COLLOQUIUM MATHEMATICUM

VOL. 104 2006 NO. 1

ON THE APPROXIMATION OF REAL CONTINUOUS FUNCTIONS BY SERIES OF SOLUTIONS OF A SINGLE SYSTEM OF PARTIAL DIFFERENTIAL EQUATIONS

BY

CARSTEN ELSNER (Hannover)

Abstract. We prove the existence of an effectively computable integer polynomial $P(x, t_0, \ldots, t_5)$ having the following property. Every continuous function $f : \mathbb{R}^s \to \mathbb{R}$ can be approximated with arbitrary accuracy by an infinite sum

$$\sum_{r=1}^{\infty} H_r(x_1, \ldots, x_s) \in C^{\infty}(\mathbb{R}^s)$$

of analytic functions H_r, each solving the same system of universal partial differential equations, namely

$$P\left(x_\sigma; H_r, \frac{\partial H_r}{\partial x_\sigma}, \ldots, \frac{\partial^5 H_r}{\partial x_\sigma^5}\right) = 0 \quad (\sigma = 1, \ldots, s).$$

1. Introduction and statement of the result. A differential equation is said to be *universal* if every continuous function (defined on the real line or on an interval) can be approximated by solutions of this single differential equation with respect to a prescribed distance function. Much work has been done to improve and to generalize L. A. Rubel's famous results concerning $C^{\infty}(\mathbb{R})$-solutions of an ordinary universal differential equation [13], [14].

THEOREM A [L. A. Rubel; 1981]. *There exists a nontrivial fourth-order algebraic differential equation* (*ADE*) *such that any real continuous function defined on the real line can be uniformly approximated by $C^{\infty}(\mathbb{R})$-solutions of this ADE. One such specific ADE is*

$$P(y', y'', y''', y'''') = 0,$$

where P denotes the polynomial

$$P(x_1, x_2, x_3, x_4) := 3x_1^4 x_2 x_4^2 - 4x_1^4 x_3^2 x_4 + 6x_1^3 x_2^2 x_3 x_4 + 24x_1^2 x_2^4 x_4 - 12x_1^3 x_2 x_3^3 - 29x_1^2 x_2^3 x_3^2 + 12x_2^7.$$

2000 *Mathematics Subject Classification*: 34A05, 26E10, 35C05.

Key words and phrases: ordinary and partial differential equations, approximation by solutions, universality theorems.

It is still an unanswered question whether such an ADE exists having *analytic* solutions approximating uniformly any continuous function on the real line. But when we restrict the approximation of functions to compact intervals, the problem of analytic solutions has been solved by M. Boshernitzan [1] in 1986.

THEOREM B [M. Boshernitzan; 1986]. *There exists a nontrivial sixth-order ADE of the form*

$$P(y', y'', \dots, y^{(6)}) = 0$$

whose real-analytic solutions (on $\mathbb{R}$*) are dense in* $C(I)$ *for any compact interval* I*.*

THEOREM C [M. Boshernitzan; 1986]. *There exists a nontrivial seventh-order ADE of the form*

$$P(y', y'', \dots, y^{(7)}) = 0$$

whose real-analytic entire solutions are dense in $C(I)$ *for any compact interval* I*.*

THEOREM D [M. Boshernitzan; 1986]. *There exists a nontrivial ADE of order* ≤ 19 *whose polynomial solutions from* $\mathbb{Q}[x]$ *are dense in* $C(I)$ *for any compact interval* I*.*

From Theorems B and C follows the one-dimensional case of the *Whitney Approximation Theorem*, which states that on compact sets $K \subset \mathbb{R}$ any continuous function can be uniformly approximated by real-analytic functions defined on $\mathbb{R}$. Moreover, by some famous results of C. E. Shannon [15] and M. B. Pour-El [12] one can identify the outputs of analog computers and the solutions of ADEs (provided that some uniqueness conditions for the solutions of the ADEs are fulfilled). Thus, by Theorems B and C, the existence of an analog computer is proved whose possible outputs are dense in the space of continuous functions.

The reader who is interested in the theory of universal equations can find various contributions to this subject:

1. There is an explicitly given ADE of order four simpler than the one from Theorem A (with six terms of weight 13) having complex-valued C^∞-solutions, whose imaginary parts approximate continuous functions on the whole real line with arbitrary accuracy [6].

2. Certain complex-valued solutions of the ADE mentioned in item 1 are also solutions of an explicitly given *algebraic functional equation* of order three, i.e. a (universal) functional equation with 39 terms involving derivatives up to order three [7]. This result is based on the concept of *local solutions of functional equations*.

3. There exists a universal ADE of order five whose $C^\infty(\mathbb{R})$-solutions approximate any continuous function on the real line with arbitrary accuracy, and the solutions additionally satisfy arithmetic conditions at algebraic points [8].

4. Some simpler universal ADE can be found when the solutions satisfy weaker conditions, i.e. for n-times differentiable solutions [4], [2].

5. There exists an algorithm which produces universal ADEs by starting from simple differential equations, for which a weak condition on a specific solution is assumed [9].

In [5] the author proves for a universal ADE that $C^\infty(\mathbb{R})$-series of analytic solutions (which are even entire functions) approximate continuous functions in the norm

$$\|g\|_\omega := \int_{-\infty}^{\infty} \omega(x)|g(x)|\,dx \qquad (g \in C(\mathbb{R})).$$

Here $\omega : \mathbb{R} \to \mathbb{R}_{>0}$ denotes a bounded continuous weight function taking positive values everywhere, such that

$$\int_{-\infty}^{\infty} \omega(x)\,dx = 1.$$

The author has proved the following result [5]:

Theorem E [C. Elsner; 2004]. *There exists a nontrivial autonomous algebraic differential equation $P = 0$ of order at most 7, where P denotes an effectively computable polynomial in at most eight variables, having the following property. Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function, and let $\varepsilon > 0$. Then there exists a series $H \in C^\infty(\mathbb{R})$ of analytic functions $H_r \in C^\omega(\mathbb{R})$,*

$$H(x) = \sum_{-\infty<r<\infty} H_r(x) \qquad (x \in \mathbb{R}),$$

such that $\|f - H\|_\omega < \varepsilon$, and each H_r solves the differential equation $P(H_r, H_r', \ldots, H_r^{(7)}) = 0$. Moreover, every analytic function H_r on $\mathbb{R}$ is an entire function on $\mathbb{C}$.

An important result concerning universal PDEs is due to R. C. Buck [3]:

Theorem F [R. C. Buck; 1981]. *For every integer $n \geq 2$ there exists a nontrivial algebraic partial differential equation in n variables whose polynomial solutions are dense in the space $C(I^n)$ of continuous functions on the unit cube I^n.*

The underlying idea in Buck's proof is the Kolmogorov–Arnold solution of Hilbert's Thirteenth Problem. In [1, Theorem 1.9], Buck's result is strengthened, among other things, by replacing the unit cube I^n by an arbitrary compact set in $\mathbb{R}^n$.

The goal of this paper is to extend the one-dimensional result from Theorem E to the s-dimensional case. Moreover, we shall describe an algorithm to compute the underlying PDE explicitly (by using a computer-algebra system). It turns out that a specific ordinary ADE of order 6 satisfying the conditions of Theorem E follows for $s = 1$ from our result. In order to state the theorem we need some preliminaries. First, let $\omega : \mathbb{R}^s \to \mathbb{R}_{>0}$ denote a bounded continuous weight function taking positive values everywhere, such that additionally

$$\int_{\mathbb{R}^s} \omega(x_1, \dots, x_s)\, d\mathbf{x} = 1. \tag{1.1}$$

For brevity we write $d\mathbf{x}$ for $dx_1 \dots dx_s$. Now let

$$\|g\|_\omega := \int_{\mathbb{R}^s} \omega(x_1, \dots, x_s)|g(x_1, \dots, x_s)|\, d\mathbf{x} \qquad (g \in C(\mathbb{R}^s)). \tag{1.2}$$

We recall the definition of real-analytic functions of several variables (as given in Definition 1.6.1 in [10]): A function f, with domain an open subset $U \subset \mathbb{R}^s$ and range $\mathbb{R}$, is called *real-analytic* if for each $\alpha \in U$ the function f may be represented by a convergent power series in some neighborhood of α. If f is a real-analytic function on $U \subset \mathbb{R}^s$ we write $f \in C^\omega(U)$. Finally, $C^\infty(\mathbb{R}^s)$ denotes the set of all functions f such that all partial derivatives $f^{(k_1,\dots,k_s)}(x_1, \dots, x_s)$ exist.

We now state the main result of the paper.

THEOREM 1. *There exists an effectively computable polynomial $P(x, t_0, \dots, t_5) \in \mathbb{Z}[x, t_0, \dots, t_5]$ having the following property. Let $s \geq 1$ be an integer, let $f : \mathbb{R}^s \to \mathbb{R}$ be a continuous function, and let $\varepsilon > 0$. Then there exists a series $H \in C^\infty(\mathbb{R}^s)$ of analytic functions $H_r \in C^\omega(\mathbb{R}^s)$,*

$$H(x_1, \dots, x_s) = \sum_{r=1}^{\infty} H_r(x_1, \dots, x_s) \qquad (x_\nu \in \mathbb{R};\ \nu = 1, \dots, s), \tag{1.3}$$

such that $\|f - H\|_\omega < \varepsilon$, and each H_r solves the system of partial differential equations

$$P\left(x_\sigma; H_r, \frac{\partial H_r}{\partial x_\sigma}, \dots, \frac{\partial^5 H_r}{\partial x_\sigma^5}\right) = 0 \qquad (\sigma = 1, \dots, s). \tag{1.4}$$

A specific polynomial $P(x, t_0, \dots, t_5)$ is homogeneous of degree 16 *in its variables $t_0, \dots, t_5$, and it consists of* 575 *terms of the form*

$$a x^b t_0^{c_0} \cdots t_5^{c_5} \qquad (a, b, c_0, \dots, c_5 \in \mathbb{Z};\ b, c_0, \dots, c_5 \geq 0;\ c_0 + \dots + c_5 = 16). \tag{1.5}$$

By standard arguments it follows easily from (1.4) that H_r also satisfies a system of autonomous partial differential equations of order six.

2. Preliminaries to the proof of Theorem 1: Notation and an auxiliary result. By $n_1, \dots, n_s$ we always denote integers. In what follows we consider any fixed continuous function $f : \mathbb{R}^s \to \mathbb{R}$. We write $f(\mathbf{x})$ for $f(x_1, \dots, x_s)$, and similarly for any function of variables $x_1, \dots, x_s$.

