COLLOQUIUM MATHEMATICUM

VOL. 104 2006 NO. 1

ON A STEADY FLOW IN A THREE-DIMENSIONAL INFINITE PIPE

BY

PAWEŁ KONIECZNY (Warszawa)

Abstract. The paper examines the steady Navier–Stokes equations in a three-dimensional infinite pipe with mixed boundary conditions (Dirichlet and slip boundary conditions). The velocity of the fluid is assumed to be constant at infinity. The main results show the existence of weak solutions with no restriction on smallness of the flux vector and boundary conditions.

1. Introduction. In this paper we examine the Navier–Stokes system

$$\begin{aligned} (v\cdot\nabla)v-\nu\Delta v+\nabla p&=0 \quad \text{in } \Omega,\\ \nabla\cdot v&=0 \quad \text{in } \Omega, \end{aligned} \tag{1.1}$$

with the boundary conditions

$$\begin{aligned} v\cdot\vec{n}&=0 && \text{on } O_P,\\ \vec{n}\cdot\mathbf{T}(v,p)\cdot\vec{\tau}_1&=0 && \text{on } O_P,\\ v\cdot\vec{\tau}_2&=0 && \text{on } O_P,\\ v&=0 && \text{on } O_D\backslash O_{D_0},\\ v&=v_* && \text{on } O_{D_0}, \end{aligned} \tag{1.2}$$

and the following behaviour at infinity:

$$\begin{aligned} v&\to v_{\infty,1} && \text{as } x_1\to-\infty,\\ v&\to v_{\infty,2} && \text{as } x_1\to+\infty, \end{aligned} \tag{1.3}$$

where the dot $\cdot$ denotes the standard scalar product in $\mathbb{R}^3$, $v=(v_1,v_2,v_3)$ is the velocity of the fluid, p the pressure, ν the positive constant viscosity coefficient; $\vec{n}$, $\vec{\tau}_1$ and $\vec{\tau}_2$ are the outer normal and tangent vectors to the boundary $\partial\Omega$ and $v_{\infty,i}$ is a constant vector field at infinity, i.e. $v_{\infty,i}=(v_{\infty,i},0,0)$. Also

$$\mathbf{T}(v,p)=\nu\mathbf{D}(v)-p\,\mathbf{Id} \tag{1.4}$$

denotes the stress tensor, where $\mathbf{Id}$ is the identity matrix and $\mathbf{D}(v)=\{v_{i,j}+v_{j,i}\}_{i,j=1,2,3}$.

2000 *Mathematics Subject Classification*: 35Q30, 76D05, 76D03.

Key words and phrases: Navier–Stokes equations, slip boundary conditions, infinite pipe, large data and flux.

The domain Ω may be represented as $\Omega = \Omega_D \cup \bigcup_{i=1}^{m} \Omega_{P_i}$, where $\Omega_D \subset \mathbb{R}^3$ is a simply connected bounded domain and Ω_{P_i} $(i = 1, \dots, m)$ are pipelike domains, which can be represented (maybe in different coordinates) as $\Omega_{P_i} = P_i \times \mathbb{R}^+$, where $P_k \subset \mathbb{R}^2$ is a bounded domain with smooth boundary. We also assume that $\Omega_{P_i} \cap \Omega_{P_j} = \emptyset$ for $i \neq j$. Furthermore $O_P = \bigcup_{i=1}^{m} O_{P_i}$, where $O_{P_i} = \partial P_i \times \mathbb{R}^+$, $O_D = \partial\Omega_D \setminus \bigcup_{i=1}^{m} \partial\Omega_{P_i}$ and O_{D_0} is the outlet/inlet for Ω_D where the velocity v is prescribed by condition $(1.2)_5$. In this paper we take $m = 2$.

The boundary data should satisfy the following compatibility condition:

$$v_{\infty,1}|P_1| = v_{\infty,2}|P_2| + \int_{\partial\Omega} \vec{v} \cdot \vec{n}. \tag{1.5}$$

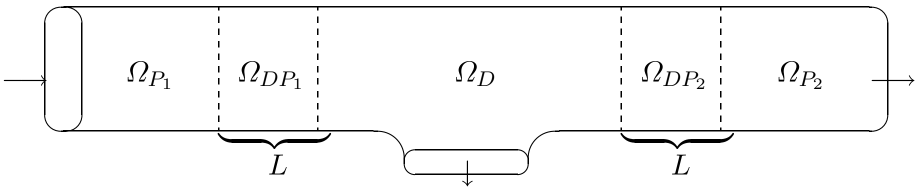

Fig. 1.1

In this paper we prove the existence of weak solutions to problem (1.1)–(1.3). We achieve this without imposing restrictions on the magnitude of data. This problem is similar to the classical Leray problem with the Dirichlet boundary condition and Poiseuille flow at infinity. The slip boundary condition $\vec{n} \cdot \mathbf{T}(v, p) \cdot \vec{\tau} = 0$ describes a model where there is no friction between the fluid and the surface of the pipe.

Slip boundary conditions are not as popular as Dirichlet conditions, however, they deserve to be given more interest. One may consider a generalization of constraints (1.2),

$$\vec{n} \cdot v = 0, \quad \vec{n} \cdot \mathbf{T}(v, p) \cdot \vec{\tau} + f v \cdot \vec{\tau} = 0 \quad \text{on } \partial\Omega_P, \tag{1.6}$$

where f is the friction coefficient. Equations (1.6) describe a model where there is a friction between the fluid and the boundary. If $f = \infty$ one gets from (1.6) the zero Dirichlet condition, and for $f = 0$ the slip boundary condition, where the friction between the fluid and the boundary is negligible.

The last case is related to modelling the flow of a perfect fluid ([12]). The general form of (1.6) is often applied to describe the motion of polymers, blood, or the liquid crystal flow (see [7]). Mixed boundary conditions may be considered as an approximation of the flow, where the velocity of the fluid in the tangent direction τ_2 is negligible, i.e. we have perfect slip boundary conditions with large fluxes (blood is an example).

From the mathematical point of view our problem can be treated as a form of Leray's problem, where the no-slip condition is taken into account. The existence of solutions is still an open question. The main difficulty is hidden in the Dirichlet integral $\int_\Omega \nabla v : \nabla v$ which can be infinite. Using slip boundary conditions we prove that this integral is finite for any data.

A similar result, i.e. without restrictions on the magnitude of the data, is obtained in the paper of Ladyzhenskaya and Solonnikov [6], but the Leray problem is modified (there is no condition on the velocity at infinity) and the existence of solutions is shown in the distribution sense, i.e. locally in space.

In this paper we obtain

$$v - v_{\infty,i} \in H^1(\Omega), \tag{1.7}$$

which allows us to control the behaviour of the fluid at infinity. In the two-dimensional case and for perfect slip boundary conditions the reader may find a similar existence result in [8], but for a three-dimensional pipe and Dirichlet boundary condition there is no such result. Here, we mix slip boundary and Dirichlet conditions to get the existence of solutions satisfying (1.7). Once it has been shown, one can obtain higher regularity and also the asymptotic structure of the vector field v using standard methods (see [9]).

Methods used to prove the existence of solutions to problem (1.1)–(1.3) are similar to those suggested by Leray and Hopf, i.e. a proper vector field a, which is meant to be a flux carrier, is constructed. For this vector field a the following conditions are fulfilled:

$$\nabla \cdot a = 0 \quad \text{in } \Omega, \tag{1.8}$$

$$a \cdot \vec{n} = 0 \quad \text{on } O_P, \tag{1.9}$$

$$a \cdot \vec{\tau}_2 = 0 \quad \text{on } O_P, \tag{1.10}$$

$$a = 0 \quad \text{on } O_D, \tag{1.11}$$

$$a = v^* \quad \text{on } O_{D_0}, \tag{1.12}$$

$$a \to v_{\infty,i} \quad \text{as } |x| \to \infty. \tag{1.13}$$

This allows us to rewrite the vector field v as

$$v = u + a, \tag{1.14}$$

and search for a vector field $u \in H^1(\Omega)$ which satisfies the following system:

$$(u \cdot \nabla)u - \nu \Delta u + \nabla p = \nu \Delta a - (u \cdot \nabla)a - (a \cdot \nabla)u - (a \cdot \nabla)u, \tag{1.15}$$

$$\begin{aligned} \nabla \cdot u &= 0 \quad \text{in } \Omega, \\ u \cdot \vec{n} &= 0 \quad \text{on } O_P, \\ \vec{n} \cdot \mathbf{T}(u,p) \cdot \vec{\tau}_1 &= -\vec{n} \cdot \mathbf{D}(a) \cdot \vec{\tau}_1 \quad \text{on } O_P, \\ u \cdot \vec{\tau}_2 &= 0 \quad \text{on } O_P, \\ u &= 0 \quad \text{on } O_D. \end{aligned} \tag{1.16}$$

Thus it is natural to consider the following space V:

DEFINITION 1.1. Set

$$V := \{f \in H^1(\Omega, \mathbb{R}^3) : \nabla \cdot f = 0,\ f_{|O_D} = 0, \\ f_{|O_P} \cdot \vec{n} = 0,\ f_{|O_P} \cdot \vec{\tau}_2 = 0\}. \tag{1.17}$$

Now we reduce our main problem to finding $u \in H^1(\Omega)$ which satisfies conditions (1.15) and (1.16).

