of X. Then the following are equivalent.*

(i) *Y is proximinal in X.*
(ii) $Y^\perp \subseteq \mathrm{NA}(X)$.
(iii) $\mathrm{NA}(X) = Q^{-1}(\mathrm{NA}(Y)) = \{f \in X^* : f_{|Y} \in \mathrm{NA}(Y)\}$.

Proof. Since X is an $\widetilde{R(1)}$ space we have (i)⇔(ii). By Proposition 2.5, we have (i)⇒(iii). We show that (iii)⇒(i).

Suppose (iii) holds true. Let $f \in Y^\perp$. Then $Q(f) = 0 \in \mathrm{NA}(Y)$. By (iii) we have $f \in \mathrm{NA}(X)$. This implies that $Y^\perp \subseteq \mathrm{NA}(X)$. ∎

REMARK 2.7. (iii)⇒(ii) is true in general but (ii)⇒(iii) can fail if $\mathrm{NA}(X)$ is not a vector space as the following example from [7] illustrates.

EXAMPLE 2.8. Let $f = (1, 1, 1/2, 1/4, \ldots) \in \ell_1$ and consider $X = \ker f$ in c_0. By Theorem 3 of [3], X is an $R(1)$ space and hence an $\widetilde{R(1)}$ space and it can be easily seen that $\mathrm{NA}(X)$ is not a vector space. Let $g = (1, 0, 0, 0, \ldots) \in \ell_1$ and consider $Y = \ker(g_{|X})$ in X. Now let $h = (0, 1, 0, 0, \ldots) \in \ell_1$. We have $h_{|X} \in \mathrm{NA}(X)$. But $h_{|Y} \notin \mathrm{NA}(Y)$, which implies that $\mathrm{NA}(X) \neq Q^{-1}(\mathrm{NA}(Y))$.

We now present another example where X is not an $\widetilde{R(1)}$ space and $\mathrm{NA}(X)$ is not a vector space.

EXAMPLE 2.9. It can be easily seen that ℓ_1 is not an $\widetilde{R(1)}$ space and $\mathrm{NA}(\ell_1)$ is not a vector space. Let $f = (1, 0, 0, \ldots) \in \ell_\infty$ and consider $Y = \ker f$ in ℓ_1. Since $f \in \mathrm{NA}(\ell_1)$, Y is proximinal in ℓ_1. Now let $g = (1, 1/2, 2/3, \ldots, n/(n+1), \ldots) \in \mathrm{NA}(\ell_1)$. Clearly $g_{|Y} = (0, 1/2, 2/3, \ldots \ldots, n/(n+1), \ldots) \notin \mathrm{NA}(Y)$. Hence $\mathrm{NA}(\ell_1) \neq Q^{-1}(\mathrm{NA}(Y))$.

We now turn our attention to the linear structure of the set of norm attaining functionals for quotient spaces by proximinal subspaces.

THEOREM 2.10. *Let Y be a proximinal subspace of X.*

(i) *If* $\mathrm{NA}(X)$ *is orthogonally linear then so is* $\mathrm{NA}(X/Y)$.
(ii) *If* $\mathrm{NA}(X)$ *is linear then so is* $\mathrm{NA}(X/Y)$.

Proof. We only need to prove (i). Suppose $\mathrm{NA}(X)$ is orthogonally linear. Let $f, g \in \mathrm{NA}(X/Y)$ with $\|f\| = \|g\| = 1$. Let $f', g' \in Y^\perp \subset X^*$ be such that $Q^*(f) = f'$ and $Q^*(g) = g'$ where $Q^* : (X/Y)^* = Y^\perp \to X^*$ is the inclusion map. Then $\|f'\| = \|g'\| = 1$ and $f', g' \in \mathrm{NA}(X)$. We also have $\|f + g\| = \|f' + g'\|$.

Suppose f is strongly orthogonal to g. Let $z \in S_{\ker g}$ be such that $f(z) = 1$. By the proximinality of Y there exists $y \in Q_Y(z)$ such that $\|y\| = 1$ and $f'(y) = 1$. It is easily seen that $y \in \ker g'$, which implies that f' is strongly orthogonal to g'. By orthogonal linearity of $\mathrm{NA}(X)$, we have $f' + g' \in \mathrm{NA}(X)$. Hence there is $x_0 \in S_X$ such that $(f' + g')(x_0) = \|f'+g'\| = \|f+g\| = (f+g)(Q_Y(x_0))$, which implies that $f+g \in \mathrm{NA}(X/Y)$. Hence $\mathrm{NA}(X/Y)$ is orthogonally linear. ∎

The following lemma is known. For completeness we give an easy proof.

LEMMA 2.11. *Let X be a Banach space and Y be a closed subspace. Let Z be a closed subspace of X/Y. If $Q_Y^{-1}(Z)$ is proximinal in X, then Z is proximinal in X/Y.*

Proof. Let $Q_Y(x_0) \in X/Y$. Since $Q_Y^{-1}(Z)$ is proximinal, there exists $z_0 \in Q_Y^{-1}(Z)$ such that $d(x_0, Q_Y^{-1}(Z)) = \|x_0 - z_0\|$. Now for any $z \in Q_Y^{-1}(Z)$ and for any $n \geq 1$, we have

$$\begin{aligned}\|Q_Y(x_0 - z)\| = d(x_0 - z, Y) &> \|x_0 - z - y_n\| - 1/n \\ &\geq d(x_0, Q_Y^{-1}(Z)) - 1/n = \|x_0 - z_0\| - 1/n \\ &\geq \|Q_Y(x_0 - z_0)\| - 1/n\end{aligned}$$

where $y_n \in Y$. So $\|Q_Y(x_0 - z)\| \geq \|Q_Y(x_0 - z_0)\|$ for every $Q_Y(z) \in Z$, which implies proximinality of Z at $Q_Y(x_0)$. Since $Q_Y(x_0)$ is arbitrary, Z is proximinal in X/Y. ∎

A consequence of the following theorem and the results proved above is that if X is a P space and $Y \subset X$ is reflexive then X/Y is a P space.

THEOREM 2.12. *Let X be an $R(1)$ space and let Y be a proximinal subspace of X. Then X/Y is an $R(1)$ space. Hence if X is a P space so is X/Y.*

Proof. Let Z be a closed subspace of finite codimension n in X/Y with $Z^\perp \subseteq \mathrm{NA}(X/Y)$. Let $f_1, \ldots, f_n \in \mathrm{NA}(X/Y)$ be such that $Z = \bigcap_{i=1}^n \ker f_i$. Then $Q_Y^{-1}(Z) = \bigcap_{i=1}^n \ker Q^*(f_i)$ and $Q^*(f_i) \in Q^*(Z^\perp) \subseteq X^*$. Since Y is proximinal, we have $Q^*(Z^\perp) = (Q_Y^{-1}(Z))^\perp \subseteq \mathrm{NA}(X)$. Since X is an $R(1)$

space, $Q_Y^{-1}(Z)$ is proximinal in X. Thus by Lemma 2.11, Z is proximinal in X/Y and this shows that X/Y is an $R(1)$ space. By Theorem 2.10(i), it follows that X/Y is a P space if X is. ∎

We do not know an answer to the following version of the "3-space" problem. If $Y \subset X$ is reflexive and X/Y is a P (or $R(1)$) space, is X a P space ($R(1)$ space)?