LEMMA 1 (Weierstrass Approximation Theorem, [11, p. 52]). *Let T be a closed bounded subset of $\mathbb{R}^s$. Then every real-valued continuous function $f(x_1, \dots, x_s)$ defined on T is the limit of a uniformly convergent series of polynomials in variables $x_1, \dots, x_s$ with real coefficients.*

For any $\varepsilon > 0$ put

$$\varepsilon_{\mathbf{n}} = \varepsilon_{n_1,\dots,n_s} := \frac{\varepsilon}{9^s \cdot 2^{|n_1|+\dots+|n_s|}} \qquad (n_1, \dots, n_s \in \mathbb{Z}). \tag{2.1}$$

Then

$$\sum_{-\infty<n_1<\infty} \cdots \sum_{-\infty<n_s<\infty} \varepsilon_{\mathbf{n}} = \frac{\varepsilon}{9^s} \cdot \prod_{\sigma=1}^{s} \sum_{-\infty<n_\sigma<\infty} \frac{1}{2^{|n_\sigma|}} = \frac{\varepsilon}{9^s} \cdot 3^s = \frac{\varepsilon}{3^s} < \varepsilon. \tag{2.2}$$

This identity will be used several times during the proof. Applying Lemma 1 to our function f on the set

$$T = \prod_{\sigma=1}^{s} [n_\sigma - 1; n_\sigma + 2],$$

we get a polynomial

$$\begin{aligned} Y_{\mathbf{n}}(\mathbf{x}) &= Y_{n_1,\dots,n_s}(x_1, \dots, x_s) \\ &= \sum_{\nu_1=0}^{N} \cdots \sum_{\nu_s=0}^{N} a_{n_1,\dots,n_s;\nu_1,\dots,\nu_s} x_1^{\nu_1} \cdots x_s^{\nu_s} = \sum_{\nu_1=0}^{N} \cdots \sum_{\nu_s=0}^{N} a_{\mathbf{n},\nu} x_1^{\nu_1} \cdots x_s^{\nu_s} \end{aligned} \tag{2.3}$$

satisfying

$$|f(\mathbf{x}) - Y_{\mathbf{n}}(\mathbf{x})| < \varepsilon_{\mathbf{n}}/16 \qquad (n_\sigma - 1 \le x_\sigma \le n_\sigma + 2;\ \sigma = 1, \dots, s). \tag{2.4}$$

Let m denote a positive integer to be defined later, depending on f, $n_1, \dots, n_s$, and ε. A weight function is given by

$$g_{\mathbf{n},m}(\mathbf{x}) = g_{n_1,\dots,n_s;m}(\mathbf{x}) := \prod_{\sigma=1}^{s} e^{-(2x_\sigma - 2n_\sigma - 1)^{2m}} \qquad (\mathbf{x} \in \mathbb{R}^s). \tag{2.5}$$

Using the real coefficients $a_{\mathbf{n},\nu}$ from (2.3), we define

$$y_{\mathbf{n},\nu,m}(\mathbf{x}) = y_{n_1,\dots,n_s;\nu_1,\dots,\nu_s;m}(\mathbf{x}) := a_{\mathbf{n},\nu} x_1^{\nu_1} \cdots x_s^{\nu_s} g_{\mathbf{n},m}(\mathbf{x}), \tag{2.6}$$

where $n_\sigma \in \mathbb{Z}$, $0 \le \nu_\sigma \le N$, $\sigma = 1, \dots, s$. Moreover, let

$$Z_{\mathbf{n},m}(\mathbf{x}) = Z_{n_1,\dots,n_s;m}(\mathbf{x}) := Y_{\mathbf{n}}(\mathbf{x}) g_{\mathbf{n},m}(\mathbf{x}), \tag{2.7}$$

and

$$H(\mathbf{x}) = H(x_1, \dots, x_s) := \sum_{-\infty<n_1<\infty} \cdots \sum_{-\infty<n_s<\infty} Z_{\mathbf{n},m}(\mathbf{x}). \tag{2.8}$$

We shall show that the infinite series (2.8) converges for every $\mathbf{x} \in \mathbb{R}^s$ and that the terms can be arranged to form the infinite series given in (1.3).

Next we define a positive number $\delta_{\mathbf{n}} = \delta_{n_1,\dots,n_s}$ corresponding to the integers $n_1, \dots, n_s$: Since it is assumed that the function $f(\mathbf{x})$ and the weight function $\omega(\mathbf{x})$ are continuous, one can find a sufficiently small number $\delta_{\mathbf{n}}$ satisfying

$$\max_{1\le\sigma\le s} \Bigg\{ \int_{n_\sigma-\delta_n/2}^{n_\sigma+\delta_n/2} \sup_{\substack{n_\tau-1\le x_\tau\le n_\tau+2\\ 1\le\tau\le s\\ \tau\ne\sigma}} \omega(\mathbf{x})(1+2|f(\mathbf{x})|)\,dx_\sigma, \tag{2.9}$$

$$\int_{n_\sigma+1-\delta_n/2}^{n_\sigma+1+\delta_n/2} \sup_{\substack{n_\tau-1\le x_\tau\le n_\tau+2\\ 1\le\tau\le s\\ \tau\ne\sigma}} \omega(\mathbf{x})(1+2|f(\mathbf{x})|)\,dx_\sigma \Bigg\} < \frac{\varepsilon_{\mathbf{n}}}{8}.$$

We use $\delta_{\mathbf{n}}$ to introduce the following closed subsets in $\mathbb{R}^s$ for $n_1, \dots, n_s \in \mathbb{Z}$:

$$I_{\mathbf{n}} = I_{n_1,\dots,n_s} := \prod_{\sigma=1}^{s} [n_\sigma - \delta_{\mathbf{n}}/2; n_\sigma + 1 + \delta_{\mathbf{n}}/2] =: \prod_{\sigma=1}^{s} I_{\mathbf{n},\sigma}, \tag{2.10}$$

$$J_{\mathbf{n}} = J_{n_1,\dots,n_s} := \prod_{\sigma=1}^{s} [n_\sigma + \delta_{\mathbf{n}}/2; n_\sigma + 1 - \delta_{\mathbf{n}}/2] =: \prod_{\sigma=1}^{s} J_{\mathbf{n},\sigma}. \tag{2.11}$$

3. Overview of the proof: the one-dimensional case, its generalization to several dimensions, and additional difficulties. Theorem 1 is a multi-dimensional generalization of the one-dimensional result of [5], but we have to overcome some additional difficulties. For the convenience of the reader we first survey the main elements of the proof in [5] and point out the difficulties appearing in the generalization.

First, the order five of the differential equations (1.4) results from their explicit computation in Section 4 below. The method of eliminating all parameters by use of resultants requires extensive computations. These are done by computer (using Maple). In [5] the general Lemma 1 has been applied to *estimate* the order of the differential equation.

The main idea in the one-dimensional case is to approximate a given continuous function $f : \mathbb{R} \to \mathbb{R}$ piecewise on the intervals

$$J_n = [n + \delta_n/2; n + 1 - \delta_n/2] \qquad (n \in \mathbb{Z},\ \delta_n > 0)$$

by polynomials $Y_n(x)$, and additionally introducing a weight function

$$g_{m,n}(x) = e^{-(2x-2n-1)^{2m}} \qquad (m \in \mathbb{Z},\ m \ge 1)$$

which takes values very close to 1 for $x \in J_n$. For sufficiently large m it bends down the polynomial $Y_n(x)$ to arbitrary small values outside the interval

$$I_n = [n - \delta_n/2; n + 1 + \delta_n/2]$$

(see Lemmas 6 and 5, eq. (3.13) in [5]). By choosing δ_n sufficiently small, the contribution of the integral

$$\int\limits_{I_n \setminus J_n} \omega(x)|f(x) - Y_n(x)g_{m,n}(x)|\, dx$$

to the whole norm integral

$$\int\limits_{-\infty}^{\infty} \omega(x)|f(x) - H(x)|\, dx$$

with

$$H(x) = \sum_{n=-\infty}^{\infty} Y_n(x)g_{m,n}(x)$$

is (roughly speaking) negligible. The detailed arguments can be found in Section 4 of [5]. The generalization to the s-dimensional case in the final Section 8 of the present paper requires more technical efforts because of the specific form of the functions $Z_{\mathbf{n},m}(\mathbf{x}) = Y_{\mathbf{n}}(\mathbf{x}) \cdot g_{\mathbf{n},m}(\mathbf{x})$, where the polynomial $Y_{\mathbf{n}}(\mathbf{x}) = Y_{\mathbf{n}}(x_1, \ldots, x_s)$ approximates the continuous function $f(\mathbf{x}) = f(x_1, \ldots, x_s)$ on the set $J_{\mathbf{n}}$ defined in (2.11). To overcome the difficulties we shall separate the integral

$$\int\limits_{n_1}^{n_1+1} \cdots \int\limits_{n_s}^{n_s+1} \omega(\mathbf{x}) \Big| f(\mathbf{x}) - \sum_{-\infty<k_1<\infty} \cdots \sum_{-\infty<k_s<\infty} Z_{\mathbf{k},m}(\mathbf{x}) \Big|\, d\mathbf{x}$$

into three parts I_1, I_2, I_3 according to different terms $Z_{\mathbf{k},m}(\mathbf{x})$. Let

$$G_{\mathbf{n}} := \prod_{i=1}^{s} [n_i; n_i + 1].$$

- For I_1 we consider the single term $Z_{\mathbf{n},m}(\mathbf{x})$ which approximates $f(\mathbf{x})$ on $J_{\mathbf{n}}$.
- For I_2 we consider the terms $Z_{\mathbf{k},m}(\mathbf{x})$ which approximate $f(\mathbf{x})$ in the neighborhood of $J_{\mathbf{n}}$ on those sets $J_{\mathbf{k}}$ where $I_{\mathbf{k}} \cap G_{\mathbf{n}} \neq \emptyset$.
- For I_3 we consider all the remaining terms $Z_{\mathbf{k},m}(\mathbf{x})$ where $I_{\mathbf{k}} \cap G_{\mathbf{n}} = \emptyset$.

The terms belonging to I_2 require the most careful investigation, since the intersections of $I_{\mathbf{k}}$ and $G_{\mathbf{n}}$ occur with different dimensions.

The parameters m of the weight functions $g_{\mathbf{n},m}(\mathbf{x})$ depend on the function $f(\mathbf{x})$ and on $n_1, \ldots, n_s$. On the one hand, m must be so large that the approximation problem is solved for $f(\mathbf{x})$ by the sum of all functions $Z_{\mathbf{n},m}(\mathbf{x})$, on the other hand the series must be infinitely differentiable with respect to each variable $x_1, \dots, x_s$. As in the one-dimensional case, we solve this problem by the Weierstrass criterion on uniformly convergent series (Lemma 8) and its application to a series of partial derivatives (Lemma 7). For $s = 1$ the inequality (3.14) in Lemma 5 of [5] plays the main role (which corresponds to Lemma 4 in the present paper): it provides very small lower bounds for the kth derivative ($k \geq 1$) of the terms $a_{n,\nu} x^\nu g_{m,n}(x)$, where $a_{n,\nu} x^\nu$ is a monomial from the polynomial $Y_n(x)$. The kth derivative becomes very small when x keeps a sufficiently large distance to the intervals J_n and I_n, in particular for $|x - n| \geq 2^k$. There are only finitely many terms from the series with $|x-n| < 2^k$, and they are k-times differentiable. The kth derivative of each of the remaining terms is small by the above mentioned results, and so the whole series is k-times differentiable for every $k \geq 1$. The proof of the bound given in Lemma 4 requires a careful treatment of all terms (as in the case $s = 1$ in [5]), since the parameter m of the weight functions may not depend on k. Nevertheless the kth derivatives at single points $\mathbf{x}$ of terms of the series increase rapidly with k. This makes it impossible to identify the series of functions as an *analytic* function. For the details of the theory of analytic functions we refer the reader to the book of Krantz and Parks [10].

With the same final argument as for $s = 1$ we finish the proof of the theorem: At every point $\mathbf{x}$ the sum of all functions $Y_{\mathbf{n}}(\mathbf{x}) g_{\mathbf{n},m}(\mathbf{x})$ forms an absolutely convergent series of countably many monomials multiplied by weight functions (i.e. a series of terms $y_{\mathbf{n},\nu,m}(\mathbf{x}) = a_{\mathbf{n},\nu} x_1^{\nu_1} \cdots x_s^{\nu_s} g_{\mathbf{n},m}(\mathbf{x})$ given by (2.6)). All the functions $y_{\mathbf{n},\nu,m}(\mathbf{x})$ satisfy the partial differential equations, and their sum can be arranged into a series of the form (1.3),

$$\sum_{r=1}^{\infty} H_r(x_1, \dots, x_s) \qquad (x_\nu \in \mathbb{R};\ \nu = 1, \dots, s),$$

as stated in our theorem.