Let us introduce a weak formulation for problem (1.15)–(1.16). Multiplying (1.15) by the test function $\Phi \in V$ and performing simple calculations leads us to the following definition:

DEFINITION 1.2. By a *weak solution* to problem (1.15)–(1.16) we mean a vector field $u \in V$ which satisfies

$$\nu \int_\Omega \mathbf{D}(u) : \nabla \Phi + \int_\Omega [(u \cdot \nabla)u + (u \cdot \nabla)a + (a \cdot \nabla)u] \cdot \Phi \\ = -\nu \int_\Omega \nabla a : \nabla \Phi - \int_\Omega (a \cdot \nabla)a \cdot \Phi - \nu \int_{\partial\Omega_P} \vec{n} \cdot \mathbf{D}(a) \cdot \vec{\tau}_1 (\Phi \cdot \vec{\tau}_1) \tag{1.18}$$

for all $\Phi \in V$.

The existence of u is shown by using the standard Galerkin method. The main difficulty is the construction of the vector field a—it should fulfill suitable conditions (see Theorem 1.4). We hope that the construction carried out in this paper will be useful for other problems.

Finally, we formulate the main theorem:

THEOREM 1.3. *Assume that $\partial\Omega$ is smooth. Then for any data satisfying the compatibility condition* (1.5) *there exists at least one weak solution $v \in H^1(\Omega, \mathbb{R}^3)$ to problem* (1.15)–(1.16) *in the sense of Definition* 1.2. *Moreover, $v - v_{\infty,i} \in H^1(\Omega_{P_i})$ and*

$$\|v - v_{\infty,i}\|_{H^1(\Omega_{P_i})} \le c(v_{\infty,1}, v_{\infty,2}, v^*, \Omega), \tag{1.19}$$

in particular

$$\int_\Omega |\nabla v|^2 \le c(v_{\infty,1}, v_{\infty,2}, v^*, \Omega). \tag{1.20}$$

The main information delivered by Theorem 1.3 is finiteness of the Dirichlet integral for arbitrarily large fluxes. This is accomplished, as mentioned before, by transferring all the flux information onto the proper smooth vector field a. Then to construct a vector field u with finite Dirichlet integral one can use the standard Galerkin method, i.e. show the existence of a solution u_k in finite-dimensional spaces, choose a proper subsequence u_{n_k} and pass to the limit.

Moreover, finiteness of the Dirichlet integral allows us to improve the regularity of solutions using standard techniques (see [8]).

In the definition of a weak solution, a vector field a is used, and to prove Theorem 1.3 we require it to fulfill some conditions which are stated in the following theorem:

THEOREM 1.4. *For every $\varepsilon > 0$ there exists a vector field a satisfying conditions* (1.8)–(1.13) *and the following estimate*:

$$\Big|\int_{\Omega} (u \cdot \nabla) a \cdot u\Big| \le \varepsilon \|u\|^2_{H^1(\Omega)} \tag{1.21}$$

for every function $u \in V$. Moreover, the following inequalities hold:

$$\|\nabla a\|_{L^2(\partial\Omega)} + \|(a \cdot \nabla)a\|_{L^2(\Omega)} + \|\nabla a\|_{L^2(\Omega)} \le c(v_{\infty,1}, v_{\infty,2}, \varepsilon, P_1, P_2) \tag{1.22}$$

and

$$\|a - v_{\infty,i}\|_{L^2(\Omega_{P_i})} \le c(v_{\infty,1}, v_{\infty,2}, \varepsilon, P_1, P_2). \tag{1.23}$$

The vector field a will be constructed in four basic steps:

1. for the bounded region Ω_D we will use the standard Leray–Hopf construction (the resulting vector field will be denoted by a_D);
2. for the unbounded region Ω_P we will adapt the Leray–Hopf construction to slip boundary conditions (vector field a_P);
3. in the bounded intersection of Ω_P and Ω_D of width L we will join a_D and a_P in such a way that the resulting field $a_{P\to D}$ will be divergence free and condition (1.21) will be satisfied;
4. in the unbounded region Ω_P we modify the previously constructed vector field $a_{P\to D}$ to be constant at infinity, i.e. $v_{\infty,i}e_1$, where e_1 is the basis vector from the standard orthogonal basis of $\mathbb{R}^3$.

Our paper is organized as follows: in the next section we give some preliminary lemmas (in Section 2.1) and prove Theorem 1.3 (in Section 2.2) assuming the validity of Theorem 1.4. In Section 3 we prove Theorem 1.4 by constructing the vector field a: in the domain Ω_D in Section 3.1, in Ω_P in Section 3.2, joining a_P and a_D in Section 3.3, and making the vector field constant at infinity in Section 3.4.

2. Proof of Theorem 1.3. This section contains the proof of Theorem 1.3. First we give some preliminaries.

2.1. *Preliminary lemmas.* In this section we show some auxiliary results, which will be necessary for the Galerkin method. The first one can be found in [11].

Lemma 2.1. *Let X be a finite-dimensional Hilbert space, let $P : X \to X$ be a continuous mapping satisfying*

$$(P(\xi), \xi) > 0 \quad \text{for all } \|\xi\| = k > 0, \tag{2.1}$$

where k is some constant, and $(\cdot,\cdot)$ is the standard inner product in X. Then there exists $\xi \in X$ with $\|\xi\| \le k$ for which

$$P(\xi) = 0. \tag{2.2}$$

The second lemma is the Korn inequality, which the reader may find in [10], however in our case we prove it as an equality:

Lemma 2.2. *For every $u \in V$ we have*

$$\int_\Omega (\mathbf{D}(u))^2 = 2\int_\Omega |\nabla u|^2. \tag{2.3}$$

Proof. Without loss of generality, we assume that $u \in V \cap C^2(\Omega)$. Then

$$\int_\Omega (\mathbf{D}(u))^2 = \int_\Omega \sum_{i,j=1}^{3} (u_{i,j} + u_{j,i})^2 = 2\int_\Omega |\nabla u|^2 + \int_\Omega 2\sum_{i,j=1}^{3} u_{i,j}u_{j,i}.$$

Let us take a closer look at the last term:

$$\begin{aligned}\int_\Omega \sum_{i,j=1}^{3} u_{i,j}u_{j,i} &= -\int_\Omega \sum_{i,j=1}^{3} u_{i,ji}u_j + \int_{\partial\Omega} \sum_{i,j=1}^{3} u_{i,j}u_j n_i \\ &= \int_\Omega (\nabla\cdot u)^2 - \int_{\partial\Omega} \sum_{i=1}^{3} u_{i,i}(u\cdot\vec{n}) + \int_{\partial\Omega} \sum_{i,j=1}^{3} u_{i,j}u_j n_i.\end{aligned} \tag{2.4}$$

Now, if we recall that $\nabla\cdot u = 0$ in Ω and $u\cdot\vec{n} = 0$ on $\partial\Omega$ we get

$$\int_\Omega \sum_{i,j=1}^{3} u_{i,j}u_{j,i} = \int_{\partial\Omega} \sum_{i,j=1}^{3} u_{i,j}u_j n_i. \tag{2.5}$$

Since $u\cdot\vec{n} = 0$ on $\partial\Omega$, we have $u_{|\partial\Omega} \in T(\partial\Omega)$ (where $T(\partial\Omega)$ is the tangent space), and so

$$0 = (u\cdot\nabla)(u\cdot\vec{n}) = \sum_{i,j=1}^{3} u_j u_{i,j} n_i + \sum_{i,j=1}^{3} u_j u_i n_{i,j}, \tag{2.6}$$

thus

$$\int_\Omega \sum_{i,j=1}^{3} u_{i,j}u_{j,i} = -\int_{\partial\Omega} \sum_{i,j=1}^{3} u_i u_j n_{i,j}. \tag{2.7}$$