For a Banach space X, let $\mathcal{P}_X = \{Y \overset{p}{\subseteq} X : \dim(X/Y) < \infty\}$. We have the following stability properties.

LEMMA 2.13. *Let X be an $R(1)$ space. Then the following statements are equivalent.*

(i) $\mathrm{NA}(X)$ *is linear.*
(ii) $\mathcal{P}_X$ *is stable under intersection.*
(iii) *For any two proximinal hyperplanes Y_1 and Y_2 of X, $Y_1 \cap Y_2$ is a proximinal subspace of X.*

Proof. (i)⇒(ii). Assume that $\mathrm{NA}(X)$ is a vector space. Let Y_1 and Y_2 be two finite-codimensional proximinal subspaces of X with codimensions n_1 and n_2 respectively. Let $Y_1 = \bigcap_{i=1}^{n_1} \ker f_i$ and $Y_2 = \bigcap_{j=1}^{n_2} \ker g_j$, where $f_1, \ldots, f_{n_1}, g_1, \ldots, g_{n_2} \in \mathrm{NA}(X)$. By Garkavi's lemma we have $Y_1^{\perp}, Y_2^{\perp} \subseteq \mathrm{NA}(X)$. Since $\mathrm{NA}(X)$ is a vector space and $Y_1^{\perp}$ and $Y_2^{\perp}$ are finite-dimensional spaces, we have $\mathrm{span}\{Y_1^{\perp}, Y_2^{\perp}\} = \mathrm{span}\{f_1, \ldots, f_{n_1}, g_1, \ldots, g_{n_2}\} \subseteq \mathrm{NA}(X)$. Since $(Y_1 \cap Y_2)^{\perp} = \mathrm{span}\{Y_1^{\perp}, Y_2^{\perp}\}$ and X is an $R(1)$ space this implies that $Y_1 \cap Y_2$ is proximinal in X.

(ii)⇒(iii) is trivial.

(iii)⇒(i). Suppose $\mathrm{NA}(X)$ is not linear. Then there exist f and g in $\mathrm{NA}(X)$ such that $f + g \notin \mathrm{NA}(X)$. Thus $\ker f \cap \ker g$ is not proximinal X (by Garkavi's lemma). ∎

PROPOSITION 2.14. *Let X be an $R(1)$ space such that* $\mathrm{NA}(X)$ *is a linear space. Let Y be a closed subspace of X. Then $\mathcal{P}_Y \subseteq \mathcal{P}_X \cap Y$. More precisely if $Z \overset{p}{\subseteq} Y$ and* $\dim(Y/Z) = n$, *then there exists a proximinal subspace Z_0 in X of codimension n such that $Z = Z_0 \cap Y$.*

Proof. Let Z be a proximinal subspace of Y of codimension n. Then by Garkavi's lemma, $Z^{\perp} \subseteq \mathrm{NA}(Y) \subseteq Y^*$. Let $\{y_i^*\}_{1 \le i \le n}$ be a basis of $Z^{\perp}$. Let x_i^* in X^* be such that $x_i^*|_Y = y_i^*$ and $\|x_i^*\| = \|y_i^*\|$. This implies that $x_i^* \in \mathrm{NA}(X)$ for every $i = 1, \ldots, n$. If $V = \mathrm{span}\{x_i^* : 1 \le i \le n\}$, then $V \subseteq \mathrm{NA}(X)$ since $\mathrm{NA}(X)$ is a vector space. Now $V_{\perp} = Z_0 \overset{p}{\subseteq} X$ since X is an $R(1)$ space. Finally, $Z_0 \cap Y = Z$. ∎

REMARK 2.15. If X is an $R(1)$ space such that $\mathrm{NA}(X)$ is a vector space and if $\mathcal{P}_Y = \mathcal{P}_X \cap Y$, then Y is also an $R(1)$ space such that $\mathrm{NA}(Y)$ is a vector space. But the converse is not true. For example, let $X = c_0$ and $Y = \ker f$ where $f = (1, 1/3, 1/4, 1/8, \ldots) \in \ell_1$. It will be shown in Proposition 5.4 that $\mathrm{NA}(Y)$ is a vector space. It was shown in [3] that every closed subspace of c_0 is an $R(1)$ space, which implies that $\mathcal{P}_Y$ is stable under intersections. Let $e_1 = (1, 0, 0, \ldots) \in \ell_1$. It is easy to see that $\ker e_1$ is proximinal in X but $\ker e_1 \cap Y$ is not in $\mathcal{P}_Y$, which implies that $\mathcal{P}_Y \subsetneq \mathcal{P}_X \cap Y$.

3. $\widetilde{R(1)}$ spaces. Recall that the *rank* of an operator $A : X \to X$ is the dimension of its image. The following proposition gives a characterization of subspaces of tensor product spaces when the ranks of the elements in the subspace are uniformly bounded.

PROPOSITION 3.1. *Let E and F be vector spaces and E' be the algebraic dual of E. Let $E' \otimes F = R(E, F)$ be the space of finite rank linear maps from E to F. Let V be a vector subspace of $R(E, F)$ such that*

$$\sup\{\mathrm{rank}(T) : T \in V\} = N < \infty.$$

Then there exist $f_1, \ldots, f_N$ in F and $e_1^*, \ldots, e_N^*$ in E' such that every T in V can be written as

$$T = \sum_{i=1}^{N} e_i^* \otimes g_i + \sum_{j=1}^{N} b_j^* \otimes f_j$$

for some $g_1, \ldots, g_N \in F$ and $b_1^*, \ldots, b_N^* \in E'$.

Proof. Let T_0 in V be such that $\mathrm{rank}(T_0) = \sup\{\mathrm{rank}(T) : T \in V\} = N < \infty$. There exists a basis $\mathcal{B}_1$ of E and a basis $\mathcal{B}_2$ of F such that the matrix of T_0 relative to $\mathcal{B}_1$ and $\mathcal{B}_2$ is

$$\begin{bmatrix} \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix}_{(N \times N)} & 0 \\ 0 & 0 \end{bmatrix}.$$

Let $V_1 = \{T \in V : x_{pq} = 0 \text{ for all } p \leq N \text{ and } q \leq N\}$ where $(x_{pq})_{p,q}$ is the matrix of T with respect to the bases $\mathcal{B}_1$ and $\mathcal{B}_2$. Since $\dim(V/V_1) < \infty$ and V consists of finite rank operators, the lemma follows easily from the following claim.