4. A differential equation for $H_r(\mathbf{x})$. For brevity we introduce the following notation:

$$y = y(x) := x^\nu e^{-(2x-2n-1)^{2m}} \qquad (n \in \mathbb{Z},\ \nu \in \mathbb{Z},\ \nu \geq 0,\ m \in \mathbb{N}). \tag{4.1}$$

Furthermore, put

$$\begin{cases} Y_1 := x^2 y y'' + \nu y^2 - x^2 y'^2, \\ Y_2 := 2xy(xy' - \nu y), \\ Y_3 := 2x^3 y'^2 - 2\nu x y^2 - 2x^3 y y'', \end{cases} \tag{4.2}$$

where ν corresponds to the function y and its parameter ν in (4.1). The functions Y_i satisfy the identities

$$NY_1 + MY_2 + Y_3 = 0, \tag{4.3}$$

$$NY_1' + MY_2' + Y_3' = 0 \tag{4.4}$$

with $N := 2n+1$ and $M := 2m-1$. The equation (4.4) follows immediately from (4.3) by differentiation. Next, eliminating the parameter N, we get

$$M(Y_1Y_2' - Y_1'Y_2) + (Y_1Y_3' - Y_1'Y_3) = 0. \tag{4.5}$$

Differentiating with respect to x, one easily proves that

$$M(Y_1Y_2'' - Y_1''Y_2) + (Y_1Y_3'' - Y_1''Y_3) = 0. \tag{4.6}$$

Now we eliminate the parameter M from (4.5), (4.6):

$$(Y_1Y_2' - Y_1'Y_2)(Y_1Y_3'' - Y_1''Y_3) - (Y_1Y_2'' - Y_1''Y_2)(Y_1Y_3' - Y_1'Y_3) = 0. \tag{4.7}$$

By differentiation it follows that

$$\begin{aligned}(Y_1Y_2' - Y_1'Y_2) \cdot (Y_1Y_3''' + Y_1'Y_3'' - Y_1''Y_3' - Y_1'''Y_3)&\\ - (Y_1Y_3' - Y_1'Y_3) \cdot (Y_1Y_2''' + Y_1'Y_2'' - Y_1''Y_2' - Y_1'''Y_2) &= 0.\end{aligned} \tag{4.8}$$

In order to express all the terms in (4.7), (4.8) by ν, x, $y, y', \ldots, y^{(5)}$, we first compute the derivatives of Y_1, Y_2, and Y_3 from (4.2):

$$\begin{aligned}
Y_1' &= 2\nu yy' + 2x(yy'' - y'^2) + x^2(yy''' - y'y''),\\
Y_1'' &= 2(\nu+1)yy'' + 2(\nu-1)y'^2 + 4x(yy''' - y'y'') + x^2(yy^{(4)} - y''^2),\\
Y_1''' &= 6(\nu-1)y'y'' + 2(\nu+3)yy''' + 6x(yy^{(4)} - y''^2)\\
&\quad + x^2(yy^{(5)} + y'y^{(4)} - 2y''y''');\\
Y_2' &= -2\nu y(y + 2xy') + 4xyy' + 2x^2(yy'' + y'^2),\\
Y_2'' &= -4\nu(2yy' + x(yy'' + y'^2)) + 4yy' + 8x(yy'' + y'^2)\\
&\quad + 2x^2(yy''' + 3y'y''),\\
Y_2''' &= -4\nu(3y'^2 + 3yy'' + x(3y'y'' + yy''')) + 12(yy'' + y'^2)\\
&\quad + 12x(yy''' + 3y'y'') + 2x^2(yy^{(4)} + 4y'y''' + 3y''^2);\\
Y_3' &= -2\nu(y^2 + 2xyy') - 6x^2(yy'' - y'^2) - 2x^3(yy''' - y'y''),\\
Y_3'' &= -4\nu(2yy' + x(yy'' + y'^2)) - 12x(yy'' - y'^2) - 12x^2(yy''' - y'y'')\\
&\quad - 2x^3(yy^{(4)} - y''^2),\\
Y_3''' &= -4\nu(3yy'' + 3y'^2 + x(yy''' + 3y'y'')) - 12(yy'' - y'^2)\\
&\quad - 36x(yy''' - y'y'') - 18x^2(yy^{(4)} - y''^2)\\
&\quad - 2x^3(yy^{(5)} + y'y^{(4)} - 2y''y''').
\end{aligned}$$

The results of the following computations can be verified by using a computer-algebra system. Putting the above terms into (4.7) and (4.8), one gets two polynomials of the third degree with respect to the variable ν:

$$r_3\nu^3 + r_2\nu^2 + r_1\nu + r_0 = 0 \quad \text{and} \quad s_3\nu^3 + s_2\nu^2 + s_1\nu + s_0 = 0.$$

The coefficients r_i, s_i $(i = 0, 1, 2, 3)$ depend on $x, y, y', \dots, y^{(5)}$. The resultant of these two polynomials having a common root ν vanishes. Hence we get

$$\begin{aligned} 0 = \ & \det \begin{pmatrix} r_3 & r_2 & r_1 & r_0 & 0 & 0 \\ 0 & r_3 & r_2 & r_1 & r_0 & 0 \\ 0 & 0 & r_3 & r_2 & r_1 & r_0 \\ s_3 & s_2 & s_1 & s_0 & 0 & 0 \\ 0 & s_3 & s_2 & s_1 & s_0 & 0 \\ 0 & 0 & s_3 & s_2 & s_1 & s_0 \end{pmatrix} \\ = \ & -16384x^{17}y^6 \cdot \left(-xyy'' + xy'^2 - yy'\right)^4 \\ & \cdot \left(2xy'^3 - 3xyy'y'' + xy^2y''' - 2yy'^2 + 2y^2y''\right)^6 \cdot P(x, y, y', \dots, y^{(5)}) \\ =: & -16384x^{17}y^6 \cdot A^4(x, y, y', y'') \cdot B^6(x, y, y', y'', y''') \cdot P(x, y, y', \dots, y^{(5)}), \end{aligned}$$

where P is the polynomial introduced in Theorem 1. We have to show that the terms $A(x, y, y', y'')$ and $B(x, y, y', y'', y''')$ do not vanish identically. For $n \in \mathbb{Z},\ m \in \mathbb{N}$ one gets

$$A(n, y(x = n), y'(x = n), y''(x = n)) = 4n^2m(4nm - 2n - 1)e^{-2},$$

which is not zero for $n \neq 0$ since $4nm - 2n - 1$ is an odd number. For $n = 0$ one has

$$A(1, y(x = 1), y'(x = 1), y''(x = 1)) = 4m \cdot (4m - 1) \cdot e^{-2} > 0.$$

The arguments for $B(x, y, y', y'', y''')$ are essentially the same. For $n \neq 0$ we have

$$\begin{aligned} & B(n, y(x = n), \dots, y'''(x = n)) \\ & \qquad = 16n^3m(4nm^2 - 6nm + 2n - 2m + 1)e^{-3} \neq 0. \end{aligned}$$

It remains to consider the case when $n = 0$. One gets

$$B(1, y(x = 1), \dots, y'''(x = 1)) = -16m(2m - 1)^2e^{-3} < 0,$$

which finally proves that

$$P(x, y, y', \dots, y^{(5)}) = 0. \tag{4.9}$$

We know from (7.1) below and from (2.5), (2.6) that any function $H_r(\mathbf{x})$ takes the form

$$H_r(\mathbf{x}) = y_{\mu_1(r),\mu_2(r),m}(\mathbf{x}) = a_{\mathbf{n},\nu} \prod_{\sigma=1}^{s} x_\sigma^{\nu_\sigma} e^{-(2x_\sigma - 2n_\sigma - 1)^{2m}} \tag{4.10}$$

for certain integers $\nu_1, \dots, \nu_s, m, n_1, \dots, n_s$. Since $P(x; t_0, \dots, t_5)$ is a homogeneous polynomial with respect to its variables $t_0, \dots, t_5$, one easily proves, by (4.1), (4.9), and (4.10), that $H_r(\mathbf{x})$ satisfies the identities (1.4).

5. Two lemmas concerning the approximation of f

LEMMA 2. *Let $f:\mathbb{R}^s\to\mathbb{R}$ be a continuous function, and let $\varepsilon>0$. Then for arbitrary integers $n_1,\dots,n_s$, there is an integer m_1, depending at most on ε, $n_1,\dots,n_s$, and f, such that*

$$(5.1)\quad |y_{\mathbf{n},\nu,m}(\mathbf{x})|\le\frac{\varepsilon_{\mathbf{n}}}{8(N+1)^s}\qquad(\mathbf{x}\notin I_{\mathbf{n}},\ m\ge m_1,\ 0\le\nu_1,\dots,\nu_s\le N).$$

LEMMA 3. *For arbitrary integers $n_1,\dots,n_s$ the polynomial $Y_{\mathbf{n}}(\mathbf{x})$ from* (2.3) *satisfies the inequality*

$$(5.2)\qquad |f(\mathbf{x})-Y_{\mathbf{n}}(\mathbf{x})g_{\mathbf{n},m}(\mathbf{x})|<\varepsilon_{\mathbf{n}}/8\qquad(\mathbf{x}\in J_{\mathbf{n}})$$

for all integers $m\ge m_2$, where m_2 depends at most on ε, $n_1,\dots,n_s$, and f.

Proof of Lemma 2. We additionally introduce the height of the polynomial $Y_{\mathbf{n}}(\mathbf{x})$:

$$(5.3)\quad H_{\mathbf{n}}=H_{n_1,\dots,n_s}:=\max_{\substack{0\le\nu_\sigma\le N\\1\le\sigma\le s}}\{1,|a_{n_1,\dots,n_s;\nu_1,\dots,\nu_s}|\}\qquad(n_1,\dots,n_s\in\mathbb{Z}).$$

Moreover, let $t_\sigma:=2x_\sigma-2n_\sigma-1$ for $1\le\sigma\le s$. From $\mathbf{x}\notin I_{\mathbf{n}}$ one knows that there is some σ with $x_\sigma\notin I_{\mathbf{n},\sigma}$ satisfying

$$(5.4)\qquad |t_\sigma|=|2x_\sigma-2n_\sigma-1|\ge1+\delta_{\mathbf{n}}.$$

The positive number

$$(5.5)\qquad M_{\mathbf{n}}:=\max_{1\le\sigma\le s}\{1,\max_{0\le\nu_\sigma\le N}\sup_{x_\sigma\in I_{n,\sigma}}|x_\sigma^{\nu_\sigma}|\}$$

depends on $n_1,\dots,n_s$, f, s and ε. Since all the functions $x_\sigma^{\nu_\sigma}e^{-t_\sigma^2}$ tend to zero for $|x_\sigma|\to\infty$ there are intervals $[A_{\mathbf{n},\sigma};B_{\mathbf{n},\sigma}]$ such that $I_{\mathbf{n},\sigma}\subset[A_{\mathbf{n},\sigma};B_{\mathbf{n},\sigma}]$, and

$$(5.6)\quad |x_\sigma^{\nu_\sigma}|e^{-t_\sigma^2}<\frac{\varepsilon_{\mathbf{n}}}{8H_{\mathbf{n}}M_{\mathbf{n}}^s(1+N)^s}\qquad(x_\sigma\notin[A_{\mathbf{n},\sigma};B_{\mathbf{n},\sigma}],\ 0\le\nu_\sigma\le N).$$