On $\partial\Omega \cap \overline{\Omega}_D$ we have $u = 0$, and on $O_P = \partial\Omega \cap \overline{\Omega}_P$ we have $u_2 = u_3 = 0$, so we can rewrite the above equality in the following way:

$$\int_\Omega \sum_{i,j=1}^{3} u_{i,j}u_{j,i} = -\int_{O_P} u_1 u_1 n_{1,1}; \tag{2.8}$$

but this equals zero, because the geometry of O_P is independent of x_1. ∎

LEMMA 2.3. *There exists a constant $c = c(\Omega)$ such that for every $u \in V$ the following inequality holds*:

$$c\|\nabla u\|_{L^2(\Omega)} \geq \|u\|_{H^1(\Omega)}. \tag{2.9}$$

Proof. It is sufficient to notice that $u_2 = u_3 = 0$ and the integral $\int_{P_1(x_1)} u \cdot \vec{\tau}_1\, dx_2\, dx_3$ equals zero where $P_1(x_1)$ is the x_1-section of P_1. This allows us to use Poincaré inequalities, the one with the zero boundary condition and the one with the mean value. ∎

Below are two useful lemmas which can be found in [5]:

LEMMA 2.4. *Let $\Omega \subset \mathbb{R}^2$ be a simply connected domain with boundary of class $C^{2+\alpha}$. For a given vector field $v^* \in C^{1+\alpha}(\partial\Omega)$ and function $g \in C^\alpha(\Omega)$ satisfying the compatibility condition*

$$\int_{\partial\Omega} v^* \cdot \vec{n} = \int_\Omega g \tag{2.10}$$

there exists a vector field $w \in C^{1+\alpha}(\partial\Omega)$ which satisfies

$$\nabla \cdot w = g \quad \text{in } \Omega, \tag{2.11}$$

$$w = v^* \quad \text{on } \partial\Omega. \tag{2.12}$$

LEMMA 2.5. *Let $\Omega \subset \mathbb{R}^3$ be a locally Lipschitz domain. For a given vector field v^* which satisfies the compatibility condition*

$$\int_{\partial\Omega} v^* \cdot \vec{n} = 0, \tag{2.13}$$

there exists a vector field w which satisfies

$$\nabla \cdot w = 0 \quad \text{in } \Omega, \tag{2.14}$$

$$w = v^* \quad \text{on } \partial\Omega. \tag{2.15}$$

LEMMA 2.6. *Let Ω be a bounded, locally Lipschitz domain. There exists a constant $c = c(\Omega)$ such that for every function $u \in H_0^1(\Omega)$ the following inequality holds*:

$$\left\|\frac{u}{\delta}\right\|_{L^2(\Omega)} \leq c\|u\|_{H_0^1(\Omega)}, \tag{2.16}$$

where $\delta(x) = \operatorname{dist}(x, \partial\Omega)$.

Proof. Without loss of generality we show the inequality in the case where $\Omega = (0, 2c)$, $\delta(x) = x$ and $u \in C_0^\infty(0, 2c)$. We have

$$\frac{u^2(x)}{x^2} = -\frac{d}{dx}\left[\frac{1}{x}\, u^2(x)\right] + \frac{1}{x}\,\frac{d}{dx} u^2(x). \tag{2.17}$$

Integrating (2.17) over the interval $(0, c)$, we get

$$\begin{aligned} \int_0^c \frac{u^2(x)}{x^2}\, dx &= \int_0^c -\frac{d}{dx}\left[\frac{1}{x}\, u^2(x)\right] dx + \int_0^c \frac{1}{x}\,\frac{d}{dx} u^2(x)\, dx \\ &= -\frac{1}{x}\, u^2(x)\bigg|_0^c + \int_0^c \frac{1}{x}\,\frac{d}{dx} u^2(x)\, dx \\ &= -\frac{1}{c}\, u^2(c) + \int_0^c \frac{1}{x}\, 2u(x)u'(x)\, dx \\ &\le \Big(4\int_0^c \frac{1}{x^2}\, u^2(x)\, dx\Big)^{1/2} \Big(\int_0^c u'^2(x)\, dx\Big)^{1/2}. \end{aligned} \tag{2.18}$$

After dividing both sides of the above inequality by $\big(\int \frac{1}{x^2} u^2(x)\, dx\big)^{1/2}$ we get the conclusion of the lemma. ∎

2.2. *Existence of a weak solution.* In this section we prove the existence of a weak solution. We use the standard Galerkin method. Let $w_1, w_2, \ldots$ be an orthonormal basis of the space V. We consider the approximation spaces $V^N = \operatorname{span}\{w_1, \ldots, w_N\}$. Solutions $u^N \in V^N$ will be looked for in the following form:

$$u^N = \sum_{i=1}^N c_i^N w_i. \tag{2.19}$$

Inserting u^N into equation (1.18), and taking w_k as test functions, we get

$$\begin{aligned} \nu \int_\Omega \mathbf{D}(u^N) : \nabla w_k &+ \int_\Omega [(u^N \cdot \nabla)u^N + (u^N \cdot \nabla)a + (a \cdot \nabla)u^N] \cdot w_k \\ &= -\nu \int_\Omega \nabla a : \nabla w_k - \int_\Omega (a \cdot \nabla)a \cdot w_k - \nu \int_{\partial\Omega_P} \vec{n} \cdot \mathbf{D}(a) \cdot \vec{\tau}_1 (w_k \cdot \vec{\tau}_1), \end{aligned} \tag{2.20}$$

which has to be valid for every $k = 1, \ldots, N$. Now we observe that (2.20) is, in fact, an equation for the unknown coefficients c_i^N. We show that this equation has a solution. In order to prove it we use Lemma 2.1. Let us introduce the following mapping P:

$$P(u^N) = \sum_{k=1}^{N} \Big(\nu \int_\Omega (\mathbf{D}(u^N) + \mathbf{D}(a)) : \nabla w_k + \int_\Omega ((u^N \cdot \nabla) u^N + (u^N \cdot \nabla) a + (a \cdot \nabla) u^N + (a \cdot \nabla) a) \cdot w_k + \int_{\partial\Omega} \vec{n} \cdot \mathbf{D}(a) \cdot \vec{\tau}_1 (w_k \cdot \vec{\tau}_1) \Big) w_k. \tag{2.21}$$

It is easily seen that the mapping $P : V^N \to V^N$ is continuous.

Now we will examine $(P(u^N), u^N)_{V^N}$, where $(\cdot,\cdot)_{V^N}$ is the inner product in V^N (derived from $H^1(\Omega)$). It is not hard to notice that the above equation is equivalent to

$$(P(u^N), u^N)_{V^N} = \nu \int_\Omega (\mathbf{D}(u^N) + \mathbf{D}(a)) : \nabla u^N + \int_\Omega ((u^N \cdot \nabla)(u^N + a) + (a \cdot \nabla)(u^N + a)) \cdot u^N + \int_{\partial\Omega} \vec{n} \cdot \mathbf{D}(a) \cdot \vec{\tau}_1 (u^N \cdot \vec{\tau}_1). \tag{2.22}$$

Let us estimate the right hand side of (2.22). We have

$$\nu \int_\Omega \mathbf{D}(u^N) : \nabla u^N = \frac{\nu}{2} \int_\Omega (\mathbf{D}(u^N))^2, \tag{2.23}$$

and using Lemma 2.2 we conclude that

$$\nu \int_\Omega \mathbf{D}(u^N) : \nabla u^N \ge \nu \|u^N\|^2_{H^1(\Omega)}. \tag{2.24}$$

The Schwarz inequality gives

$$\Big| \nu \int_\Omega \mathbf{D}(a) : \nabla u^N \Big| \le C_1 \|\nabla a\|_{L^2(\Omega)} \|u^N\|_{H^1(\Omega)},$$
$$\Big| \int_\Omega (a \cdot \nabla) a \cdot u^N \Big| \le C_2 \|(a \cdot \nabla) a\|_{L^2(\Omega)} \|u^N\|_{H^1(\Omega)};$$

in the first of these inequalities we used Lemma 2.3, namely

$$\|\nabla u\|_{L^2(\Omega)} \ge c \|u\|_{H^1(\Omega)} \tag{2.25}$$

for every $u \in V$.

From the basic properties of the functional $\int_\Omega u \cdot \nabla v \cdot w$ we also have

$$\int_\Omega (u^N \cdot \nabla) u^N \cdot u^N = 0 \quad \text{and} \quad \int_\Omega (a \cdot \nabla) u^N \cdot u^N = 0. \tag{2.26}$$

We choose the constant $\eta = \nu/2$ and apply Theorem 1.4 to get

$$\Big|\int_\Omega (u^N\cdot\nabla)a\cdot u^N\Big| \le \frac{\nu}{2}\,\|u^N\|^2_{H^1(\Omega)}. \tag{2.27}$$

There is also another part to estimate:

$$\begin{aligned}\Big|\int_{\partial\Omega} \vec{n}\cdot\mathbf{D}(a)\cdot\vec{\tau}_1(u^N\cdot\vec{\tau}_1)\Big| &\le C_3\|\nabla a\|_{L^2(\partial\Omega)}\|u^N\|_{L^2(\partial\Omega)}\\ &\le C_4\|\nabla a\|_{L^2(\partial\Omega)}\|u^N\|_{H^1(\Omega)},\end{aligned} \tag{2.28}$$

where we used the trace theorem.