CLAIM. *If $T \in V_1$ and $i, j \notin \{1, \ldots, N\}$, then $x_{ij} = 0$.*

Proof of the Claim. Pick $0 \neq \lambda \in \mathbb{C}$ and let $S = T_0 + \lambda T$. We consider the determinant of the $(N+1) \times (N+1)$ submatrix of S whose rows are $\{1, \cdots, N\} \cup \{i\}$ and columns are $\{1, \cdots, N\} \cup \{j\}$ with respect to the bases

$\mathcal{B}_1$ and $\mathcal{B}_2$. This submatrix has the following form:

$$\begin{bmatrix} 1 & 0 & \cdots & 0 & \lambda b_1 \\ 0 & 1 & \cdots & 0 & \lambda b_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & \lambda b_N \\ \lambda a_1 & \lambda a_2 & \cdots & \lambda a_N & \lambda x_{ij} \end{bmatrix}_{(N+1)\times(N+1)}$$

for some scalars $a_1, \dots, a_N$ and $b_1, \dots, b_N$. Since $\mathrm{rank}(S) \le N$, this determinant is 0. But by a direct computation, this implies that

$$\lambda x_{ij} = \lambda^2(a_1 b_1 + \cdots + a_N b_N)$$

and since λ is arbitrary, it follows that $x_{ij} = 0$. ∎

The following proposition along with Proposition 3.1 gives the structure of closed subspaces of $E \otimes E^*$.

PROPOSITION 3.2. *Let E be a Banach space and M be a closed subspace of $E \otimes E^*$. Then there exists $n_0 \in \mathbb{N}$ such that* $\mathrm{rank}(T) \le n_0$ *for every $T \in M$.*

Proof. Let $V_n = \{T \in E \otimes E^* : \mathrm{rank}(T) \le n\}$ for every $n \in \mathbb{N}$. We have $M = \bigcup_{n\in\mathbb{N}}(M \cap V_n)$. Since M is a Banach space, by the Baire category theorem there exists k_0 such that $\mathrm{int}(M \cap V_{k_0}) \ne \emptyset$. Let $m \in M$ and $\varepsilon > 0$ be such that $B_M(m,\varepsilon) \subset V_{k_0}$. Now $B_M(m,\varepsilon) - B_M(m,\varepsilon) \subset V_{2k_0}$. So $B_M(0,\varepsilon) \subset B_M(m,\varepsilon) - B_M(m,\varepsilon) \subset V_{2k_0}$, which implies that $M \subset V_{2k_0}$. Hence the ranks of the operators of M are uniformly bounded. ∎

REMARK 3.3. We recall from [4] that $\mathrm{NA}(\mathcal{K}(\ell_2)) = \ell_2 \otimes \ell_2$. Let Y be a closed subspace of $\mathcal{K}(\ell_2)$ such that $Y^\perp \subseteq \mathrm{NA}(\mathcal{K}(\ell_2))$. Now Propositions 3.1 and 3.2 imply that there exist $f_1, \dots, f_N$ and $e_1^*, \dots, e_N^*$ in ℓ_2 such that every T in $Y^\perp$ can be written as

$$T = \sum_{i=1}^{N} e_i^* \otimes g_i + \sum_{j=1}^{N} b_j^* \otimes f_j$$

for some $g_1, \dots, g_N$ and $b_1^*, \dots, b_N^* \in \ell_2$.

We now study proximinality questions for factor reflexive subspaces of $\mathcal{K}(\ell_2)$. Let V be a finite-dimensional subspace of ℓ_2 and let

$$Z_V = \{S \in \mathcal{K}(\ell_2) : S(\ell_2) \subseteq V^\perp \text{ and } S^*(\ell_2) \subseteq V^\perp\}. \tag{1}$$

In other words, in an orthonormal basis $\mathcal{B} = \mathcal{B}_1 \cup \mathcal{B}_2$ where $\mathcal{B}_1$ is a basis of V and $\mathcal{B}_2$ is a basis of $V^{\perp}$, the matrix of S has the form

$$\begin{bmatrix} [0]_{d\times d} & \vdots & 0 \\ \cdots & \vdots & \cdots \\ 0 & \vdots & [\alpha_{ij}] \end{bmatrix}$$

if and only if $S \in Z_V$ where d is the dimension of V.

Proposition 3.4. *For a finite-dimensional subspace V of ℓ_2 let Z_V be defined as in* (1). *Then Z_V is a proximinal subspace of $\mathcal{K}(\ell_2)$.*

Proof. It suffices to show that every operator whose matrix relative to $\mathcal{B}$ has the form

$$\begin{bmatrix} [\beta^{(1)}_{kl}]_{d\times d} & \vdots & [\beta^{(2)}_{mn}] \\ \cdots & \vdots & \cdots \\ [\beta^{(3)}_{pq}] & \vdots & 0 \end{bmatrix}$$

has a nearest point in Z_V (since we can translate by a vector in Z_V). Such an operator has finite rank. Let

$$W = \operatorname{span}\{V \cup S(V) \cup S^*(V)\}$$

and let W' be a finite-dimensional subspace of $V^{\perp}$ such that

$$W \subseteq V \oplus W'.$$

Let $\mathcal{B}' = \mathcal{B}_1 \cup \mathcal{B}_2' \cup \mathcal{B}_3$ be an orthonormal basis of ℓ_2 such that $\mathcal{B}_1$ (as before) is an orthonormal basis of V, $\mathcal{B}_2'$ is an orthonormal basis of W' and $\mathcal{B}_3$ is an orthonormal basis of $(V \oplus W')^{\perp}$. The matrix of S relative to $\mathcal{B}'$ is of the following form:

$$\begin{bmatrix} [\beta^{(1)}_{kl}]_{d\times d} & \vdots & [\beta^{(2)}_{mn}] & \vdots & 0 \\ \cdots\cdots\cdots & \vdots & \cdots\cdots & \vdots & \cdots \\ [\beta^{(3)}_{pq}] & \vdots & [0]_{d'\times d'} & \vdots & 0 \\ \cdots & \vdots & \cdots & \vdots & \cdots \\ & \vdots & & \vdots & \\ 0 & \vdots & 0 & \vdots & 0 \\ & \vdots & & \vdots & \end{bmatrix}$$

where d' is the dimension of W'. Let $P : \ell_2 \to V \oplus W'$ be the orthogonal projection. If $L \in Z_V$, then $P(S-L)P = S - PLP$ and we have $PLP \in Z'$ with

$$Z' = \{L' \in \mathcal{K}(\ell_2) : L'(\ell_2) \subseteq W' \text{ and } L'^*(\ell_2) \subseteq W'\}.$$