This inequality remains true when $e^{-t_\sigma^2}$ on the left side is replaced by $e^{-t_\sigma^{2m}}$ for any positive integer m, since $x_\sigma\notin I_{\mathbf{n},\sigma}$ implies by (5.4) that $|t_\sigma|>1$. Moreover, the value

$$M_\sigma=\max_{0\le\nu_\sigma\le N}\sup_{x_\sigma\in[A_{\mathbf{n},\sigma};B_{\mathbf{n},\sigma}]}|x_\sigma^{\nu_\sigma}|\qquad(1\le\sigma\le s)$$

is clearly finite, and it depends on $n_1,\dots,n_s$, f, s, σ, ε, and on the interval $[A_{\mathbf{n},\sigma};B_{\mathbf{n},\sigma}]$, but it does not depend on m. Thus there is a positive integer $m_1=m_1(n_1,\dots,n_s,f,s,\varepsilon)$ such that

$$M_\sigma e^{-(1+\delta_{\mathbf{n}})^{2m}}<\frac{\varepsilon_{\mathbf{n}}}{8H_{\mathbf{n}}M_{\mathbf{n}}^s(1+N)^s}\qquad(m\ge m_1,\ 1\le\sigma\le s).$$

(Any term depending on f may also depend on the chosen approximation polynomial $Y_{\mathbf{n}}$ and on its parameters N, $H_{\mathbf{n}}$.) For $x_\sigma\in[A_{\mathbf{n},\sigma};B_{\mathbf{n},\sigma}]\setminus I_{\mathbf{n},\sigma}$ it follows by (5.4) that

$$|x_\sigma^{\nu_\sigma}|e^{-t_\sigma^{2m}} < \frac{\varepsilon_{\mathbf{n}}}{8H_{\mathbf{n}}M_{\mathbf{n}}^s(1+N)^s} \tag{5.7}$$

for all $m \geq m_1$, $0 \leq \nu_1, \dots, \nu_s \leq N$.

For $\mathbf{x} \notin I_{\mathbf{n}}$, in particular for $x_{\sigma_0} \notin I_{\mathbf{n},\sigma_0}$, by application of (5.5)–(5.7) we now have

$$\begin{aligned}
|y_{\mathbf{n},\nu,m}(\mathbf{x})| &\leq H_{\mathbf{n}} \prod_{\sigma=1}^{s} |x_\sigma^{\nu_\sigma}|e^{-t_\sigma^{2m}} \\
&= H_{\mathbf{n}}\Big(\prod_{\substack{1\leq\sigma\leq s \\ x_\sigma \in I_{n,\sigma}}} |x_\sigma^{\nu_\sigma}|e^{-t_\sigma^{2m}} \Big)\Big(\prod_{\substack{1\leq\sigma\leq s \\ x_\sigma \notin I_{n,\sigma}}} |x_\sigma^{\nu_\sigma}|e^{-t_\sigma^{2m}} \Big) \\
&\leq H_{\mathbf{n}}\Big(\prod_{\substack{1\leq\sigma\leq s \\ x_\sigma \in I_{n,\sigma}}} |x_\sigma^{\nu_\sigma}| \Big)\Bigg(\prod_{\substack{1\leq\sigma\leq s \\ x_\sigma \notin I_{n,\sigma}}} \frac{\varepsilon_{\mathbf{n}}}{8H_{\mathbf{n}}M_{\mathbf{n}}^s(1+N)^s} \Bigg) \\
&\leq H_{\mathbf{n}}M_{\mathbf{n}}^s \frac{\varepsilon_{\mathbf{n}}}{8H_{\mathbf{n}}M_{\mathbf{n}}^s(1+N)^s} = \frac{\varepsilon_{\mathbf{n}}}{8(1+N)^s}
\end{aligned}$$

for all $m \geq m_1$, $0 \leq \nu_1, \dots, \nu_s \leq N$.

Here we have used the fact that there exists x_{σ_0} satisfying $x_{\sigma_0} \notin I_{n,\sigma_0}$, and that $H_{\mathbf{n}} \geq 1$, $M_{\mathbf{n}} \geq 1$. ∎

Proof of Lemma 3. Let $\mathbf{x} \in J_{\mathbf{n}}$. Then we know by (2.11) that $|t_\sigma| = |2x_\sigma - 2n_\sigma - 1| \leq 1 - \delta_{\mathbf{n}}$ for $\sigma = 1, \dots, s$. Let

$$F_{\mathbf{n}} := 1 + \sup_{\substack{n_\sigma \leq x_\sigma \leq n_\sigma+1 \\ 1\leq\sigma\leq s}} |f(x_1, \dots, x_s)|. \tag{5.8}$$

Obviously there exists an integer m_2 satisfying

$$1 - \frac{\varepsilon_{\mathbf{n}}}{2^{s+4}F_{\mathbf{n}}} < e^{-(1-\delta_{\mathbf{n}})^{2m}} < 1 \quad (m \geq m_2); \tag{5.9}$$

it depends on $n_1, \dots, n_s$ and on f, s, ε. It follows immediately from (5.9) and from $|t_\sigma| \leq 1 - \delta_{\mathbf{n}}$ that for every σ there exists a real number α_σ such that

$$e^{-t_\sigma^{2m}} = 1 + \alpha_\sigma, \quad \text{where} \quad |\alpha_\sigma| < \frac{\varepsilon_{\mathbf{n}}}{2^{s+4}F_{\mathbf{n}}} < 1 \quad (1 \leq \sigma \leq s). \tag{5.10}$$

Of course, α_σ depends on $n_1, \dots, n_s$ and f, s, x_σ, ε. Then one has

$$\prod_{\sigma=1}^{s} e^{-t_\sigma^{2m}} = \prod_{\sigma=1}^{s}(1+\alpha_\sigma) = \sum_{\tau=0}^{s} \prod_{1\leq\mu_1<\cdots<\mu_\tau\leq s} \alpha_{\mu_1}\cdots\alpha_{\mu_\tau} =: 1 + \alpha.$$

The number α depends on $n_1, \dots, n_s$, f, s, ε and $x_1, \dots, x_s$. For $|\alpha|$ we get an upper bound from (5.10), namely

$$|\alpha| = \Big|\sum_{\tau=1}^{s} \prod_{1\le\mu_1<\cdots<\mu_\tau\le s} \alpha_{\mu_1}\cdots\alpha_{\mu_\tau}\Big| \le (2^s-1)\max_{1\le\sigma\le s}|\alpha_\sigma| \tag{5.11}$$
$$< 2^s\,\frac{\varepsilon_{\mathbf{n}}}{2^{s+4}F_{\mathbf{n}}} = \frac{\varepsilon_{\mathbf{n}}}{16F_{\mathbf{n}}}.$$

We now estimate $|Y_{\mathbf{n}}(\mathbf{x})|$ on $J_{\mathbf{n}}$ by (2.4) and (5.8):

$$\sup_{\mathbf{x}\in J_{\mathbf{n}}}|Y_{\mathbf{n}}| \le \sup_{\substack{n_\sigma\le x_\sigma\le n_\sigma+1\\ 1\le\sigma\le s}} |Y_{\mathbf{n}}(x_1,\ldots,x_s)|$$
$$\le \frac{\varepsilon_{\mathbf{n}}}{16} + \sup_{\substack{n_\sigma\le x_\sigma\le n_\sigma+1\\ 1\le\sigma\le s}} |f(x_1,\ldots,x_s)| < F_{\mathbf{n}}.$$

Applying (2.4) for a second time, one finally gets, by application of (5.11),

$$|f(\mathbf{x}) - Y_{\mathbf{n}}(\mathbf{x})g_{\mathbf{n},m}(\mathbf{x})| = \Big|f(\mathbf{x}) - \Big(\prod_{\sigma=1}^{s} e^{-t_\sigma^{2m}}\Big)Y_{\mathbf{n}}(\mathbf{x})\Big| = |f(\mathbf{x}) - (1+\alpha)Y_{\mathbf{n}}(\mathbf{x})|$$
$$\le |f(\mathbf{x}) - Y_{\mathbf{n}}(\mathbf{x})| + |\alpha|\cdot|Y_{\mathbf{n}}(\mathbf{x})| < \frac{\varepsilon_{\mathbf{n}}}{16} + \frac{\varepsilon_{\mathbf{n}}}{16F_{\mathbf{n}}}F_{\mathbf{n}}$$
$$= \frac{\varepsilon_{\mathbf{n}}}{8} \qquad (\mathbf{x}\in J_{\mathbf{n}},\ m\ge m_2).$$

Thus Lemma 3 is proved. ∎

6. On a bound for partial derivatives of $y_{\mathbf{n},\nu,m}(\mathbf{x})$. Throughout this section the continuous function f, the real number $\varepsilon > 0$ and the approximating polynomial $Y_{\mathbf{n}}(\mathbf{x})$ are as in the preceding section.

LEMMA 4. *Let $n_1,\ldots,n_s$ and $0\le\nu_1,\ldots,\nu_s\le N$ be arbitrary integers. Then there is an integer m_3 depending on $s,n_1,\ldots,n_s,\nu_1,\ldots,\nu_s,f$ and ε such that*

$$|y_{\mathbf{n},\nu,m}^{(k_1,\ldots,k_s)}(\mathbf{x})| < \frac{\varepsilon_{\mathbf{n}}}{8(1+N)^s} \tag{6.1}$$

for $\mathbf{x}\in\mathbb{R}^s$, $m\ge m_3$, and nonnegative integers $k_1,\ldots,k_s$ such that $|x_\sigma-n_\sigma| \ge 2^{1+k_\sigma}$ for at least one σ with $1\le\sigma\le s$.

Proof. We express $y_{\mathbf{n},\nu,m}(\mathbf{x})$ by (2.5), (2.6), and apply (5.3). Then we get

$$|y_{\mathbf{n},\nu,m}^{(k_1,\ldots,k_s)}(\mathbf{x})| = \Big|a_{\mathbf{n},\nu}\prod_{\sigma=1}^{s}\frac{\partial^{k_\sigma}}{\partial x_\sigma^{k_\sigma}}(x_\sigma^{\nu_\sigma}e^{-(2x_\sigma-2n_\sigma-1)^{2m}})\Big| \tag{6.2}$$
$$\le H_{\mathbf{n}}\Big|\Big\{\prod_{\substack{\sigma=1\\ |x_\sigma-n_\sigma|\ge 2^{1+k_\sigma}}}^{s} \cdot \prod_{\substack{\sigma=1\\ |x_\sigma-n_\sigma|<2^{1+k_\sigma}}}^{s}\Big\}\frac{\partial^{k_\sigma}}{\partial x_\sigma^{k_\sigma}}(x_\sigma^{\nu_\sigma}e^{-(2x_\sigma-2n_\sigma-1)^{2m}})\Big|$$

In what follows we fix the integers $n_1, \dots, n_s, k_1, \dots, k_s$ from the lemma. We first estimate the term $\left|\frac{\partial^{k_\sigma}}{\partial x_\sigma^{k_\sigma}}(x_\sigma^{\nu_\sigma} e^{-(2x_\sigma - 2n_\sigma - 1)^{2m}})\right|$ corresponding to $|x_\sigma - n_\sigma| < 2^{1+k_\sigma}$. For brevity we put

$$x = x_\sigma, \quad k = k_\sigma, \quad \nu = \nu_\sigma, \quad n = n_\sigma, \quad t := 2x - 2n - 1. \tag{6.3}$$