Summing up the above estimates we get

$$\begin{aligned}(P(u^N),u^N)_{V^N} \ge \|u^N\|_{H^1(\Omega)}\cdot\bigg(\frac{\nu}{2}\,\|u^N\|_{H^1(\Omega)} - C(\|\nabla a\|_{L^2(\Omega)}\\ + \|(a\cdot\nabla)a\|_{L^2(\Omega)} + \|\nabla a\|_{L^2(\partial\Omega)})\bigg).\end{aligned} \tag{2.29}$$

Now, if only some constant K satisfies the inequality

$$\frac{\nu}{2}\,K - C(\|\nabla a\|_{L^2(\Omega)} + \|(a\cdot\nabla)a\|_{L^2(\Omega)} + \|\nabla a\|_{L^2(\partial\Omega)}) > 0, \tag{2.30}$$

then also

$$(P(u^N),u^N)_{V^N} > 0 \quad \text{for every } \|u^N\|_{V^N} = K, \tag{2.31}$$

and, from Lemma 2.1, there exists $u_*^N \in V^N$ for which $P(u_*^N) = 0$. Furthermore, the existence of this element and the definition of $P : V^N \to V^N$ (see (2.21)) imply the existence of the coefficients c_i^N in (2.20), and $u_*^N = \sum_{k=1}^N c_k^N w_k$. Additionally, we get an estimate for $\|u_*^N\|_{H^1(\Omega)}$ independent of N:

$$\|u_*^N\|_{H^1(\Omega)} \le K. \tag{2.32}$$

The sequence $\{u_*^i\}_{i=1}^\infty$ is bounded in the Hilbert space V, so there exists a subsequence $\{u_*^{i_k}\}_{k=1}^\infty$ weakly convergent to, say, $u_* \in V$:

$$u_*^{i_k} \rightharpoonup u_* \quad \text{weakly in } V \text{ as } k\to\infty. \tag{2.33}$$

However, we need strong convergence. We cannot use the Rellich theorem, because the region Ω is unbounded. Let us consider the bounded regions

$$\Omega_k = \Omega \cap \{x\in\mathbb{R}^3 : |x_1| \le k\} \quad \text{for } k = 1,2,\ldots. \tag{2.34}$$

From the sequence $\{u_*^{i_k}\}$ we choose a subsequence (still denoted by $u_*^{i_k}$) strongly convergent in $L^4(\Omega_1)$ to u_*; this subsequence has a further subsequence strongly convergent in $L^4(\Omega_2)$, and so forth. By a diagonal procedure we choose a subsequence $u_*^{i_k}$ strongly convergent in L^4 to u_* in every region Ω_k. Let us denote this sequence by $\{u^k\}$. From strong convergence in Ω_k we deduce the existence of a function $f:\mathbb{N}\to\mathbb{N}$ for which

$$\|u^{f(k)} - u_*\|_{L^4(\Omega_k)} \le 1/k. \tag{2.35}$$

Our aim is to deal with the nonlinear term in the weak formulation of our problem. We claim that

$$\int_\Omega (u^{f(k)} \cdot \nabla) u^{f(k)} \cdot \Phi \to \int_\Omega (u_* \cdot \nabla) u_* \cdot \Phi \tag{2.36}$$

for every test function $\Phi \in V$. Let us calculate:

$$\int_\Omega (u^{f(k)} \cdot \nabla) u^{f(k)} \cdot \Phi = \int_\Omega ((u^{f(k)} - u_*) \cdot \nabla) u^{f(k)} \cdot \Phi + \int_\Omega (u_* \cdot \nabla) u^{f(k)} \cdot \Phi.$$

In $\int_\Omega (u_* \cdot \nabla) u^{f(k)} \cdot \Phi$ we can justify the passage to the limit directly from the definition of weak convergence. For the other integral we have

$$\Big| \int_\Omega ((u^{f(k)} - u_*) \cdot \nabla) u^{f(k)} \cdot \Phi \Big| \to 0 \quad \text{as } k \to \infty. \tag{2.37}$$

Indeed, using the Schwarz inequality and triangle inequality we get

$$\begin{aligned} &\Big| \int_\Omega ((u^{f(k)} - u_*) \cdot \nabla) u^{f(k)} \cdot \Phi \Big| \\ &\quad \le \Big| \int_{\Omega_k} ((u^{f(k)} - u_*) \cdot \nabla) u^{f(k)} \cdot \Phi \Big| + \Big| \int_{\Omega'_k} ((u^{f(k)} - u_*) \cdot \nabla) u^{f(k)} \cdot \Phi \Big| \\ &\quad \le C \|u^{f(k)} - u_*\|_{L^4(\Omega_k)} K \|\Phi\|_{L^4(\Omega_k)} + CK^2 \|\Phi\|_{L^4(\Omega'_k)}, \end{aligned}$$

where K is a constant estimating the norms $\|u^k\|_{H^1(\Omega)}$ (see (2.32)), and $\Omega'_k = \Omega \setminus \Omega_k$ is the complementary region to Ω_k. Recalling (2.35), we can write

$$\Big| \int_\Omega ((u^{f(k)} - u_*) \cdot \nabla) u^{f(k)} \cdot \Phi \Big| \le CK \|\Phi\|_{H^1(\Omega)}/k + CK^2 \|\Phi\|_{H^1(\Omega'_k)}.$$

We have

$$\|\Phi\|_{H^1(\Omega'_k)} \to 0 \quad \text{as } k \to \infty, \tag{2.38}$$

and so

$$\int_\Omega (u^{f(k)} \cdot \nabla) u^{f(k)} \cdot \Phi \to \int_\Omega (u_* \cdot \nabla) u_* \cdot \Phi \quad \text{as } k \to \infty. \tag{2.39}$$

Let us also look at the term

$$\int_\Omega \mathbf{D}(u^{f(k)}) : \nabla \Phi. \tag{2.40}$$

From the definition of weak convergence we find that as $k \to \infty$,

$$\int_\Omega \mathbf{D}(u^{f(k)}) : \nabla \Phi \to \int_\Omega \mathbf{D}(u_*) : \nabla \Phi. \tag{2.41}$$

The calculations justifying the passage to the limit in the other terms are simpler than for the nonlinear term.

We proved that there exists a function $u_* \in V$ which satisfies

$$\nu \int_{\Omega} \mathbf{D}(u_*) : \nabla \Phi + \int_{\Omega} [(u_* \cdot \nabla) u_* + (u_* \cdot \nabla) a + (a \cdot \nabla) u_*] \cdot \Phi = -\nu \int_{\Omega} \nabla a : \nabla \Phi - \int_{\Omega} (a \cdot \nabla) a \cdot \Phi - \nu \int_{\partial\Omega_P} \vec{n} \cdot \mathbf{D}(a) \cdot \vec{\tau}_1 (\Phi \cdot \vec{\tau}_1) \tag{2.42}$$

for every test function $\Phi \in V$.

Additionally, we have an estimate on the norm of u_*:

$$\|u_*\|_{H^1(\Omega)} \le K, \tag{2.43}$$

where the constant K depends only on a, Ω and the conditions imposed on the solution.

Consequently, we have

$$\begin{aligned} \|v - v_{\infty,i}\|_{H^1(\Omega_{P_i})} &= \|u + a - v_{\infty,i}\|_{H^1(\Omega_{P_i})} \\ &\le \|u\|_{H^1(\Omega_{P_i})} + \|a - v_{\infty,i}\|_{H^1(\Omega_{P_i})} \le C, \end{aligned} \tag{2.44}$$

where the constant C is as in the assertion of Theorem 1.3.

3. Proof of Theorem 1.4. As mentioned in the introduction, our construction will be divided into four steps, each described in a separate subsection.

3.1. *A vector field in Ω_D.* In this section we will prove the following theorem:

THEOREM 3.1. *For every $\varepsilon_D > 0$ and for every vector field v^* given on $\partial\Omega$ and satisfying the compatibility condition* (1.5) *there exists a vector field a_D which satisfies the following conditions*:

$$\nabla \cdot a_D = 0 \quad \text{in } \Omega_D,$$
$$a_D = v^* \quad \text{on } \partial\Omega_D$$

and

$$\int_{\Omega_D} |u \cdot \nabla a_D \cdot u| \le \varepsilon_D \|u\|^2_{H^1_0(\Omega_D)} \tag{3.1}$$

for every $u \in H^1_0(\Omega_D)$.