Clearly Z' is a finite-dimensional vector subspace of Z_V consisting of operators whose matrix in $\mathcal{B}'$ has the form

$$\begin{bmatrix} 0 & \vdots & 0 & \vdots & 0 \\ \cdots & \vdots & \cdots & \vdots & \cdots \\ 0 & \vdots & [\gamma_{ij}] & \vdots & 0 \\ \cdots & \vdots & \cdots & \vdots & \cdots \\ & \vdots & & \vdots & \\ 0 & \vdots & 0 & \vdots & 0 \\ & \vdots & & \vdots & \end{bmatrix}$$

Moreover since $\|P\| = 1$, we have

$$\|S - PLP\| = \|P(S - L)P\| \leq \|S - L\|.$$

Therefore

$$\inf\{\|S - L\| : L \in Z_V\} = \inf\{\|S - L'\| : L' \in Z'\},$$

and this infimum is attained since $\dim(Z') < \infty$, which completes the proof of the proposition. ■

We are now ready to state the main theorem of this section.

THEOREM 3.5. *Let Y be a closed subspace of $\mathcal{K}(\ell_2)$ such that $Y^\perp \subseteq$* $\mathrm{NA}(\mathcal{K}(\ell_2))$. *Then Y is a proximinal subspace of $\mathcal{K}(\ell_2)$. In particular $\mathcal{K}(\ell_2)$ is an $\widetilde{R(1)}$ space.*

Proof. By Proposition 3.1, there is a finite-dimensional subspace V of ℓ_2 such that $Z_V \subseteq Y$. By Proposition 3.4, the space Z_V is a proximinal subspace. Also $\mathcal{K}(\ell_2)/Z_V$ is reflexive. Hence by Proposition 2.3, Y is a proximinal subspace of $\mathcal{K}(\ell_2)$. ■

REMARK 3.6. If $M \subseteq \mathrm{NA}(\mathcal{K}(\ell_2)) \subseteq \mathcal{K}(\ell_2)^*$ is a norm-closed subspace, then M is necessarily reflexive. Indeed, since the dual unit ball of $\mathcal{K}(\ell_2)$ is weakly sequentially complete, M^* is a quotient of X (see [1, Lemma 2.1]). Now being an M-embedded dual space, M^* and thus M is reflexive (see [5, Chapter III]).

We now prove that any $\widetilde{R(1)}$ space with orthogonal linearity of norm attaining functionals is a $\widetilde{P}$ space. For a proximinal subspace Y of X let $P_Y^{-1}(0) = \{x \in X : d(x, Y) = \|x\|\}$.

PROPOSITION 3.7. *Let X be an $\widetilde{R(1)}$ space such that* $\mathrm{NA}(X)$ *is orthogonally linear. Then X is a $\widetilde{P}$ space.*

Proof. Let $Z \overset{p}{\subseteq} Y \overset{p}{\subseteq} X$ be such that X/Z is reflexive. We have to show that $Z \overset{p}{\subseteq} X$. Since X is an $\widetilde{R(1)}$ space, it suffices to show that $Z^\perp \subset \mathrm{NA}(X)$.

The space $Y^\perp$ is proximinal in $Z^\perp$ and thus $Z^\perp = (P_{Y^\perp}^{-1}(0) \cap Z^\perp) + Y^\perp$. We have $Y \overset{p}{\subseteq} X$ and this implies that $Y^\perp \subseteq \mathrm{NA}(X)$. Also $Z \overset{p}{\subseteq} Y$ and so we have $P_{Y^\perp}^{-1}(0) \cap Z^\perp \subseteq \mathrm{NA}(X)$ and each functional in $P_{Y^\perp}^{-1}(0) \cap Z^\perp$ is strongly orthogonal to $Y^\perp$. Since $\mathrm{NA}(X)$ is orthogonally linear this implies that $Z^\perp \subseteq \mathrm{NA}(X)$. ∎

REMARK 3.8. Since $\mathrm{NA}(\mathcal{K}(\ell_2))$ is a vector space, it follows by Proposition 3.7 that $\mathcal{K}(\ell_2)$ is a $\widetilde{P}$ space.

We next show that c_0-direct sums of reflexive spaces are $\widetilde{R(1)}$ spaces.

LEMMA 3.9. *Let $\{X_i : i \in \mathbb{N}\}$ be a family of reflexive spaces and consider its c_0-direct sum $X = (\bigoplus X_i)_{c_0}$. Let M be a closed subspace of* $\mathrm{NA}(X)$. *Then there exists a finite set A such that* $\mathrm{supp}(f) \subset A$ *for every $f \in M$.*

Proof. Let $V_n = \{f = (f_i) \in \mathrm{NA}(X) : f_i = 0 \ \forall i > n_0\}$. Then $M = \bigcup_{n\in\mathbb{N}}(V_n \cap M)$. Using the Baire category theorem arguments as in Proposition 3.2, we can get $\varepsilon > 0$ and n_0 such that $B_M(0, \varepsilon) \subset V_{n_0}$, which implies that $M \subseteq V_{n_0}$ and this completes the proof. ∎

It is easy to see that $\mathrm{NA}(X) = \{f = (f_i) \in X^* : f$ has only finitely many non-zero coordinates$\}$ and thus is a vector space.

PROPOSITION 3.10. *Let $\{X_i : i \in \mathbb{N}\}$ be a family of reflexive spaces and $X = (\bigoplus X_i)_{c_0}$. Let Y be a factor reflexive subspace of X. Then the following are equivalent.*

(i) *Y is proximinal in X.*
(ii) $Y^\perp \subseteq \mathrm{NA}(X)$.
(iii) *there exists a finite set A such that* $\mathrm{supp}(f) \subset A$ *for every $f \in Y^\perp$.*

Proof. (i)⇒(ii) by Lemma 2.2; (ii)⇒(iii) follows by 3.9; (iii)⇒(ii) is easy to see.