Applying the Leibniz rule, we get

$$\begin{aligned}
&\left|\frac{\partial^k}{\partial x^k}(x^\nu e^{-(2x-2n-1)^{2m}})\right| \\
&= \left|\sum_{\kappa=\max\{0,k-\nu\}}^{k} \binom{k}{\kappa} \nu(\nu-1)\cdots(\nu-k+\kappa+1) x^{\nu-k+\kappa} 2^\kappa \frac{\partial}{\partial t^\kappa}(e^{-t^{2m}})\right| \\
&\le 2^k N! \sum_{\kappa=\max\{0,k-\nu\}}^{k} \binom{k}{\kappa} (|n| + 2^{1+k})^N \left|\frac{\partial}{\partial t^\kappa}(e^{-t^{2m}})\right|.
\end{aligned} \tag{6.4}$$

Here we have used the inequality $|x| < |n| + 2^{1+k}$, which follows from $|x - n| < 2^{1+k}$, and from the fact that $\nu(\nu-1)\cdots(\nu-k+\kappa+1) \le \nu! \le N!$ for $0 < \nu - k + \kappa + 1$. Below we refer to the proof of Lemma 5, formula (3.14), in [5]. We have

$$\left|\frac{\partial}{\partial t^\kappa}(e^{-t^{2m}})\right| = |P_{\kappa(2m-1)}(t) e^{-t^{2m}}| \le |P_{\kappa(2m-1)}(t)|$$

for some integer polynomial $P_{\kappa(2m-1)}(t)$ of degree $\kappa(2m-1)$ and of height bounded by $\kappa!(2m)^\kappa$ (see Lemma 4 in [5]). Since $\kappa \le k$ and $k(2m-1)+1 \le 2m(k+1)$ and $|t| = |2x - 2n - 1| \le 1 + 2|x - n| \le 1 + 2^{k+2} \le 2^{k+3}$ we get

$$\begin{aligned}
|P_{\kappa(2m-1)}(t)| &\le (k(2m-1)+1) k! (2m)^k \max\{1, |t|^{k(2m-1)}\} \\
&\le 2m(k+1) k! (2m)^k \max\{1, |t|^{2km}\} \\
&\le (k+1)! (2m)^{k+1} 2^{2mk(k+3)}.
\end{aligned}$$

Putting this inequality into the right side of (6.4), we get the inequality

$$\begin{aligned}
&\left|\frac{\partial^k}{\partial x^k}(x^\nu e^{-(2x-2n-1)^{2m}})\right| \\
&\qquad \le 2^k N! (|n| + 2^{1+k})^N \sum_{\kappa=0}^{k} \binom{k}{\kappa} (k+1)! (2m)^{k+1} 2^{2mk(k+3)} \\
&\qquad = 4^k N! (|n| + 2^{1+k})^N (k+1)! (2m)^{k+1} 2^{2mk(k+3)};
\end{aligned}$$

using it we estimate the terms corresponding to $|x_\sigma - n_\sigma| < 2^{1+k_\sigma}$ on the right side of (6.2). Putting $K := 1 + \max\{k_1, \dots, k_s\}$ and $M := \max\{|n_1|, \dots, |n_s|\}$, we have proved the following result.

LEMMA 5. *Under the conditions of Lemma* 4,

$$
|y_{\mathbf{n},\nu,m}^{(k_1,\dots,k_s)}(\mathbf{x})| \le H_{\mathbf{n}} 4^{sK} N!^s \left(M+2^{1+K}\right)^{sN} (K+1)!^s (2m)^{Ks+s} \times 2^{2msK(K+3)} \prod_{\substack{\sigma=1\\ |x_\sigma-n_\sigma|\ge 2^{1+k_\sigma}}}^{s} \left|\frac{\partial^{k_\sigma}}{\partial x_\sigma^{k_\sigma}}\left(x_\sigma^{\nu_\sigma} e^{-(2x_\sigma-2n_\sigma-1)^{2m}}\right)\right|. \tag{6.5}
$$

The hypothesis of Lemma 4 guarantees the existence of at least one index σ such that $|x_\sigma - n_\sigma| \ge 2^{1+k_\sigma}$.

LEMMA 6. *Under the conditions of Lemma* 4,

$$
W{:}= H_{\mathbf{n}} \cdot 4^{sK} N!^s \left(M+2^{1+K}\right)^{sN} (K+1)!^s (2m)^{Ks+s} 2^{2msK(K+3)} \times \left|\frac{\partial^{k_\sigma}}{\partial x_\sigma^{k_\sigma}}\left(x_\sigma^{\nu_\sigma} e^{-(2x_\sigma-2n_\sigma-1)^{2m}}\right)\right| \le \frac{\varepsilon_{\mathbf{n}}}{8(1+N)^s} \qquad (m \ge m_3) \tag{6.6}
$$

for any σ satisfying $|x_\sigma - n_\sigma| \ge 2^{1+k_\sigma}$. In particular, for such σ we have

$$
\left|\frac{\partial^{k_\sigma}}{\partial x_\sigma^{k_\sigma}}\left(x_\sigma^{\nu_\sigma} e^{-(2x_\sigma-2n_\sigma-1)^{2m}}\right)\right| \le 1 \qquad (m \ge m_3).
$$

Obviously, Lemma 4 follows from Lemmas 5 and 6. It remains to prove Lemma 6. For brevity we shall use again the abbreviations given in (6.3). The main idea is to keep the constant m_3 independent of $k_1, \dots, k_s$. For this purpose it is necessary to distinguish several cases.

CASE 1: $1 \le K \le N$. From the binomial theorem one easily deduces that

$$
\left(M+2^{1+K}\right)^{sN} \le 2^{sN}(1+M)^{sN} 2^{sN(1+K)} \le (2+2M)^{sN} 2^{sN(N+1)}. \tag{6.7}
$$

We follow the lines of the proof of Lemma 5 in [5] and omit some details. Put $p_{n,\nu}(x) := H_{\mathbf{n}} N!^s (2+2M)^{sN} 2^{sN(N+1)} x^\nu$. Expanding that polynomial at $n + 1/2$ we write

$$
p_{n,\nu}(x) = \sum_{\mu=0}^{\nu} A_{n,\nu,\mu} 2^\mu (x-n-1/2)^\mu = \sum_{\mu=0}^{\nu} A_{n,\nu,\mu} t^\mu =: T_{n,\nu}(t).
$$

By $h(T_{n,\nu}) := \max_{0\le\mu\le\nu} |A_{n,\nu,\mu}|$ we denote the height of the polynomial $T_{n,\nu}(t)$ which depends on $\mathbf{n}$ (in particular on n), and on s, N, f, ε, but not on m. Proceeding as in the proof of Lemma 5 in [5], we now get, instead of [5, (3.19)]:

$$
W \le \{2(N+1)!h(T_{n,\nu})|t|^N e^{-t^{2m}/2}\} \times \{16^K (K+2)! m^{K+1} |t|^{2Km} 4^{sK} (K+1)!^s (2m)^{Ks+s} 2^{2msK(K+3)} e^{-t^{2m}/2}\}. \tag{6.8}
$$

As in [5], one can find a positive integer m_4 depending on N and $\delta_{\mathbf{n}}$ satisfying

$$
2N < (1+\delta_{\mathbf{n}})^{2m} \qquad (m \ge m_4),
$$

where $\delta_{\mathbf{n}}$ is defined by (2.9). Therefore one has, following the arguments from [5],

$$(6.9)\quad 2(N+1)!h(T_{n,\nu})|t|^N e^{-t^{2m}/2} \le 2(N+1)!h(T_{n,\nu})(1+\delta_{\mathbf{n}})^{2N} e^{-(1+\delta_{\mathbf{n}})^{2m}/2} \\ \le 2(N+1)! \max_{0\le\nu\le N} h(T_{n,\nu}) \cdot (1+\delta_{\mathbf{n}})^{2N} e^{-(1+\delta_{\mathbf{n}})^{2m}/2} \le \frac{\varepsilon_{\mathbf{n}}}{8(1+N)^s},$$

which holds for $m \ge m_5$ for some positive integer $m_5 \ge m_4$ not depending on $k_1, \dots, k_s$, and for $|t| \ge 1+\delta_{\mathbf{n}}$. Note that the hypothesis $|x-n| \ge 2^{1+k} > 2^k + 1/2$ implies that $|2x-2n|-1 \ge 2^{1+k}$. This gives $|t| = |2x-2n-1| \ge 2^{1+k} \ge 2 \ge 1+\delta_{\mathbf{n}}$ for all integers $k \ge 0$. It remains to prove the inequality

$$(6.10)\quad 16^K (K+2)! m^{K+1} t^{2Km} 4^{sK} (K+1)!^s (2m)^{Ks+s} 2^{2msK(K+3)} e^{-t^{2m}/2} < 1$$

for $K \ge 1$, $t = 2^{1+K}$, $m \ge m_6 := \max\{s, 16\}$ (see the corresponding arguments for (3.22) and (3.23) in [5]). Since $m \ge s$, (6.10) follows immediately from

$$(6.11)\quad 16^K (K+2)! m^{K+1} t^{2Km} 4^{mK} (K+1)!^m (2m)^{m(K+1)} 2^{2m^2K(K+3)} e^{-t^{2m}/2} \\ < 1,$$

where $K \ge 1$, $t = 2^{1+K}$, $m \ge m_6$. We shall see below that the inequality (6.11) is fulfilled by proving a stronger one stated in case 2.

Case 2: $N < K$. The arguments are essentially the same as in case 1. Put

$$p_{n,\nu}(x) := H_{\mathbf{n}} N!^s (2+2M)^{sN} x^\nu,$$

and let W be given as in (6.6). Since $N < K$ we replace the inequality (6.7) by $(M+2^{1+K})^{sN} \le (2+2M)^{sN} 2^{sK(K+1)}$. Then (6.8) takes the form

$$(6.12)\quad W \le \{2(N+1)!h(T_{n,\nu})|t|^N e^{-t^{2m}/2}\} \\ \times \{16^K (K+2)! m^{K+1} |t|^{2Km} 4^{sK} (K+1)!^s (2m)^{Ks+s} 2^{2msK(K+3)} \\ \times 2^{sK(K+1)} e^{-t^{2m}/2}\}$$

with a modified integer polynomial $T_{n,\nu}$. As in (6.9) we get

$$(6.13)\qquad 2(N+1)!h(T_{n,\nu})|t|^N e^{-t^{2m}/2} \le \frac{\varepsilon_{\mathbf{n}}}{8(1+N)^s},$$

which again holds for $m \ge m_7$ for some positive integer m_7 not depending on $k_1, \dots, k_s$, and for $|t| \ge 1+\delta_{\mathbf{n}}$. For $m \ge m_6$ we have $m \ge s$, and therefore it suffices to show that

$$(6.14)\quad 16^K (K+2)! m^{K+1} t^{2Km} 4^{mK} (K+1)!^m (2m)^{m(K+1)} 2^{2m^2K(K+3)} \\ \times 2^{mK(K+1)} e^{-t^{2m}/2} < 1$$

with $K \ge 1$, $t = 2^{1+K}$, $m \ge m_3 := \max\{m_5, m_6, m_7\}$. Obviously, (6.14) implies (6.10). When (6.14) is proved, we will have deduced the inequality from (6.6) by (6.8), (6.9) and (6.11)–(6.14). This will prove the lemma.

In order to verify (6.14) we again distinguish two cases.