A method first introduced by Leray and then clarified by Hopf is adapted here. We will need a function $\varphi_\varepsilon(t)$ described as follows:

DEFINITION 3.2. Let $\varphi_\varepsilon : \mathbb{R}^+ \to \mathbb{R}$ be defined by

$$\varphi_\varepsilon(t) = \begin{cases} 1 & \text{for } t < \gamma^2(\varepsilon), \\ \varepsilon \ln(\gamma(\varepsilon)/t) & \text{for } \gamma^2(\varepsilon) \le t < \gamma(\varepsilon), \\ 0 & \text{for } t \ge \gamma^2(\varepsilon), \end{cases} \tag{3.2}$$

where $\gamma(\varepsilon) = \exp(-1/\varepsilon)$.

Using φ_ε we can define a function Φ_ε:

DEFINITION 3.3. Let $\Phi_\varepsilon : \Omega \to \mathbb{R}$ be defined as follows:

$$\Phi_\varepsilon(x) = (m * \varphi_\varepsilon)(\delta(x)), \tag{3.3}$$

where $\delta(x) = \operatorname{dist}(x, \partial\Omega_D)$, m is a standard mollifier with support in the interval $(-\gamma^2/2, \gamma^2/2)$, and φ_ε is the function from Definition 3.2.

The following properties of Φ_ε will be used:

FACT 3.4.

(1) $\Phi_\varepsilon \in C^\infty(\Omega_D)$,
(2) $\Phi_\varepsilon \le 1$ *for every* $x \in \Omega_D$,
(3) $\Phi_\varepsilon = 1$ *for* $x \in \Omega_D$ *such that* $\delta(x) \le \gamma^2(\varepsilon)/2$,
(4) $\Phi_\varepsilon = 0$ *for* $x \in \Omega_D$ *such that* $\delta(x) \ge 2\gamma(\varepsilon)$,
(5) $|\nabla\Phi_\varepsilon| \le \varepsilon/\delta(x)$.

Now let a'_D be the vector field from Lemma 2.5 with the vector field v^* given on the boundary. Since Ω_D is simply connected, the vector field a'_D can be represented as

$$a'_D = \nabla \times A \tag{3.4}$$

for some vector field A which also satisfies

$$\|A\|_{H^2(\Omega_D)} \le c\|v^*\|_{H^{1/2}(\partial\Omega)}. \tag{3.5}$$

Consider the vector field $a_D(\varepsilon)$ defined as follows:

$$a_D(\varepsilon) = \nabla \times (\Phi_\varepsilon A). \tag{3.6}$$

Using the properties of Φ_ε we have $a_{D|\partial\Omega} = v^*$, and also $\nabla \cdot a_D = 0$. Furthermore

$$|a_D| = |\nabla \times (\Phi_\varepsilon A)| \le 2(|\nabla\Phi_\varepsilon|\,|A| + |\Phi_\varepsilon \nabla \times A|). \tag{3.7}$$

From the Schwarz inequality and the properties of the form $\int_\Omega v \cdot \nabla u \cdot w$ we compute:

$$\Big|\int_\Omega u \cdot \nabla a_D \cdot u\Big| = \Big|\int_\Omega u \cdot \nabla u \cdot a_D\Big| \le \|u\|_{H^1(\Omega)}\Big|\int_\Omega |u|^2|a_D|^2\Big|^{1/2}.$$

Next, we have

$$\begin{aligned}\Big|\int_\Omega |u|^2|a_D|^2\Big|^{1/2} &\le \Big|\int_\Omega |u|^2 2(|\nabla\Phi_\varepsilon|\,|A| + |\Phi_\varepsilon\nabla \times A|)|^2\Big|^{1/2}\\ &\le c(A)\Big|\int_\Omega \varepsilon\Big|\frac{u}{\delta}\Big|^2 + \Big(\int_\Omega |u|^4\Big)^{1/2}\Big(\int_{\Omega_\varepsilon} |\nabla \times A|^4\Big)^{1/2}\Big|^{1/2}\end{aligned}$$

$$\leq c(A)\varepsilon\|u\|_{H_0^1(\Omega)} + c(A)\|u\|_{L^4(\Omega)}\Big(\int\limits_{\Omega_\varepsilon}|\nabla\times A|^4\Big)^{1/2}$$
$$\leq \|u\|_{H_0^1(\Omega)}\sigma(\varepsilon),$$

where $\sigma(\varepsilon)$ is a continuous real function with $\sigma(\varepsilon)\to 0$ as $\varepsilon\to 0$, and

$$\Omega_\varepsilon = \{x\in\Omega_D : \delta(x) < 2\gamma(\varepsilon)\}. \tag{3.8}$$

Our claim then follows after taking ε small enough.

3.2. *A vector field in Ω_P.* In this subsection we will prove the following theorem:

THEOREM 3.5. *Given ε_P, in $\Sigma := P_1\times\mathbb{R}$ there exists a vector field a_P which satisfies the following conditions*:

$$\begin{aligned} \nabla\cdot a_P &= 0 \quad \textit{in } \Sigma,\\ a_P\cdot\vec{n} &= 0 \quad \textit{on } \partial\Sigma,\\ a_P\cdot\vec{\tau}_2 &= 0 \quad \textit{on } \partial\Sigma,\\ \int\limits_{I(x_1)} a_P\cdot\vec{n} &= v_{\infty,1}|P_1|. \end{aligned}$$

Moreover

$$\operatorname{supp} a_P \subset \Sigma_{\varepsilon_P} = \{x\in\Sigma : \operatorname{dist}(x,\partial\Sigma)\leq\varepsilon_P\}$$

and

$$\Big|\int\limits_{\Sigma} u\cdot\nabla a_P\cdot u\Big| \leq \varepsilon_P\|u\|^2_{H^1(\Sigma)} \tag{3.9}$$

for every $u\in H^1(\Sigma)$ for which

$$\nabla\cdot u = 0 \quad \textit{in } \Sigma,\qquad u\cdot\vec{\tau}_2 = 0\quad \textit{on } \partial\Sigma,\qquad u\cdot\vec{n}=0\quad \textit{on } \partial\Sigma. \tag{3.10}$$

First, let us introduce the following definition:

DEFINITION 3.6. Let $\Psi_\varepsilon:\Sigma\to\mathbb{R}$ be defined as follows:

$$\Psi_\varepsilon(x) = (m*\varphi'_\varepsilon)(\delta(x)), \tag{3.11}$$

where $\delta(x)=\operatorname{dist}(x,\partial\Sigma)$ and m is the standard mollifier with support in $(-\gamma^2(\varepsilon)/2,\gamma^2(\varepsilon))$.

It is obvious from the definition that

(1) $\Psi_\varepsilon\in C^\infty(\Sigma)$,
(2) $\Psi_\varepsilon(x)=0$ for $x\in\Omega$ such that $\delta(x)\geq 2\gamma(\varepsilon)$ or $\delta(x)\leq\gamma^2(\varepsilon)$.

Now, denoting by $I(x_1)=P_1(x_1)$ the x_1-section of the region P_1 at some point x_1 and defining

$$K_\varepsilon = \int\limits_{I(x_1)} \Psi_\varepsilon(x)\,dx_2\,dx_3, \tag{3.12}$$

we find that

(1) K_ε is independent of x_1,
(2) $K_\varepsilon > c$, where c is a constant independent of ε.