(ii)⇒(i). By Lemma 3.9 we can get n_0 such that for all $f = (f_i) \in Y^\perp$, $f_i = 0$ if $i > n_0$. Let $I = \{i : 1 \le i \le n_0\}$,

$$Y_1 = \left\{x = (x_i) \in (\textstyle\bigoplus_\infty X_i)_I : \displaystyle\sum_{i\in I} f_i(x_i) = 0 \ \forall f = (f_i) \in Y^\perp\right\}$$

and $Y_2 = (\bigoplus_{c_0} X_i)_{\mathbb{N}\setminus I}$. Then clearly $Y = Y_1 \oplus_\infty Y_2$ and Y_1 is a closed subspace in a reflexive space $(\bigoplus_\infty X_i)_I$. So $Y_1 \overset{p}{\subseteq} (\bigoplus_\infty X_i)_I$. We have now $Y = Y_1 \oplus_\infty Y_2 \overset{p}{\subseteq} X$, which completes the proof. ∎

THEOREM 3.11. *Let $\{X_i : i \in \mathbb{N}\}$ be a family of reflexive spaces and $X = (\bigoplus X_i)_{c_0}$. Then X is a $\widetilde{P}$ space.*

Proof. Proposition 3.10 implies that X is an $\widetilde{R(1)}$ space and so by Proposition 3.7, X is a $\widetilde{P}$ space (since $\mathrm{NA}(X)$ is a vector space). ∎

REMARK 3.12. Let X be an $\widetilde{R(1)}$ space such that $\mathrm{NA}(X)$ is a vector space. Let Y_1 be a factor reflexive proximinal subspace of X and Y_2 be a finite-codimensional proximinal subspace of X. Observe that $Y_1^{\perp}$ is a reflexive subspace of $\mathrm{NA}(X)$ and $Y_2^{\perp}$ is a finite-dimensional subspace of $\mathrm{NA}(X)$. So $Y_1^{\perp}+Y_2^{\perp}=(Y_1\cap Y_2)^{\perp}\subseteq \mathrm{NA}(X)$. Since X is an $\widetilde{R(1)}$ space, we conclude that $Y_1\cap Y_2$ is a factor reflexive proximinal subspace of X.

REMARK 3.13. It is interesting to see whether the analogue of Lemma 2.13 holds true for factor reflexive spaces.

It follows from the discussion on $\mathcal{K}(\ell_2)$ that if $Y_1,\dots,Y_n$ are factor reflexive proximinal subspaces of $\mathcal{K}(\ell_2)$, then $Y_1\cap\dots\cap Y_n$ is also proximinal. Moreover, the following shows that for c_0-direct sums of reflexive spaces, the analogue of Lemma 2.13 holds true for factor reflexive spaces.

Let X be the c_0-direct sum of a family $\{X_i : i\in\mathbb{N}\}$ of reflexive spaces. Let N be a closed subspace of $\mathrm{NA}(X)$. Then there is a finite set A of $\mathbb{N}$ such that $N\subseteq M=(\bigoplus_{\ell^1} X_i^*)_{i\in A}$. But M is a reflexive space. Hence so is N. Now by Propositions 3.10 and 2.3, $N_\perp$ is proximinal in X.

Let Y_1 and Y_2 be two factor reflexive proximinal subspaces of X. As before there exist finite subsets A_1 and A_2 of $\mathbb{N}$ such that $Y_1^{\perp}\subseteq M_1=(\bigoplus_{\ell^1} X_i^*)_{i\in A_1}$ and $Y_2^{\perp}\subseteq M_2=(\bigoplus_{\ell^1} X_i^*)_{i\in A_2}$. Now by duality $(M_1\cap M_2)_\perp\subseteq Y_1\cap Y_2\subseteq(\bigoplus_{c_0} X_i)_{i\in\mathbb{N}\setminus(A_1\cup A_2)}$. But $(M_1\cap M_2)_\perp$ is proximinal in X. Thus by Proposition 2.3 again, $Y_1\cap Y_2$ is proximinal in X.

We conclude this section with the following questions.

(i) Is X a $\widetilde{P}$ space only if it is an $\widetilde{R(1)}$ space and $\mathrm{NA}(X)$ is orthogonally linear?
(ii) Is there any example of an $R(1)$ space X and $Y\subset X$ such that the quotient is infinite-dimensional and reflexive, every finite-codimensional subspace containing Y is proximinal in X, but Y itself is not proximinal in X?
(iii) We do not know whether $\mathcal{K}(\ell_p)$ for $1<p<\infty$ and $p\neq 2$ is at least a P space.

4. Renorming of $\widetilde{R(1)}$ spaces. It is known that given a separable space there is an equivalent smooth norm with the same set of norm attaining functionals, i.e., proximinal hyperplanes are the same (see [2]). A natural question then is to know whether proximinal factor reflexive subspaces remain the same. In this section, we answer this question affirmatively. We start with a crucial and simple lemma which applies in particular to all separable spaces.

LEMMA 4.1. *Let $(X, \|\cdot\|)$ be a normed linear space. Let L be any weakly compact convex symmetric subset of X. Let $|||\cdot|||$ be the norm whose unit ball satisfies $B_X(|||\cdot|||) = B_X(\|\cdot\|) + L$. Let Y be a closed subspace of $(X, \|\cdot\|)$. If Y is proximinal in $(X, \|\cdot\|)$ then Y is proximinal in $(X, |||\cdot|||)$.*

Proof. Let $x \in X$ be such that $d_{|||\cdot|||}(x, Y) = 1$. Then for every $n \in \mathbb{N}$, we have $Y \cap (B_{(X,\|\cdot\|)}(x, 1 + 1/n) + (1 + 1/n)L) \neq \emptyset$. Let $y_n = t_n + l_n \in Y \cap (B_{(X,\|\cdot\|)}(x, 1 + 1/n) + (1 + 1/n)L)$, where $y_n \in Y$ and $l_n \in L$. Let $\{l_{n_i}\}$ be a weakly converging subsequence of $\{l_n\}$ and let $x + l = w\text{-}\lim(x + l_{n_i})$. We have $d_{\|\cdot\|}(x + l, Y) = 1$. Since Y is proximinal in $(X, \|\cdot\|)$, we have $d_{\|\cdot\|}(x + l, Y) = \|x + l - y\| = 1$ for some $y \in Y$. If $v = x + l - y$, one has $x + l - v = y \in Y$ and thus $d_{|||\cdot|||}(x, Y) = 1 = |||x - y|||$ and Y is proximinal in $(X, |||\cdot|||)$. ∎

We now prove the main theorem of this section which shows that a separable $\widetilde{R(1)}$ space can be smoothly renormed preserving its proximinality properties. In particular these arguments also hold for $R(1)$ spaces.