CASE 2.1: $m \le K$. Let $t = 2^{K+1}$ and $m \ge 9$. Then we get, using $K^{4/K} < 5$ for $K \ge 1$,

$$4^m \ge 143360 \ge 28672 \cdot K^{4/K} \ge 28672^{1/K} K^{4/K}$$

and therefore

$$4^{mK} \ge 28672 K^4.$$

Since $K+3 \le 4K$ for $K \ge 1$, we get $4^{m(K+1)} > 4^{mK} > 112(K+3)^4$, or

$$7 \cdot 8 \cdot (K+3)^4 - 2^{2m(K+1)-1} < 0.$$

This gives

$$\begin{aligned} K \log 16 + (K+2)^2 + (K+1)K + 2K^2(K+1)\log 2 + 8K^2(K+1)^2 \\ + 2K^3(K+3)\log 2 + K^2(K+1)\log 2 - 2^{2m(K+1)-1} < 0, \end{aligned}$$

since the first seven terms on the left side are bounded by $8(K+3)^4$ each. Using the hypothesis $m \le K$ and $A \ge \log A$ for $A \ge 1$, it follows that

$$\begin{aligned} K \log 16 + (K+2)\log(K+2) + (K+1)\log m + 2mK(K+1)\log 2 \\ + m(K+1)\log(8m(K+1)) + 2m^2K(K+3)\log 2 \\ + mK(K+1)\log 2 - 2^{2m(K+1)-1} < 0. \end{aligned} \tag{6.15}$$

Another form of this inequality is

$$\begin{aligned} 16^K (K+2)^{K+2} m^{K+1} t^{2Km} 4^{m(K+1)} (K+1)^{m(K+1)} (2m)^{m(K+1)} \\ \times 2^{2m^2K(K+3)} 2^{mK(K+1)} e^{-t^{2m}/2} < 1. \end{aligned}$$

Since $A^A \ge A!$ for all integers $A \ge 1$, (6.14) follows immediately.

CASE 2.2: $K < m$. Now $m \ge 16$ implies that $4^{mK} \ge 4^m \ge 28672 m^4$. Using the same arguments as in case 2.1, we get

$$7 \cdot 8 \cdot (m+3)^4 - 2^{2m(K+1)-1} < 0,$$

from which one can deduce that

$$\begin{aligned} m \log 16 + (m+2)^2 + (m+1)m + 2m^2(m+1)\log 2 + 8m^2(m+1)^2 \\ + 2m^3(m+3)\log 2 + m^2(m+1)\log 2 - 2^{2m(K+1)-1} < 0. \end{aligned}$$

By the hypothesis $K < m$ one easily estimates the left side in order to obtain (6.15) again. As shown in case 2.1, (6.14) follows from (6.15). Lemmas 4 and 6 are proved. ■

7. Definition of the approximation function $H(\mathbf{x})$. We recall that the degree N of the polynomial $Y_{\mathbf{n}}(\mathbf{x})$ depends on $n_1, \ldots, n_s$, f, s, ε. If we keep f, s and ε fixed, the set of functions

$$Y := \{y_{n_1,\ldots,n_s;\nu_1,\ldots,\nu_s;m}(\mathbf{x}) : n_1, \ldots, n_s \in \mathbb{Z},\ 0 \le \nu_1, \ldots, \nu_s \le N\}$$

is countable. The parameter m is chosen when $n_1, \dots, n_s$ are given, and depends on them. There exist two $1:1$ mappings

$$\mu_i : \mathbb{N} \to \mathbb{Z}^s \quad (i = 1,2), \qquad \mu_1(r) = (n_1, \dots, n_s), \quad \mu_2(r) = (\nu_1, \dots, \nu_s)$$

such that

$$H_r(\mathbf{x}) = y_{\mu_1(r);\mu_2(r);m}(\mathbf{x}) \quad (r \in \mathbb{N}), \qquad Y = \{H_r(\mathbf{x}) : r \in \mathbb{N}\}. \tag{7.1}$$

Since $g_{\mathbf{n},m}(\mathbf{x})$ and $a_{\mathbf{n},\nu} x_1^{\nu_1} \cdots x_s^{\nu_s}$ are analytic functions on $\mathbb{R}^s$ for arbitrary $m \geq 1$, $n_1, \dots, n_s \in \mathbb{Z}$, $0 \leq \nu_1, \dots, \nu_s \leq N$, it follows that $H_r(\mathbf{x})$ is an analytic function on $\mathbb{R}^s$ for any integer $r \in \mathbb{N}$. In what follows let m be given by

$$m := \max\{m_1, m_2, m_3\}, \tag{7.2}$$

so that m depends on $n_1, \dots, n_s, f, s$, and ε. Finally, put

$$H(\mathbf{x}) = \sum_{r=1}^{\infty} H_r(\mathbf{x}) \qquad (\mathbf{x} \in \mathbb{R}^s). \tag{7.3}$$

Now we shall first show that for every $\mathbf{x} \in \mathbb{R}^s$ and for arbitrary non-negative integers $k_1, \dots, k_s$ the series $\sum_r H_r^{(k_1,\dots,k_s)}(\mathbf{x})$ converges absolutely. Then we shall prove that $H \in C^\infty(\mathbb{R}^s)$. Finally, it remains to investigate how the function H approximates the given continuous function f on $\mathbb{R}^s$ with respect to the norm $\|\cdot\|_\omega$. This will be done in the following section.

In order to prove the absolute convergence of the series (7.3) we introduce the set

$$L_{\mathbf{n},\mathbf{k}} = L_{n_1,\dots,n_s;k_1,\dots,k_s} := \prod_{\sigma=1}^{s} [n_\sigma - 2^{k_\sigma+1}; n_\sigma + 2^{k_\sigma+1}]. \tag{7.4}$$

Then, by the inequality (6.1) from Lemma 4 and the identity (2.2), we get

$$\begin{aligned}
\sum_{r=1}^{\infty} |H_r^{(k_1,\dots,k_s)}(\mathbf{x})| - \sum_{\substack{r=1\\ \mathbf{x} \in L_{\mu_1(r);k_1,\dots,k_s}}}^{\infty} |H_r^{(k_1,\dots,k_s)}(\mathbf{x})| \\
&= \sum_{\substack{r=1\\ \mathbf{x} \notin L_{\mu_1(r);k_1,\dots,k_s}}}^{\infty} |H_r^{(k_1,\dots,k_s)}(\mathbf{x})| = \sum_{\substack{r=1\\ \mathbf{x} \notin L_{\mu_1(r);k_1,\dots,k_s}}}^{\infty} |y_{\mu_1(r);\mu_2(r);m}^{(k_1,\dots,k_s)}(\mathbf{x})| \\
&= \sum_{\substack{-\infty < n_1,\dots,n_s < \infty\\ \mathbf{x} \notin L_{\mathbf{n},\mathbf{k}}}} \sum_{0 \leq \nu_1,\dots,\nu_s \leq N} |y_{\mathbf{n},\nu,m}^{(k_1,\dots,k_s)}(\mathbf{x})| \\
&\leq \sum_{\substack{-\infty < n_1,\dots,n_s < \infty\\ \mathbf{x} \notin L_{\mathbf{n},\mathbf{k}}}} \sum_{0 \leq \nu_1,\dots,\nu_s \leq N} \frac{\varepsilon_{\mathbf{n}}}{8(1+N)^s} \leq \sum_{-\infty < n_1,\dots,n_s < \infty} \frac{\varepsilon_{\mathbf{n}}}{8} < \frac{\varepsilon}{8}.
\end{aligned}$$

Hence

$$\sum_{r=1}^{\infty} |H_r^{(k_1,\dots,k_s)}(\mathbf{x})| < \sum_{\substack{r=1 \\ \mathbf{x}\in L_{\mu_1(r);k_1,\dots,k_s}}}^{\infty} |H_r^{(k_1,\dots,k_s)}(\mathbf{x})| + \frac{\varepsilon}{8} < \infty.$$

Obviously the last sum has finitely many terms. We have proved that the series $\sum_r H_r^{(k_1,\dots,k_s)}(\mathbf{x})$ converges absolutely. Therefore we may rearrange the terms arbitrarily. To prove that H belongs to $C^\infty(\mathbb{R}^s)$ we shall need two lemmas from real analysis.

LEMMA 7. *For any real numbers $a < b$ let $f_n : [a;b]^s \to \mathbb{R}$ $(n = 1,2,\dots)$ be a sequence of partially differentiable functions such that the series $\sum_{n=1}^{\infty} f_n(\mathbf{x})$ converges for at least one $\mathbf{x} = \mathbf{x}_0$. Additionally, assume that all the series*

$$\sum_{n=1}^{\infty} \frac{\partial f_n(\mathbf{x})}{\partial x_\sigma} \qquad (1 \le \sigma \le s)$$

converge uniformly on $[a;b]^s$. Then $\sum_{n=1}^{\infty} f_n(\mathbf{x})$ converges uniformly on $[a;b]^s$ to a function $f : [a;b]^s \to \mathbb{R}$ which is partially differentiable with respect to each variable $x_1,\dots,x_s$. Furthermore,

$$\frac{\partial f(\mathbf{x})}{\partial x_\sigma} = \sum_{n=1}^{\infty} \frac{\partial f_n(\mathbf{x})}{\partial x_\sigma} \qquad (1 \le \sigma \le s). \tag{7.5}$$

LEMMA 8 (The Weierstrass criterion). *For any $A \subset \mathbb{R}^s$ let $f_n : A \to \mathbb{R}$ $(n = 1,2,\dots)$ be such that $|f_n(\mathbf{x})| \le c_n$ for all $n \ge 1$ and $\mathbf{x} \in A$, where c_n are some positive numbers not depending on $\mathbf{x}$. Additionally, assume that the series $\sum_{n=1}^{\infty} c_n$ converges. Then the series $\sum_{n=1}^{\infty} f_n(\mathbf{x})$ converges uniformly on A.*

Let $K := k_1 + \dots + k_s$. Now we shall show by an inductive argument that $\sum_{r=1}^{\infty} H_r^{(k_1,\dots,k_s)}(\mathbf{x})$ represents a partially differentiable function $H^{(k_1,\dots,k_s)}(\mathbf{x})$, and

$$\frac{\partial H^{(k_1,\dots,k_s)}(\mathbf{x})}{\partial x_\sigma} = \sum_{r=1}^{\infty} \frac{\partial H_r^{(k_1,\dots,k_s)}(\mathbf{x})}{\partial x_\sigma} \qquad (1 \le \sigma \le s) \tag{7.6}$$

for all $\mathbf{x} \in \mathbb{R}^s$ and $K \ge 0$. Proceeding step by step, we put $K = 0$ at the beginning and then repeat the arguments for $K = 1,2,\dots$. For brevity we introduce the notation

$$H_r^{(K)}(\mathbf{x}) := H_r^{(k_1,\dots,k_s)}(\mathbf{x}).$$

Let $a < b$. For $\mathbf{x} \in [a;b]^s$ and $1 \le \sigma \le s$ we have

$$\sum_{r=1}^{\infty} \frac{\partial H_r^{(K)}(\mathbf{x})}{\partial x_\sigma} = \sum_{\substack{r=1 \\ L_{\mu_1(r);k_1,\dots,k_{\sigma-1},1+k_\sigma,k_{\sigma+1},\dots,k_s} \cap [a;b]^s \neq \emptyset}}^{\infty} \frac{\partial H_r^{(K)}(\mathbf{x})}{\partial x_\sigma} + \sum_{\substack{r=1 \\ L_{\mu_1(r);k_1,\dots,k_{\sigma-1},1+k_\sigma,k_{\sigma+1},\dots,k_s} \cap [a;b]^s = \emptyset}}^{\infty} \frac{\partial H_r^{(K)}(\mathbf{x})}{\partial x_\sigma}. \tag{7.7}$$