We can now define the vector field a_P:

DEFINITION 3.7. Set

$$a_P(x) = \left(\frac{|P_1| v_{\infty,1}}{K_\varepsilon}\, \Psi_\varepsilon(x), 0, 0\right). \tag{3.13}$$

Before we start the proof of Theorem 3.5 we should remark that the compatibility condition at infinity is satisfied:

$$\int_{I(x_1)} a_P \cdot \vec{n} = v_{\infty,1} |P_1|, \tag{3.14}$$

and also

$$\nabla \cdot a_P = 0. \tag{3.15}$$

Proof of Theorem 3.5. First, we notice that for u satisfying

$$u \cdot \vec{\tau}_2 = 0, \qquad u \cdot \vec{n} = 0 \qquad \text{on } \partial\Sigma, \tag{3.16}$$

one has

$$u_2, u_3 \in H_0^1(\Sigma). \tag{3.17}$$

Denote the ith coordinate of the vector field a_P by $a_P^{(i)}$. We calculate

$$\begin{aligned} \int_\Sigma u \cdot \nabla a_P \cdot u &= \int_\Sigma u \cdot \nabla a_P \cdot u = \int_\Sigma u_2 \frac{\partial a_P^{(1)}}{\partial x_2} u_1 + \int_\Sigma u_3 \frac{\partial a_P^{(1)}}{\partial x_3} u_1 \\ &= -\int_\Sigma u_{2,2} a_P^{(1)} u_1 - \int_\Sigma u_2 a_P^{(1)} u_{1,2} + \int_{\partial\Sigma} u_2 a_P^{(1)} u_1 \\ &\quad - \int_\Sigma u_{3,3} a_P^{(1)} u_1 - \int_\Sigma u_3 a_P^{(1)} u_{1,3} + \int_{\partial\Sigma} u_3 a_P^{(1)} u_1 \\ &= -\int_\Sigma u_2 a_P^{(1)} u_{1,2} - \int_\Sigma u_3 a_P^{(1)} u_{1,3}. \end{aligned} \tag{3.18}$$

Now, we get

$$\left|\int_\Sigma u \cdot \nabla a_P \cdot u\right| \le \left|\int_\Sigma u_2 a_P^{(1)} u_{1,2}\right| + \left|\int_\Sigma u_3 a_P^{(1)} u_{1,3}\right|. \tag{3.19}$$

Recalling that $|P_1| v_{\infty,1}/K_\varepsilon \le M(v_\infty, |P_1|)$ for a constant M independent of ε and using the properties of $\Psi_\varepsilon(x)$, we estimate the first integral on the right hand side of (3.19):

$$\left|\int_\Sigma u_2 a_P^{(1)} u_{1,2}\right| \le K\varepsilon \left|\int_\Sigma \frac{u_2}{\delta}\, u_{1,2}\right| \le K\varepsilon \left(\int_\Sigma \left(\frac{u_2}{\delta}\right)^2\right)^{1/2} \left(\int_\Sigma u_{1,2}^2\right)^{1/2}. \tag{3.20}$$

From (3.17) and Lemma 2.6 we get

$$\Big|\int_{\Sigma} u_2 a_P^{(1)} u_{1,2}\Big| \le K\varepsilon \|u\|^2_{H^1(\Sigma)}.$$

In a similar way we estimate the second integral in (3.19). Choosing ε small enough we get the claim of our theorem. ■

3.3. *Joining the vector fields a_P and a_D.* In this section we will show how to join the previously constructed fields a_P and a_D in such a way that the resulting vector field $a_{P\to D}$ is divergence free and preserves proper estimates.

THEOREM 3.8. *For every constant $\varepsilon_{PD} > 0$ and the vector field v^* from Theorem 3.1 there exists a vector field $a_{P\to D}$ defined in Ω which satisfies the following conditions*:

$$\begin{aligned} \nabla\cdot a_{P\to D} &= 0 && \textit{in } \Omega,\\ a_{P\to D} &= v^* && \textit{on } O_D,\\ a_{P\to D}\cdot \vec{n} &= 0 && \textit{on } O_P,\\ a_{P\to D}\cdot \vec{\tau}_2 &= 0 && \textit{on } O_P, \end{aligned}$$

and the estimate

$$\Big|\int_{\Omega} u\cdot \nabla a_{P\to D}\cdot u\Big| \le \varepsilon_{PD}\|u\|^2_{H^1(\Omega)}. \tag{3.21}$$

Let us assume, without loss of generality, that our construction is carried out in the domain $P_1\times(0,L)$ (see Fig. 1.1 in the introduction).

Our aim is to construct a vector field which smoothly joins the vector fields a_D and a_P. In this construction a function π described below will be helpful.

DEFINITION 3.9. Let $\theta < L/2$ be a constant and $\pi : (0,L)\to\mathbb{R}$ be a smooth function satisfying the following conditions:

$$|\pi'| < 2/(L-2\theta),\quad \pi_{|(0,\theta)}\equiv 1,\quad \pi_{|(L-\theta,L)}\equiv 0. \tag{3.22}$$

Then we also have

$$\pi'_{|(0,\theta)}\equiv 0,\qquad \pi'_{|(L-\theta,L)}\equiv 0. \tag{3.23}$$

Let us take a closer look at the vector field w defined as follows:

$$w(x_1,x_2,x_3) := \pi(x_1)a_P(x_1,x_2,x_3) + (1-\pi(x_1))a_D(x_1,x_2,x_3). \tag{3.24}$$

It satisfies

$$w = 0 \quad \text{on } O_D\cap O_P, \tag{3.25}$$

$$\begin{aligned} \nabla\cdot w &= \pi\nabla\cdot a_P + (1-\pi)\nabla\cdot a_D + \pi'(a_P^{(1)} - a_D^{(1)})\\ &= \pi'(a_P^{(1)} - a_D^{(1)}) \end{aligned} \tag{3.26}$$

and we need to correct it to make it divergence free in the domain $\Sigma := \Omega_D \cap \Omega_P = P_1 \times (0, L)$. In the following we use the notation

$$g := a_P^{(1)} - a_D^{(1)}. \tag{3.27}$$

LEMMA 3.10. *There exists* $\overline{w}$ *such that*

$$\nabla \cdot \overline{w} = g \quad \text{in } \Sigma$$

and

$$\Big| \int_\Sigma u \cdot \nabla \pi' \overline{w} \cdot u \Big| \le c(\Sigma, \varepsilon_D, \varepsilon_P) \|u\|^2_{H^1(\Sigma)},$$

$$\nabla \cdot \pi' \overline{w} = \pi' g \quad \text{in } \Sigma,$$

where ε_D *and* ε_P *are the constants from Theorems* 3.1 *and* 3.5. *Additionally*

$$c(\Sigma, \varepsilon_D, \varepsilon_P) \to 0 \quad \text{as } \varepsilon_D, \varepsilon_P \to 0. \tag{3.28}$$

To prove this lemma we will need the following:

LEMMA 3.11. *There exists a covering of the boundary* ∂P_1 *with open rectangles whose sides are parallel to the axes of the basic coordinate system and with open rectangles whose sides form a* $\pi/4$ *angle with these basic axes, such that two corners of these rectangles are inside* P_1, *and the other two are outside* P_1, *determining a direction—from the inside to the outside of the domain. Furthermore, in each rectangle the distance function* $\delta(x, \partial P_1)$ *is decreasing in the direction determined above.*

REMARK 3.12. For an illustration of the above situation see the following pictures:

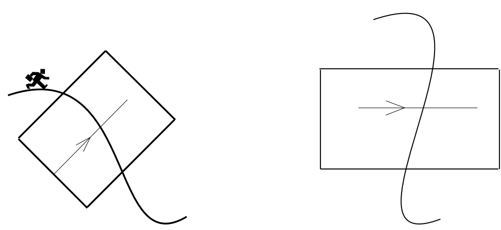

Fig. 3.1

Proof of Lemma 3.10. The rectangles from Lemma 3.11 cover not only the boundary ∂P_1, but also some of its neighbourhood. Let us assume that $\operatorname{supp} g$ is a subset of this neighbourhood. This assumption can be made, by the properties of the fields a_P and a_D (see the construction).

From this covering we choose a finite subcovering. Let η_k be a smooth partition of unity and $g_k := \eta_k g$.

We will carry out the construction of the field $\overline{w}$ in each rectangle separately; then we will add all $\overline{w}_k$'s to get a vector field

$$\overline{w} = \sum_k \overline{w}_k. \tag{3.29}$$

The following properties will hold:

$$\int_\Omega (u\cdot\nabla)\overline{w}\cdot u = \sum_k \int_\Omega (u\cdot\nabla)\overline{w}_k\cdot u, \tag{3.30}$$

$$\nabla\cdot\overline{w} = \sum_k \nabla\cdot\overline{w}_k = \sum_k g_k = g. \tag{3.31}$$

Without loss of generality we assume that the construction for each $\overline{w}_k$ is done in coordinates where the origin is as in one of the pictures below:

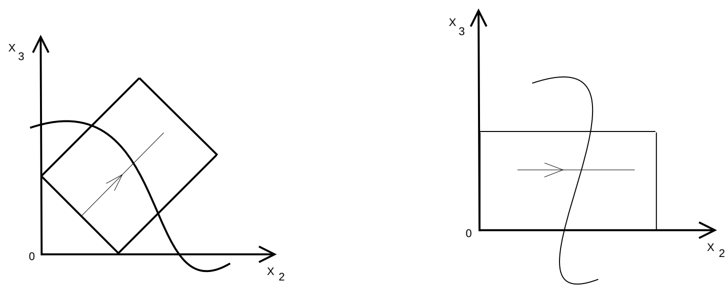

Fig. 3.2

In the first case the vector field $\overline{w}_k$ is described in the following way:

$$\overline{w}_k^{(1)}(x) = 0, \tag{3.32}$$

$$\overline{w}_k^{(2)}(x) = \int_0^{x_3} g_k(x_1, x_2 - x_3 + t, t)\,dt, \tag{3.33}$$

$$\overline{w}_k^{(3)}(x) = \int_0^{x_3} g_k(x_1, x_2 - x_3 + t, t)\,dt. \tag{3.34}$$

This field has its support included in the rectangle. As the support of the function g gets closer to the boundary, the support of w_k also gets closer to the boundary.