THEOREM 4.2. *Let $(X, \|\cdot\|)$ be a separable $\widetilde{R(1)}$ space. Then there exists an equivalent Gateaux smooth norm $|||\cdot|||$ on X such that X with this new norm is again $\widetilde{R(1)}$.*

Proof. By Theorem 9(iv) from [2] there exists an equivalent Gateaux smooth norm $|||\cdot|||$ on X such that $\mathrm{NA}((X, \|\cdot\|)) = \mathrm{NA}((X, |||\cdot|||))$. Indeed, let $\{x_n\}$ be a dense subset of B_X, define $T : \ell_2 \to X$ by $T(\alpha) = \sum_{n=1}^{\infty} 2^{-n}\alpha_n x_n$, and let $K = T(B_{\ell_2})$. The set K is convex, symmetric and norm compact. Let $|||\cdot|||$ be the norm whose unit ball satisfies $B_X(|||\cdot|||) = B_X(\|\cdot\|) + K$. Let $X = (X, \|\cdot\|)$ and $X_1 = (X, |||\cdot|||)$. By Lemma 4.1, $f \in \mathrm{NA}(X)$ if and only if $f \in \mathrm{NA}(X_1)$. Moreover, for $f \in X^*$,

$$
\begin{aligned}
(2)\qquad |||f|||^* &= \sup\{|f(x_1)| : x_1 \in B_{X_1}\}\\
&= \sup\{|f(x + k)| : x \in B_X,\ k \in K\}\\
&= \sup\{|f(x)| : x \in B_X\} + \{|f(k)| : k \in K\}\\
&= \|f\|^* + \sup\{|f(T(\alpha))| : \alpha \in B_{\ell_2}\} = \|f\|^* + \|T^*(f)\|_2.
\end{aligned}
$$

Since T^* is one-to-one and $\|\cdot\|_2$ is strictly convex, it follows that $|||\cdot|||^*$ is strictly convex and thus $|||\cdot|||$ is Gateaux smooth.

Let Y be a factor reflexive subspace of X. Suppose that $Y^\perp \subseteq \mathrm{NA}(X) = \mathrm{NA}(X_1)$. Since X is an $\widetilde{R(1)}$ space, Y is proximinal in X. Let $Y_1 = (Y, |||\cdot|||)$. Then by Lemma 4.1, Y is proximinal in $(X, |||\cdot|||)$, which completes the proof. ∎

REMARK 4.3. By the above results, c_0 and more generally c_0-direct sums of sequences of reflexive spaces admit Gateaux smooth norms such that with these new norms these spaces are still $\widetilde{R(1)}$ spaces.

5. Linearity of $\mathrm{NA}(Y)$ for a hyperplane Y in c_0. We first recall that $\mathrm{NA}(c_0)$ is a vector space and for any proximinal hyperplane Y in c_0, $\mathrm{NA}(Y)$ is a vector space (by Proposition 2.5). However when Y is not proximinal, $\mathrm{NA}(Y)$ can fail to be linear. In this direction we present an example which shows that if $f = (f_i) \in \ell_1$ is not norm attaining then $\mathrm{NA}(\ker f)$ need not even be orthogonally linear.

EXAMPLE 5.1. Let $f = (1/2, 1/2, 1/4, 1/8, \ldots) \in \ell_1$. Let $X = \ker f$. It can be easily seen that $\mathrm{NA}(X)$ is not a vector space ([3]). We show that it is not even orthogonally linear. Indeed, let $g = (1, 0, 0, \ldots)$ and $H = (0, 0, 1, 0, 0, \ldots)$. Now $x = (0, -1/2, 1, 0, 0, \ldots) \in S_{\ker g}$ is such that $H(x) = \|H\| = \|H_{|\ker g \cap X}\| = 1$. So H is strongly orthogonal to g in X^*. But $g + H = (1, 0, 1, 0, 0, \ldots)$ and $\|g + H\|_{X^*} = 2 = 1 + \sum_{i=1}^{\infty} 2^{-i}$. Let $x^{(n)} = (1, -1, \sum_{i=1}^{n} 2^{-i}, -1_4, \ldots, -1_{n+4}, 0, \ldots)$, where $1_i = 1$ for $4 \le i \le n + 4$. Then $x^{(n)}$ is in B_X and $(g + H)(x^{(n)}) \to 2$ but there is no $x \in B_X$ such that $(g + H)(x) = 2$; this implies that $g + H \notin \mathrm{NA}(X)$. Hence $\mathrm{NA}(X)$ is not orthogonally linear. Thus by Theorem 3 of [3] and Corollary 5 of [7], X is an $R(1)$-space but not a P space.

In view of the above example, one can ask the following questions.

QUESTION 5.2. *Are there any non-proximinal hyperplanes of c_0 such that the set of all norm attaining functionals is a vector space?*

QUESTION 5.3. *Do linearity and orthogonal linearity coincide in hyperplanes of c_0? This is a particular case of Question* 1 *from* [7].

We answer affirmatively the above questions.

To state the next result we need the following notation.

Let $f = (f_i) \in S_{\ell_1}$. Suppose $f \notin \mathrm{NA}(c_0)$. Let $|f_{i_1}| = \sup\{|f_i| : i \in \mathbb{N}\}$ and $|f_{i_j}| = \sup\{|f_i| : i \in \mathbb{N} \setminus \{i_1, \ldots, i_{j-1}\}\}$ for $j \ge 2$. Then $\{|f_{i_n}|\}$ is a decreasing sequence. Let $Y = \ker f$.

PROPOSITION 5.4. *Suppose $|f_{i_1}| \ge \sum_{i=1, i \ne i_1}^{\infty} |f_i|$. Then Y is isometric to c_0 and thus $\mathrm{NA}(Y)$ is a vector space. Moreover $\mathrm{NA}(Y) = \{g_{|Y} : g \in \mathrm{NA}(c_0)$ with the i_1th coordinate zero$\}$.*

Proof. Let $y = (y_i) \in Y$ and let $T : Y \to c_0(\mathbb{N} \setminus \{i_1\})$ be defined by $T(y) = (y_i)_{i \in \mathbb{N} \setminus \{i_1\}}$. We have $\|T(y)\|_\infty = \|y\|_\infty$ and T is onto $c_0(\mathbb{N} \setminus \{i_1\})$. Thus we have

$$\begin{aligned} \mathrm{NA}(Y) &= T^*(\mathrm{NA}(c_0(\mathbb{N} \setminus \{i_1\}))) \\ &= \{g_{|Y} : g \in \mathrm{NA}(c_0(\mathbb{N})) \text{ with the } i_1\text{th coordinate zero}\}. \ \blacksquare \end{aligned}$$

First we prove the converse for a particular hyperplane. Let $f = (f_i) \in S_{\ell_1} \setminus \mathrm{NA}(c_0)$ be such that each f_i has a constant sign for $i \in \mathbb{N}$. As above, let $|f_{i_1}| = \max\{|f_i| : i \in \mathbb{N}\}$ and $|f_{i_j}| = \sup\{|f_i| : i \in \mathbb{N} \setminus \{i_1, i_2, \dots, i_{j-1}\}\}$ for $j \geq 2$. Let $Y = \ker f$.

PROPOSITION 5.5. *If* $\mathrm{NA}(Y)$ *is a vector space then* $|f_{i_1}| \geq \sum_{i=1,\, i\neq i_1}^{\infty} |f_i|$.