The first sum on the right side of (7.7) consists of finitely many terms. The terms of the second sum can be estimated by Lemma 4: The condition $L_{\mu_1(r);k_1,\dots,k_{\sigma-1},1+k_\sigma,k_{\sigma+1},\dots,k_s} \cap [a;b]^s = \emptyset$ and the hypothesis $\mathbf{x} \in [a;b]^s$ imply that $\mathbf{x} \notin L_{\mu_1(r);k_1,\dots,k_{\sigma-1},1+k_\sigma,k_{\sigma+1},\dots,k_s}$. This means that for some σ_0 with $1 \le \sigma_0 \le s$, either

$$|x_{\sigma_0} - n_{\sigma_0}| \ge 2^{k_{\sigma_0}+1} \quad (\text{if } \sigma_0 \neq \sigma) \qquad \text{or} \qquad |x_{\sigma_0} - n_{\sigma_0}| \ge 2^{k_{\sigma_0}+2} \quad (\text{if } \sigma_0 = \sigma).$$

Additionally we need (7.2). Therefore the conditions of Lemma 4 are satisfied when the second sum in (7.7) is estimated by (6.1), (7.1), and (2.2). For $\mathbf{x} \in [a;b]^s$ and for integers r satisfying $L_{\mu_1(r);k_1,\dots,k_{\sigma-1},1+k_\sigma,k_{\sigma+1},\dots,k_s} \cap [a;b]^s = \emptyset$ we have

$$\left| \frac{\partial H_r^{(K)}(\mathbf{x})}{\partial x_\sigma} \right| < \frac{\varepsilon_{\mu_1(r)}}{8(1 + N(\mu_1(r)))^s},$$

and

$$\begin{aligned} \sum_{\substack{r=1 \\ L_{\mu_1(r);k_1,\dots,k_{\sigma-1},1+k_\sigma,k_{\sigma+1},\dots,k_s} \cap [a;b]^s = \emptyset}}^{\infty} \frac{\varepsilon_{\mu_1(r)}}{8(1 + N(\mu_1(r)))^s} &\le \sum_{r=1}^{\infty} \frac{\varepsilon_{\mu_1(r)}}{8(1 + N(\mu_1(r)))^s} \\ &= \sum_{-\infty < n_1 < \infty} \cdots \sum_{-\infty < n_s < \infty} \sum_{0 \le \nu_1 \le N} \cdots \sum_{0 \le \nu_s \le N} \frac{\varepsilon_{\mathbf{n}}}{8(1 + N(n_1, \dots, n_s))^s} \\ &= \sum_{-\infty < n_1 < \infty} \cdots \sum_{-\infty < n_s < \infty} \frac{\varepsilon_{\mathbf{n}}}{8} < \frac{\varepsilon}{8} < \infty. \end{aligned}$$

Hence, by Lemma 8, the second sum on the right side of (7.7) converges uniformly on $[a;b]^s$, and therefore the sum on the left side has the same property. Thus we may apply Lemma 7, which proves that the function defined by

$$H^{(K)}(\mathbf{x}) = \sum_{r=1}^{\infty} H_r^{(K)}(\mathbf{x}) \qquad (\mathbf{x} \in [a;b]^s) \tag{7.8}$$

is partially differentiable with respect to each variable $x_1, \dots, x_s$, and that (7.6) holds. In particular the sum on the right side of (7.8) converges absolutely on $[a;b]^s$, which has been shown before. Since the set $[a;b]^s$ can be chosen arbitrarily large, we have proved that $H \in C^\infty(\mathbb{R}^s)$.

8. Approximation of $f(\mathbf{x})$ by $H(\mathbf{x})$. The goal of this section is to estimate $\|f-H\|_\omega$. For this purpose we first express $H(\mathbf{x})$ by the multiple sum (2.8):

$$
\begin{aligned}
(8.1)\quad \|f-H\|_\omega &= \int_{\mathbb{R}^s} \omega(\mathbf{x})|f(\mathbf{x})-H(\mathbf{x})|d\mathbf{x} \\
&= \sum_{-\infty<n_1<\infty} \cdots \sum_{-\infty<n_s<\infty} \int_{n_1}^{n_1+1} \cdots \int_{n_s}^{n_s+1} \omega(\mathbf{x}) \\
&\qquad \times \Big| f(\mathbf{x}) - \sum_{-\infty<k_1<\infty} \cdots \sum_{-\infty<k_s<\infty} Z_{\mathbf{k},m}(\mathbf{x}) \Big| d\mathbf{x} \\
&\le \sum_{-\infty<n_1<\infty} \cdots \sum_{-\infty<n_s<\infty} \int_{n_1}^{n_1+1} \cdots \int_{n_s}^{n_s+1} \omega(\mathbf{x}) \Big\{ |f(\mathbf{x}) - Z_{\mathbf{n},m}(\mathbf{x})| \\
&\qquad + \sum_{\substack{-1\le i_1,\dots,i_s\le +1 \\ i_1^2+\cdots+i_s^2>0}} |Z_{n_1+i_1,\dots,n_s+i_s,m}(\mathbf{x})| \\
&\qquad + \sum_{\substack{-\infty<k_1<\infty \\ k_\sigma\ne n_\sigma-1,n_\sigma,n_\sigma+1 \\ (\sigma=1,\dots,s)}} \cdots \sum_{-\infty<k_s<\infty} |Z_{\mathbf{k},m}(\mathbf{x})| \Big\} \, d\mathbf{x} \\
&=: I_1 + I_2 + I_3.
\end{aligned}
$$

The numbers I_1, I_2, I_3 correspond to the three terms inside the curly brackets.

a) *An upper bound for I_1.* The following inequalities follow from (2.4), (2.5), and (2.7):

$$
(8.2)\qquad |Z_{\mathbf{k},m}(\mathbf{x})| \le |Y_{\mathbf{k}}(\mathbf{x})| \le 1 + |f(\mathbf{x})|
$$

for $k_\sigma - 1 \le x_\sigma \le k_\sigma + 2$, $\sigma = 1,\dots,s$. Hence

$$
\begin{aligned}
I_1 &= \sum_{-\infty<n_1<\infty} \cdots \sum_{-\infty<n_s<\infty} \int_{n_1}^{n_1+1} \cdots \int_{n_s}^{n_s+1} \omega(\mathbf{x})|f(\mathbf{x}) - Z_{\mathbf{n},m}(\mathbf{x})| \, d\mathbf{x} \\
&= \sum_{-\infty<n_1<\infty} \cdots \sum_{-\infty<n_s<\infty} \sideset{}{^*}\sum_{\substack{(t_\sigma,u_\sigma) \\ \sigma=1,\dots,s}} \int_{n_1+t_1}^{n_1+u_1} \cdots \int_{n_s+t_s}^{n_s+u_s} \omega(\mathbf{x})|f(\mathbf{x}) - Z_{\mathbf{n},m}(\mathbf{x})| \, d\mathbf{x},
\end{aligned}
$$

where every pair (t_σ, u_σ) for $\sigma = 1,\dots,s$ in the sum $\sum^*$ is one of the three pairs

$$
(0, \delta_{\mathbf{n}}/2), \qquad (\delta_{\mathbf{n}}/2, 1-\delta_{\mathbf{n}}/2), \qquad (1-\delta_{\mathbf{n}}/2, 1).
$$

Therefore the sum $\sum^*$ consists of 3^s terms. For convenience we introduce two polynomials:

$$p(t) := \left(\frac{1}{2} - \frac{3\delta_{\mathbf{n}}}{4}\right) t^2 + \left(\frac{1}{2} - \frac{\delta_{\mathbf{n}}}{4}\right) t + \frac{\delta_{\mathbf{n}}}{2},$$
$$q(t) := -\left(\frac{1}{2} - \frac{3\delta_{\mathbf{n}}}{4}\right) t^2 + \left(\frac{1}{2} - \frac{\delta_{\mathbf{n}}}{4}\right) t + \left(1 - \frac{\delta_{\mathbf{n}}}{2}\right).$$

In particular one has

(8.3)

t	$p(t)$	$q(t)$
-1	0	$\delta_n/2$
0	$\delta_n/2$	$1-\delta_n/2$
$+1$	$1-\delta_n/2$	1

Thus we get

$$I_1 = \sum_{-\infty<n_1<\infty} \cdots \sum_{-\infty<n_s<\infty} \sum_{v_1=-1}^{1} \cdots \sum_{v_s=-1}^{1} \int\limits_{n_1+p(v_1)}^{n_1+q(v_1)} \cdots \int\limits_{n_s+p(v_s)}^{n_s+q(v_s)} \omega(\mathbf{x}) \times |f(\mathbf{x}) - Z_{\mathbf{n},m}(\mathbf{x})|\, d\mathbf{x}$$
$$= \Bigg\{ \sum_{-\infty<n_1<\infty} \cdots \sum_{-\infty<n_s<\infty} \sum_{\substack{v_1=-1 \\ v_1^2+\cdots+v_s^2>0}}^{1} \cdots \sum_{v_s=-1}^{1} \int\limits_{n_1+p(v_1)}^{n_1+q(v_1)} \cdots \int\limits_{n_s+p(v_s)}^{n_s+q(v_s)}$$
$$+ \sum_{-\infty<n_1<\infty} \cdots \sum_{-\infty<n_s<\infty} \int\limits_{n_1+\delta_n/2}^{n_1+1-\delta_n/2} \cdots \int\limits_{n_s+\delta_n/2}^{n_s+1-\delta_n/2} \Bigg\} \omega(\mathbf{x}) |f(\mathbf{x}) - Z_{\mathbf{n},m}(\mathbf{x})|\, d\mathbf{x}.$$

The integrands of the first multiple sum in {} can be estimated trivially by (8.2), whereas an upper bound for the integrands of the right multiple sum is given by Lemma 3, since $\mathbf{x} \in J_{\mathbf{n}}$ by (2.11). So we have

$$I_1 \le \sum_{-\infty<n_1<\infty} \cdots \sum_{-\infty<n_s<\infty} \sum_{\substack{v_1=-1 \\ v_1^2+\cdots+v_s^2>0}}^{1} \cdots \sum_{v_s=-1}^{1} \int\limits_{n_1+p(v_1)}^{n_1+q(v_1)} \cdots \int\limits_{n_s+p(v_s)}^{n_s+q(v_s)} \omega(\mathbf{x}) \times (1 + 2|f(\mathbf{x})|)\, d\mathbf{x}$$
$$+ \sum_{-\infty<n_1<\infty} \cdots \sum_{-\infty<n_s<\infty} \int\limits_{n_1+\delta_n/2}^{n_1+1-\delta_n/2} \cdots \int\limits_{n_s+\delta_n/2}^{n_s+1-\delta_n/2} \omega(\mathbf{x})\, \frac{\varepsilon_{\mathbf{n}}}{8}\, d\mathbf{x}. \tag{8.4}$$