LEMMA 3.13.
(1) $\overline{w}_k \in C^\infty(\Omega)$,
(2) $\nabla\cdot\overline{w}_k = g_k$.

Now we get the estimates from the claim of Lemma 3.10.

Let P_k be the kth rectangle multiplied by the interval $(0, L)$, and P'_k be the kth rectangle multiplied by $(\theta, L-\theta)$, where $\theta > 0$ is the constant from Definition 3.9. Let us calculate:

$$\int_{P_k} |u|^2|\pi' w_k|^2\,dx = \int_{P_k} |u|^2\Big|\int_0^{x_3} \pi' g_k(x_1,x_2-x_3+t,t)\,dt\Big|^2\,dx \tag{3.35}$$

$$= \int_{P_k} |u|^2\Big|\int_0^{x_3} \eta_k\pi'(x_1)(a_P^{(1)}-a_D^{(2)})(x_1,x_2-x_3+t,t)\,dt\Big|^2\,dx$$

$$\le \int_{P_k} \pi'|u|^2\Big|\int_0^{x_3} (a_P^{(1)}-a_D^{(2)})(x_1,x_2-x_3+t,t)\,dt\Big|^2\,dx$$

$$\le \int_{P_k} \pi'|u|^2\Big|\int_0^{x_3} (|a_P^{(1)}|+|a_D^{(2)}|)(x_1,x_2-x_3+t,t)\,dt\Big|^2\,dx$$

$$\le c(\pi')\int_{P'_k} |u|^2\Big|\int_0^{x_3} p_1(x_1,x_2-x_3+t,t)\,dt\Big|^2\,dx$$

$$\le c(\pi')\int_{P'_k} |u|^2\Big|\int_0^{x_3} p_2(x_1,x_2-x_3+t,t)\,dt\Big|^2\,dx,$$

where

$$p_1 = |\varepsilon/\delta| + 2(|\nabla\Phi_\varepsilon|\,|A| + |\Phi_\varepsilon\nabla\times A|), \tag{3.36}$$

$$p_2 = |\varepsilon/\delta| + 2(|\varepsilon/\delta|\,|A| + |\Phi_\varepsilon\nabla\times A|). \tag{3.37}$$

From the properties of our rectangles we get

$$\int_0^{x_3} \frac{1}{\delta(x_1,x_2-x_3+t,t)}\,dt \le x_3\,\frac{1}{\delta(x_1,x_2,x_3)}, \tag{3.38}$$

We also have

$$\int_{P'_k} |u|^2|\pi' w_k|\,dx \le c(L,A,K)\Big(\int_{P'_k} |u|^2|\varepsilon/\delta|^2\,dx + \int_{P'_k} |u|^2\int_0^{x_3} |\Phi_\varepsilon\nabla\times A|^2(x_1,x_2-x_3+t,t)\,dt\,dx\Big). \tag{3.39}$$

We estimate the second integral on the right hand side of (3.39) in a similar way to that in the Dirichlet case. Namely, for ε small enough (depending on η)

$$\int_{P'_k} |u|^2\int_0^{x_3} |\Phi_\varepsilon\nabla\times A|(x_1,x_2-x_3+t,t)\,dt\,dx \le \eta\|u\|^2_{H^1(\Omega)}. \tag{3.40}$$

An estimate for the former integral is achieved in a similar way to the previous one. We balance the term $1/\delta(x)$ with the function u, but the function u may not be zero near $P_1 \times \{0\} \cup P_1 \times \{L\}$. However, in our case we are away from the singularities near $P_1 \times \{0\} \cup P_1 \times \{L\}$ (thanks to θ).

Setting ε small enough we get the required estimate.

The case with the rectangle sides not parallel to the axes is almost identical to that described above. We will just give the definition for $\overline{w}_k$:

$$\overline{w}_k^{(1)}(x) = 0, \tag{3.41}$$

$$\overline{w}_k^{(2)}(x) = \int_0^{x_2} g_k(x_1, t, x_3)\, dt, \tag{3.42}$$

$$\overline{w}_k^{(3)}(x) = 0. \tag{3.43}$$

In that way we get a vector field

$$\overline{w} = \sum_k \overline{w}_k, \tag{3.44}$$

which satisfies the assertion of Lemma 3.10 (the number of rectangles is finite). ■

Now we have a divergence free vector field

$$w = \pi a_P + (1-\pi)a_D - \pi'\overline{w}. \tag{3.45}$$

Unfortunately, it may not be zero on the boundary, so we need to correct it in a suitable way. Let us formulate this in the following lemma:

LEMMA 3.14. *For*

$$w = \pi a_P + (1-\pi)a_D - \pi'\overline{w} \tag{3.46}$$

there exists a vector field $\overline{v}$ *which satisfies*

$$\nabla \cdot \overline{v} = 0 \quad \textit{in } \Sigma,$$
$$\overline{v} = w \quad \textit{on } O_D \cap O_P.$$

Moreover, for every function $u \in H^1(\Sigma)$ *with its trace on* $\partial\Sigma \backslash (P_1 \times \{0\} \cup P_1 \times \{L\})$ *equal to zero, the estimate*

$$\Big| \int_\Sigma u \cdot \nabla\overline{v} \cdot u \Big| \le c(\varepsilon) \|u\|^2_{H^1(\Sigma)} \tag{3.47}$$

holds, where $c(\varepsilon) \to 0$ *as* $\varepsilon \to 0$.

Proof. Let $v(x_1)$ be the two-dimensional vector field from Lemma 2.4 for the boundary conditions $v^* = \overline{w}(x_1)$ and the function $g \equiv 0$. Then $\overline{v}(x_1) = (0, v(x_1))$ is a well defined divergence free vector field. We also have

$$\nabla \cdot (\pi'\overline{v}) = 0. \tag{3.48}$$

The vector field $\overline{v}$ can be written as

$$\overline{v} = \nabla \times W \tag{3.49}$$

for some vector field W.

REMARK 3.15. Of course the compatibility condition (2.10) for v is satisfied.

Let

$$\overline{v}_\varepsilon := \pi'(\nabla \times \psi_\varepsilon W). \tag{3.50}$$

Let us recall that $\operatorname{supp} \pi' \subset (\theta, L-\theta)$. As in the previous sections (see Lemmas 3.1 and 3.5), singularities are removed by u or π'. ■

Finally, the vector field

$$a_{P\to D} = \pi a_P + (1-\pi)a_D - \pi'\overline{w} - \overline{v}_\varepsilon, \tag{3.51}$$

for ε_P and ε_D small enough, satisfies the following conditions:

$$\begin{aligned} \nabla \cdot a_{P\to D} &= 0 \quad \text{ in } \Omega,\\ a_{P\to D} &= 0 \quad \text{ on } \partial\Omega,\\ \Big|\int_\Omega u\cdot \nabla a_{P\to D}\cdot u\Big| &\le \varepsilon_{PD}\|u\|^2_{H^1(\Omega)}. \end{aligned} \tag{3.52}$$

Thus, Theorem 3.8 has been proved.

3.4. *Transition to a constant vector field.* Now we would like to make the vector field a constant at infinity. This is the last step in the construction of the vector field a from Theorem 1.4.