Proof. Suppose $\mathrm{NA}(Y)$ is a vector space. We argue by contradiction. Assume that there exists a finite subset J_1 of $\mathbb{N} \setminus \{i_1\}$ such that $|f_{i_1}| \leq \sum_{i\in J_1} |f_i|$. Then there exist $\alpha^{(1)} = (\alpha_i^{(1)})$ in $[-1, 1]^{|J_1|}$ and $\alpha^{(2)} = (\alpha_j^{(2)})$ in $[-1, 1]^{|(J_1\cup\{i_1\})\setminus\{i_2\}|}$ such that

$$-f_{i_1} = \sum_{i\in J_1} \alpha_i^{(1)} f_i \quad \text{and} \quad -f_{i_2} = \sum_{j\in(J_1\cup\{i_1\})\setminus\{i_2\}} \alpha_j^{(2)} f_j.$$

Let $g_1 = e_{i_1}$ and $g_2 = e_{i_2}$. It is easy to see that $g_1|_Y, g_2|_Y \in \mathrm{NA}(Y)$. Indeed, let $y^{(1)} = (y_i^{(1)})$ and $y^{(2)} = (y_i^{(2)})$ in S_Y, where

$$y_i^{(1)} = \begin{cases} \alpha_i^{(1)} & \text{if } i \in J_1, \\ 1 & \text{if } i = i_1, \\ 0 & \text{otherwise,} \end{cases} \qquad y_i^{(2)} = \begin{cases} \alpha_i^{(2)} & \text{if } i \in (J_1 \cup \{i_1\}) \setminus \{i_2\}, \\ 1 & \text{if } i = i_2, \\ 0 & \text{otherwise.} \end{cases}$$

Then $g_1|_Y(y^{(1)}) = 1 = \|g_1|_Y\|_{Y^*}$ and $g_2|_Y(y^{(2)}) = 1 = \|g_2|_Y\|_{Y^*}$. We now have

LEMMA 5.6. *The following are equivalent.*

(i) $g_1|_Y + g_2|_Y \in \mathrm{NA}(Y)$.

(ii) *There exists a finite subset* J_2 *of* $\mathbb{N} \setminus \{i_1, i_2\}$ *such that* $|f_{i_1}| + |f_{i_2}| \leq \sum_{i\in J_2} |f_i|$.

Proof of Lemma 5.6. (i)$\Rightarrow$(ii). Suppose $g_1|_Y + g_2|_Y \in \mathrm{NA}(Y)$ but there is no finite subset J_2 of $\mathbb{N} \setminus \{i_1, i_2\}$ such that $|f_{i_1}| + |f_{i_2}| \leq \sum_{i\in J_2} |f_i|$. Let $y = (y_i) \in S_Y$ be such that $(g_1 + g_2)(y) = |y_{i_1} + y_{i_2}| = \|(g_1 + g_2)|_Y\|_{Y^*}$. It is easy to see that y_{i_1} and y_{i_2} have the same sign. We have $f(y) = 0$, so $-(y_{i_1} f_{i_1} + y_{i_2} f_{i_2}) = \sum_{i=1,\, i\neq i_1,i_2}^{\infty} y_i f_i$, which implies that

$$|y_{i_1} f_{i_1} + y_{i_2} f_{i_2}| = \Big|\sum_{i\neq i_1,i_2} y_i f_i\Big| \leq \sum_{i\neq i_1,i_2} |y_i f_i| < \sum_{i\neq i_1,i_2} |f_i|.$$

Let $\alpha_{i_1}, \alpha_{i_2} \in [-1, 1]$ be such that $\mathrm{sign}(\alpha_{i_1}) = \mathrm{sign}(\alpha_{i_2}) = \mathrm{sign}(y_{i_1}) = \mathrm{sign}(y_{i_2})$, $|y_{i_1}| < |\alpha_{i_1}|$, $|y_{i_2}| < |\alpha_{i_2}|$ and

$$-(\alpha_{i_1} f_{i_1} + \alpha_{i_2} f_{i_2}) = \sum_{i=1,\, i\neq i_1,i_2}^{\infty} |f_i|.$$

Let $\alpha_{i_1}^{(n)}$ and $\alpha_{i_2}^{(n)}$ in $[-1, 1]$ be such that

$$-(\alpha_{i_1}^{(n)} f_{i_1} + \alpha_{i_2}^{(n)} f_{i_2}) = \sum_{i=1,\, i\neq i_1,i_2}^{n} |f_i|,$$

$\alpha_{i_1}^{(n)} \to \alpha_{i_1}$ and $\alpha_{i_2}^{(n)} \to \alpha_{i_2}$. Now let $y^{(n)} = (y_i^{(n)})$, where

$$y_i^{(n)} = \begin{cases} -\operatorname{sign}(f_i) & \text{if } i \in \{1,\dots,n\} \setminus \{i_1, i_2\}, \\ \alpha_{i_1}^{(n)} & \text{if } i = i_1, \\ \alpha_{i_2}^{(n)} & \text{if } i = i_2. \end{cases}$$

Then $(g_1 + g_2)(y^{(n)}) = \alpha_{i_1}^{(n)} + \alpha_{i_2}^{(n)}$ and $(g_1 + g_2)(y^{(n)}) \to \alpha_{i_1} + \alpha_{i_2}$. This contradicts the fact that $\|(g_1 + g_2)|_Y\| = |y_{i_1} + y_{i_2}|$. So there exists a finite subset J_2 of $\mathbb{N} \setminus \{i_1, i_2\}$ such that $|f_{i_1}| + |f_{i_2}| \le \sum_{i \in J_2} |f_i|$.

(ii)⇒(i). Assume there exists a finite subset J_2 of $\mathbb{N} \setminus \{i_1, i_2\}$ such that $|f_{i_1}| + |f_{i_2}| \le \sum_{i \in J_2} |f_i|$. Then there exists $\alpha_i \in [-1,1]^{|J_2|}$ such that $|f_{i_1}| + |f_{i_2}| = -\sum_{i \in J_2} \alpha_i f_i$. Consider $y = (y_i)$, where

$$y_i = \begin{cases} \alpha_i & \text{if } i \in J_2, \\ \operatorname{sign}(f_{i_1}) & \text{if } i = i_1, \\ \operatorname{sign}(f_{i_2}) & \text{if } i = i_2, \\ 0 & \text{otherwise.} \end{cases}$$

Then $|(g_1 + g_2)(y)| = 2$ and so $g_1|_Y + g_2|_Y \in \mathrm{NA}(Y)$.

End of proof of Proposition 5.5. If $g_1 + g_2 \notin \mathrm{NA}(Y)$ we are done. Otherwise consider $g_3 = e_{i_3}$. Then as in Lemma 5.6 we can show that $g_1 + g_2 + g_3 \in \mathrm{NA}(Y)$ if and only if there exists a finite subset J_3 of $\mathbb{N} \setminus \{i_1, i_2, i_3\}$ such that $|f_{i_1}| + |f_{i_2}| + |f_{i_3}| \le \sum_{i \in J_3} |f_i|$. Since $f \in S_{\ell_1}$, there exists n_0 such that $\sum_{j=1}^{n_0} |f_{i_j}| \ge 2/3$. So this process has to stop, and we get $n < n_0$ such that $\sum_{j=1}^{n} g_j$ and g_{n+1} are in $\mathrm{NA}(Y)$ but $\sum_{j=1}^{n+1} g_j$ is not, contrary to the assumption that $\mathrm{NA}(Y)$ is a vector space.