The construction is stated in the following lemma:

LEMMA 3.16. *For every constant D and vector fields a_P, $a_{\infty,1}$ (defined in $P_1\times\mathbb{R}$) there exists a smooth vector field $a_{P\to\infty}$ which satisfies the following conditions*:

$$\begin{aligned} \nabla\cdot a_{P\to\infty} &= 0 && \textit{in } P_1\times\mathbb{R},\\ a_{P\to\infty} &= 0 && \textit{on } \partial\Sigma,\\ a_{P\to\infty} &= a_P && \textit{for } x_1\ge D,\\ a_{P\to\infty} &= a_{\infty,1} && \textit{for } x_1\le 0. \end{aligned}$$

Proof. Let us take a closer look at the vector field $a_\pi = \pi a_P + (1-\pi)a_{\infty,1}$. Its divergence equals

$$\nabla\cdot a_\pi = \pi'(v_{\infty,1} - c\Psi_\varepsilon), \tag{3.53}$$

for a constant c and a function $\pi' : (0,D)\to(0,1)$ satisfying

$$\pi'(0) = \pi'(1) = 0, \qquad |\pi'| \le 2/D. \tag{3.54}$$

Let

$$g = v_{\infty,1} - c\Psi_\varepsilon. \tag{3.55}$$

We will try to find a vector field $\overline{w}$ satisfying

$$\nabla\cdot\overline{w} = -g \quad \text{ in } \Sigma. \tag{3.56}$$

We also have

$$\pi'\nabla\cdot\overline{w} = \pi'\nabla\cdot(0,w) = \nabla\cdot(\pi'(0,w)) = \nabla\cdot(\pi'\overline{w}). \tag{3.57}$$

Notice that the function g does not depend on x_1, and neither does $\overline{w}$. Let $\overline{v}$ be a vector field satisfying

$$\nabla\cdot\overline{v} = 0 \quad \text{in } \Sigma, \tag{3.58}$$

$$\overline{v} = -\overline{w} \quad \text{on } \partial\Sigma\setminus(I(0)\cup I(D)). \tag{3.59}$$

The compatibility condition is satisfied:

$$\int\limits_{\partial\Omega(x_1)} \overline{v}\cdot\vec{n} = 0. \tag{3.60}$$

Now, let $a_{P\to\infty}$ be defined by

$$a_{P\to\infty} = a_\pi + \pi'\overline{w} + \pi'\overline{v}. \tag{3.61}$$

This field satisfies

$$\begin{aligned} \nabla\cdot a_{P\to\infty} &= 0 && \text{in } P_1\times\mathbb{R},\\ a_{P\to\infty} &= 0 && \text{on } \partial\Sigma,\\ a_{P\to\infty} &= a_P && \text{for } x_1 \ge D,\\ a_{P\to\infty} &= a_{\infty,1} && \text{for } x_1 \le 0. \ \blacksquare \end{aligned}$$

LEMMA 3.17. *Given $\eta > 0$, there exists D (the constant from Lemma 3.16) such that the vector field $a_{P\to\infty}$ defined above satisfies*

$$\Big|\int\limits_\Omega u\cdot\nabla a_{P\to\infty}\cdot u\Big| \le 5\eta\|u\|^2_{H^1(\Omega)} \tag{3.62}$$

for every function $u\in H^1(\Sigma)\cap V$.

Proof. Using (3.61) we have

$$\begin{aligned} \Big|\int\limits_\Omega u\cdot\nabla a_{P\to\infty}\cdot u\Big| \le \Big|\int\limits_\Omega u\cdot\nabla a_\pi\cdot u\Big| + \Big|\int\limits_\Omega u\cdot\nabla\pi'\overline{w}\cdot u\Big| \\ + \Big|\int\limits_\Omega u\cdot\nabla\pi'\overline{v}\cdot u\Big|. \end{aligned} \tag{3.63}$$

We estimate the first integral:

$$\begin{aligned} \Big|\int\limits_\Omega u\cdot\nabla a_\pi\cdot u\Big| &= \Big|\int\limits_\Sigma u\cdot\nabla(\pi a_P + (1-\pi)a_{\infty,1})u\Big| \\ &= \Big|\int\limits_\Sigma u_1\pi'(a_P - a_{\infty,1})u_1 + \int\limits_\Sigma \pi u\cdot\nabla a_P\cdot u \\ &\quad + \int\limits_\Sigma (1-\pi)u\cdot\nabla a_2\cdot u\Big| \end{aligned} \tag{3.64}$$

$$\leq (2/D)\|a_P - a_{\infty,1}\|_{C(\Sigma)} \int_\Sigma |u|^2 + \Big|\int_\Sigma u\cdot\nabla a_P\cdot u\Big| + \Big|\int_\Sigma u\cdot\nabla a_2\cdot u\Big|$$
$$\leq (2/D)\|a_P - a_{\infty,1}\|_{C(\Sigma)}\|u\|^2_{H^1(\Sigma)} + 2\eta\|u\|^2_{H^1(\Sigma)}.$$

The vector fields a_P and $a_{\infty,1}$ are smooth with a finite $\sup_\Sigma$ norm, so, taking D large enough, we get

$$\Big|\int_\Omega u\cdot\nabla a_\pi\cdot u\Big| \leq 3\eta\|u\|^2_{H^1(\Sigma)}. \tag{3.65}$$

We can estimate the next integral by

$$\begin{aligned}\Big|\int_\Sigma u\cdot\nabla\pi'\overline{w}\cdot u\Big| &= \Big|\int_\Sigma u\cdot\nabla u\cdot\pi'\overline{w}\Big| \leq c\|\nabla u\|_{L^2(\Sigma)}\Big(\int_\Sigma |u|^2|\pi'\overline{w}|^2\Big)^{1/2}\\ &\leq c\|\nabla u\|_{L^2(\Sigma)}(2/D)\|\overline{w}\|_{C(\Sigma)}\Big(\int_\Sigma |u|^2\Big)^{1/2}\\ &\leq (2c/D)\|\overline{w}\|_{C(\Sigma)}\|u\|^2_{H^1(\Sigma)}.\end{aligned} \tag{3.66}$$

The vector field $\overline{w}$ is smooth and has a finite norm $\|\overline{w}\|_{C(\Sigma)}$, so, taking D large enough, we obtain the desired estimate

$$\Big|\int_\Sigma u\cdot\nabla\pi'\overline{w}\cdot u\Big| \leq \eta\|u\|^2_{H^1(\Sigma)}. \tag{3.67}$$

In a similar way we get

$$\Big|\int_\Sigma u\cdot\nabla\pi'\overline{v}\cdot u\Big| \leq \eta\|u\|^2_{H^1(\Sigma)}. \tag{3.68}$$

Summing inequalities (3.65)–(3.68), we obtain (3.62). ■

To finish the proof of Theorem 1.4 we notice that all parts of the vector field a have been constructed. They are gathered in the following definition of the vector field a:

$$a(x) := \begin{cases} a_D(x) & \text{for } x\in\Omega_D\setminus\Omega_P,\\ a_{P\to D,i}(x) & \text{for } x\in\Omega_{DP_i},\\ a_{P\to\infty,i}(x) & \text{for } x\in\Omega_{P_i},\end{cases} \tag{3.69}$$

Estimates (1.21) and (1.22) also hold if proper constants, occurring in the construction, were chosen small enough. Thus, the proof of Theorem 1.4 is finished.

Acknowledgements. The results presented in this paper were obtained during my work on my master thesis. I would like to thank my thesis supervisor Piotr Bogusław Mucha for his encouragement and guidance. I would also like to thank Piotr Rybka for helpful comments which improved the readability of the paper.

REFERENCES

[1] R. A. Adams, *Sobolev Spaces*, Academic Press, New York, 1975.
[2] C. J. Amick and L. E. Fraenkel, *Steady solutions of the Navier–Stokes equations representing plane flow in channels of various types*, Acta Math. 144 (1980), 83–151.
[3] G. P. Galdi, *An Introduction to the Mathematical Theory of the Navier–Stokes Equations*, *Vols. I*, *II*, Springer, New York, 1994.
[4] D. Gilbarg and N. S. Trudinger, *Elliptic Partial Differential Equations of Second Order*, Springer, New York, 1977.
[5] O. A. Ladyzhenskaya, *The Mathematical Theory of Viscous Incompressible Flow*, Gordon and Breach, New York, 1966.
[6] O. A. Ladyzhenskaya and V. A. Solonnikov, *Determination of solutions of boundary value problems for stationary Stokes and Navier–Stokes equations having an unbounded Dirichlet integral*, Zap. Nauchn. Sem. Leningrad. Otdel. Mat. Inst. Steklova (LOMI) 96 (1980), 117–160 (in Russian).
[7] C. Liu and J. Shen, *On liquid crystal flows with free-slip boundary conditions*, Discrete Contin. Dynam. Systems 7 (2001), 307–318.
[8] P. B. Mucha, *On Navier–Stokes equations with slip boundary conditions in an infinite pipe*, Acta Appl. Math. 76 (2003), 1–15.
[9] —, *Asymptotic behavior of a steady flow in a two-dimensional pipe*, Studia Math. 158 (2003), 39–58.
[10] V. A. Solonnikov and V. E. Shchadilov, *On a boundary value problem for a stationary system of Navier–Stokes equations*, Trudy Mat. Inst. Steklova 125 (1973), 186–199.
[11] R. Temam, *Navier–Stokes Equations*, North-Holland, Amsterdam, 1977.
[12] W. M. Zajączkowski, *On global special solutions for Navier–Stokes equations with boundary slip conditions in a cylindrical domain*, GAKUTO Ser. in Math. 21, 2004.

Institute of Applied Mathematics and Mechanics
Warsaw University
Banacha 2
02-097 Warszawa, Poland
E-mail: konieczny@hydra.mimuw.edu.pl

Received 25 January 2005;
revised 28 April 2005 (4558)