Remark 5.7. Lemma 5.6 is not true if f_i's do not have constant sign. Indeed, let $f = (1, -1, 1/2, 1/4, 1/8, \dots)$. Then both $e_1 = (1, 0, 0, \dots)$ and $e_2 = (0, 1, 0, 0, \dots)$ are in $\mathrm{NA}(\ker f)$ and also $e_1 + e_2 \in \mathrm{NA}(\ker f)$ but Lemma 5.6(ii) is not satisfied. But here $e_1 + e_2 + e_3 \notin \mathrm{NA}(\ker f)$.

As usual let $f = (f_i) \in S_{\ell_1} \setminus \mathrm{NA}(c_0)$. Let $|f_{i_1}| = \max\{|f_i| : i \in \mathbb{N}\}$ and $|f_{i_j}| = \max\{|f_i| : i \in \mathbb{N} \setminus \{i_1, \dots, i_{j-1}\}\}$ for $j \ge 2$. Let $Y = \ker f$. Then we have

Theorem 5.8. $\mathrm{NA}(Y)$ *is a vector space if and only if* $|f_{i_1}| \ge \sum_{i=1,\, i \ne i_1}^{\infty} |f_i|$. *Moreover if* $\mathrm{NA}(Y)$ *is a vector space, then* $\mathrm{NA}(Y) = \{h_{|Y} :$ $h \in \mathrm{NA}(c_0)$ *with the* i_1*th coordinate zero*$\}$.

Proof. Let $f = (f_i) \in S_{\ell_1}$, $|f| = (|f_i|)$ and let

$$\operatorname{sign}(f_i) = \begin{cases} 1 & \text{if } f_i \ge 0, \\ -1 & \text{if } f_i < 0. \end{cases}$$

Now we define a map $T : c_0 \to c_0$ by $T(x) = (\operatorname{sign}(f_i) x_i)$. Then T is an invertible isometry and $T(\ker f) = \ker |f|$. Hence $\mathrm{NA}(\ker |f|) = T^*(\mathrm{NA}(\ker f))$.

If $\mathrm{NA}(\ker f)$ is a vector space, then so is $\mathrm{NA}(\ker |f|)$. By Proposition 5.5, $|f_{i_1}| \geq \sum_{i=1,\, i\neq i_1}^{\infty} |f_i|$. The converse follows again by Proposition 5.4. The second part is a consequence of Proposition 5.4. ∎

THEOREM 5.9. *Let $f = (f_i) \in S_{\ell_1}$. Then $\mathrm{NA}(\ker f)$ is orthogonally linear if and only if it is linear.*

Proof. Suppose $\mathrm{NA}(\ker f)$ is orthogonally linear. Let T be an isometry from c_0 to c_0 defined by $T(x) = (\mathrm{sign}(f_i)x_i)$ as in the previous proof. Then $\mathrm{NA}(\ker f)$ is orthogonally linear if and only if $\mathrm{NA}(\ker |f|)$ is. Now it is enough to prove that, if $\mathrm{NA}(\ker |f|)$ is orthogonally linear, then $|f_{i_1}| \geq \sum_{i=1,\, i\neq i_1}^{\infty} |f_i|$ where $|f_{i_j}| = \sup\{|f_i| : i \in \mathbb{N} \setminus \{i_1, \ldots, i_{j-1}\}\}$ for $j \geq 1$ and $i_0 = \{0\}$. Suppose not. Let $g_j = e_{i_j}$ for $j \geq 1$. Then $g_j \in \mathrm{NA}(\ker |f|)$ for $j \geq 1$. It is easy to see that g_3 is strongly orthogonal to g_1. Thus $g_1 + g_3 \in \mathrm{NA}(\ker |f|)$ by orthogonal linearity. Now as in the proof of Lemma 5.6, there exists a finite subset $J_2 \subset \mathbb{N} \setminus \{i_1, i_3\}$ such that $|f_{i_1}| + |f_{i_3}| \leq \sum_{i\in J_2} |f_i|$. There exists $\alpha_i \in [-1, 1]$ for $i \in J_2 \cup \{i_2, i_4\}$ such that

$$-(|f_{i_1}| - |f_{i_3}|) = \sum_{i\in J_2\cup\{i_2,i_4\}} \alpha_i |f_i| \quad \text{and} \quad \alpha_{i_4} \in \{-1, 1\}.$$

Now let $y = (y_i)$, where

$$y_i = \begin{cases} \alpha_i & \text{if } i \in J_2 \cup \{i_2, i_4\}, \\ 1 & \text{if } i = i_1, \\ -1 & \text{if } i = i_3, \\ 0 & \text{otherwise.} \end{cases}$$

Then $y \in S_{\ker(g_1+g_3)}$ and $|g_4(y)| = 1$, which implies that g_4 is strongly orthogonal to $g_1 + g_3$. Thus $g_1 + g_3 + g_4 \in \mathrm{NA}(X)$ by orthogonal linearity. Proceeding as in Proposition 5.5, we show that there exists $n \in \mathbb{N}$ such that g_{n+1} is strongly orthogonal to $g_1 + g_3 + g_4 + \cdots + g_n$ but $g_1 + g_3 + g_4 + \cdots + g_{n+1}$ is not in $\mathrm{NA}(X)$, which contradicts the orthogonal linearity of $\mathrm{NA}(X)$. The converse is trivial. ∎

COROLLARY 5.10. *Let Y be a non-proximinal hyperplane in c_0. Let $f = (f_i) \in \ell_1$ be such that $Y = \ker f$. Then the following are equivalent.*

(i) *Y is a P space.*
(ii) *$|f_{i_1}| \geq \sum_{i=1,\, i\neq i_1}^{\infty} |f_i|$, where $|f_{i_1}| = \max\{|f_i| : i \in \mathbb{N}\}$ and $|f_{i_j}| = \sup\{|f_i| : i \in \mathbb{N} \setminus \{i_1, \ldots, i_{j-1}\}\}$ for $j \geq 2$.*
(iii) *$\mathrm{NA}(Y) = \{h_{|Y} : h \in \mathrm{NA}(c_0)$ with the i_1th coordinate zero$\}$.*

Proof. (ii)⇔(iii) follows from Theorem 5.8; (i)⇒(iii) follows by Corollary 5 of [7], Theorem 5.9 and Theorem 5.8; (iii)⇒(i) follows by Theorem 3 of [3] and Corollary 5 of [7]. ∎

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