If $v_1^2 + \cdots + v_s^2 > 0$, then there exists σ with $v_\sigma \neq 0$ and, by (8.3),

$$q(v_\sigma) - p(v_\sigma) = \delta_{\mathbf{n}}/2.$$

Moreover, for every τ $(1 \le \tau \le s)$ we have

$$0 < (n_\tau + q(v_\tau)) - (n_\tau + p(v_\tau)) = q(v_\tau) - p(v_\tau) < 1.$$

Assuming additionally $v_\sigma = -1$, we have

$$\begin{aligned}
&\int\limits_{n_1+p(v_1)}^{n_1+q(v_1)} \cdots \int\limits_{n_s+p(v_s)}^{n_s+q(v_s)} \omega(\mathbf{x})(1+2|f(\mathbf{x})|)\,d\mathbf{x} \\
&\qquad \le \int\limits_{n_1+p(v_1)}^{n_1+q(v_1)} \cdots \int\limits_{n_s+p(v_s)}^{n_s+q(v_s)} \sup_{\substack{n_\tau-1\le x_\tau\le n_\tau+2\\ 1\le\tau\le s\\ \tau\ne\sigma}} \omega(\mathbf{x})(1+2|f(\mathbf{x})|)\,d\mathbf{x} \\
&\qquad \le \int\limits_{n_\sigma}^{n_\sigma+\delta_n/2} \sup_{\substack{n_\tau-1\le x_\tau\le n_\tau+2\\ 1\le\tau\le s\\ \tau\ne\sigma}} \omega(\mathbf{x})(1+2|f(\mathbf{x})|)\,dx_\sigma.
\end{aligned}$$

For $v_\sigma = +1$ the arguments are the same, leading to a similar integral with lower limit $n_\sigma + 1 - \delta_n/2$ and upper limit $n_\sigma + 1$. Applying the definition (2.9) of $\delta_{\mathbf{n}}$, we have proved that

$$\int\limits_{n_1+p(v_1)}^{n_1+q(v_1)} \cdots \int\limits_{n_s+p(v_s)}^{n_s+q(v_s)} \omega(\mathbf{x})(1+2|f(\mathbf{x})|)\,d\mathbf{x} < \frac{\varepsilon_{\mathbf{n}}}{8}$$

for all $v_1^2 + \cdots + v_s^2 > 0$, $v_\tau = -1, 0, +1$. Now we easily find an upper bound for I_1 from (8.4), namely

$$\begin{aligned}
I_1 &\le \sum_{-\infty<n_1<\infty} \cdots \sum_{-\infty<n_s<\infty} \Bigg\{ \sum_{\substack{v_1=-1\\ v_1^2+\cdots+v_s^2>0}}^{1} \cdots \sum_{v_s=-1}^{1} \frac{\varepsilon_{\mathbf{n}}}{8} + \int_{\mathbb{R}^s} \omega(\mathbf{x})\,\frac{\varepsilon_{\mathbf{n}}}{8}\,d\mathbf{x} \Bigg\} \\
&= \sum_{-\infty<n_1<\infty} \cdots \sum_{-\infty<n_s<\infty} \{(3^s-1)\varepsilon_{\mathbf{n}}/8 + \varepsilon_{\mathbf{n}}/8\} \\
&= \sum_{-\infty<n_1<\infty} \cdots \sum_{-\infty<n_s<\infty} 3^s\,\varepsilon_{\mathbf{n}}/8 = \varepsilon/8.
\end{aligned}$$

Here we have applied the identities (1.1) and (2.2). Thus we have proved the following lemma:

LEMMA 9. *We have*

$$I_1 = \sum_{-\infty<n_1<\infty} \cdots \sum_{-\infty<n_s<\infty} \sum_{v_1=-1}^{1} \cdots \sum_{v_s=-1}^{1} \int\limits_{n_1+p(v_1)}^{n_1+q(v_1)} \cdots \int\limits_{n_s+p(v_s)}^{n_s+q(v_s)} \omega(\mathbf{x}) \times |f(\mathbf{x}) - Z_{\mathbf{n},m}(\mathbf{x})|\,d\mathbf{x} \le \varepsilon/8.$$

b) *An upper bound for I_2.* Next we investigate the second term in (8.1):

$$I_2 = \sum_{-\infty<n_1<\infty} \cdots \sum_{-\infty<n_s<\infty} \int_{n_1}^{n_1+1} \cdots \int_{n_s}^{n_s+1} \omega(\mathbf{x}) \times \sum_{\substack{-1\le i_1,\dots,i_s\le +1\\ i_1^2+\cdots+i_s^2>0}} |Z_{n_1+i_1,\dots,n_s+i_s,m}(\mathbf{x})|\, d\mathbf{x}$$
$$= \sum_{-\infty<n_1<\infty} \cdots \sum_{-\infty<n_s<\infty} \sum_{\substack{k_1=n_1-1\\ (k_1,\dots,k_s)\neq(n_1,\dots,n_s)}}^{n_1+1} \cdots \sum_{k_s=n_s-1}^{n_s+1} \int_{n_1}^{n_1+1} \cdots \int_{n_s}^{n_s+1} \omega(\mathbf{x}) \times |Z_{k_1,\dots,k_s,m}(\mathbf{x})|\, d\mathbf{x}. \tag{8.5}$$

In what follows we consider all the 3^s-1 domains $I_\mathbf{k}$ for $n_i-1 \le k_i \le n_i+1$, $i=1,\dots,s$, $(k_1,\dots,k_s)\neq(n_1,\dots,n_s)$, where $I_\mathbf{k}$ is given by (2.10) for $\mathbf{n}=\mathbf{k}$. Let

$$G_\mathbf{n} := \prod_{i=1}^{s} [n_i; n_i+1].$$

We separate the domain $G_\mathbf{n}$ of integration from (8.5) into parts overlapping with $I_\mathbf{k}$ and the remaining ones. For this purpose we define

$$J_{\mathbf{n},\mathbf{k}} := \int_{n_1}^{n_1+1} \cdots \int_{n_s}^{n_s+1} \omega(\mathbf{x})|Z_{\mathbf{k},m}(\mathbf{x})|\, d\mathbf{x} = \Big\{ \int_{G_\mathbf{n}\cap I_\mathbf{k}} + \int_{G_\mathbf{n}\setminus I_\mathbf{k}} \Big\} \omega(\mathbf{x})|Z_{\mathbf{k},m}(\mathbf{x})|\, d\mathbf{x}.$$

First we deal with $G_\mathbf{n}\cap I_\mathbf{k}$. From the hypothesis $n_i-1 \le k_i \le n_i+1$ for $i=1,\dots,s$ we know that this is a nonempty set. Then one deduces from $(k_1,\dots,k_s)\neq(n_1,\dots,n_s)$ and from (2.10) the existence of at least one σ such that the projection of $G_\mathbf{n}\cap I_\mathbf{k}$ onto the x_σ-axis gives an interval of length $\delta_\mathbf{k}/2$. It follows that $k_\sigma \neq n_\sigma$. Consequently, we have either $n_\sigma = k_\sigma+1$ or $n_\sigma = k_\sigma-1$. There exists a subset $L_{\mathbf{n},\mathbf{k}}$ (of dimension $s-1$) such that either

$$G_\mathbf{n}\cap I_\mathbf{k} = [n_\sigma; n_\sigma+\delta_\mathbf{k}/2] \times L_{\mathbf{n},\mathbf{k}} \quad \text{with } k_\sigma = n_\sigma - 1 \tag{8.6}$$

or

$$G_\mathbf{n}\cap I_\mathbf{k} = [n_\sigma+1-\delta_\mathbf{k}/2; n_\sigma+1] \times L_{\mathbf{n},\mathbf{k}} \quad \text{with } k_\sigma = n_\sigma + 1. \tag{8.7}$$

Any point $\mathbf{x}=(x_1,\dots,x_s)$ from the domain of integration in $J_{\mathbf{n},\mathbf{k}}$ satisfies $n_i \le x_i \le n_i+1$ for $i=1,\dots,s$. Moreover $k_i \in \{n_i-1, n_i, n_i+1\}$ for each i. Hence we get the inequalites

$$k_i - 1 \le n_i \le x_i \le n_i+1 = (n_i-1)+2 \le k_i+2 \quad (i=1,\dots,s), \tag{8.8}$$

which allow us (by application of (2.4), (2.5), and (2.7)) to estimate $|f(\mathbf{x}) - Y_{\mathbf{k}}(\mathbf{x})|$ for $\mathbf{x} \in G_{\mathbf{n}} \cap I_{\mathbf{k}}$:

$$|Z_{\mathbf{k},m}(\mathbf{x})| \le |Y_{\mathbf{k}}(\mathbf{x})| \le 1 + |f(\mathbf{x})| \quad (\mathbf{x} \in G_{\mathbf{n}} \cap I_{\mathbf{k}}). \tag{8.9}$$

First assume that (8.6) holds. Applying (8.9) and (8.8), we then have

$$\begin{aligned}\int\limits_{G_{\mathbf{n}}\cap I_{\mathbf{k}}} \omega(\mathbf{x})|Z_{\mathbf{k},m}(\mathbf{x})|\,d\mathbf{x} &= \int\limits_{n_\sigma}^{n_\sigma+\delta_k/2} \int\limits_{L_{\mathbf{n},\mathbf{k}}} \omega(\mathbf{x})|Z_{\mathbf{k},m}(\mathbf{x})|\,d\mathbf{x} \\ &\le \int\limits_{n_\sigma}^{n_\sigma+\delta_k/2} \int\limits_{L_{\mathbf{n},\mathbf{k}}} \omega(\mathbf{x})(1+|f(\mathbf{x})|)\,d\mathbf{x} \\ &\le \int\limits_{n_\sigma}^{n_\sigma+\delta_k/2} \int\limits_{L_{\mathbf{n},\mathbf{k}}} \sup_{\substack{k_\tau-1\le x_\tau\le k_\tau+2\\ 1\le\tau\le s\\ \tau\ne\sigma}} \omega(\mathbf{x})(1+|f(\mathbf{x})|)\,d\mathbf{x} \\ &\le \int\limits_{n_\sigma}^{n_\sigma+\delta_k/2} \sup_{\substack{k_\tau-1\le x_\tau\le k_\tau+2\\ 1\le\tau\le s\\ \tau\ne\sigma}} \omega(\mathbf{x})(1+|f(\mathbf{x})|)\,dx_\sigma < \frac{\varepsilon_{\mathbf{k}}}{8}.\end{aligned}$$

Here the upper bound $\varepsilon_{\mathbf{k}}/8$ follows from (2.9) (by setting $n_\sigma = k_\sigma + 1$), whereas the last but one term results from $L_{\mathbf{n},\mathbf{k}} \subset G_{\mathbf{n}}$, $n_i \le x_i \le n_i + 1$ and $d\mathbf{x} = dx_1 \cdots dx_s$. When (8.7) holds, one gets the same upper bound by using similar arguments:

$$\begin{aligned}&\int\limits_{G_{\mathbf{n}}\cap I_{\mathbf{k}}} \omega(\mathbf{x})|Z_{\mathbf{k},m}(\mathbf{x})|\,d\mathbf{x} \\ &\qquad\le \int\limits_{n_\sigma+1-\delta_k/2}^{n_\sigma+1} \sup_{\substack{k_\tau-1\le x_\tau\le k_\tau+2\\ 1\le\tau\le s\\ \tau\ne\sigma}} \omega(\mathbf{x})(1+|f(\mathbf{x})|)\,dx_\sigma < \frac{\varepsilon_{\mathbf{k}}}{8}\end{aligned}$$

with $n_\sigma = k_\sigma - 1$. Altogether we have proved that

$$\int\limits_{G_{\mathbf{n}}\cap I_{\mathbf{k}}} \omega(\mathbf{x})|Z_{\mathbf{k},m}(\mathbf{x})|\,d\mathbf{x} \le \frac{\varepsilon_{\mathbf{k}}}{8} \tag{8.10}$$

for $\mathbf{k} = (k_1, \ldots, k_s)$ with $n_i - 1 \le k_i \le n_i + 1$ and $(k_1, \ldots, k_s) \ne (n_1, \ldots, n_s)$. Next we treat the domain $G_{\mathbf{n}} \setminus I_{\mathbf{k}}$ from the second integral of $J_{\mathbf{n},\mathbf{k}}$. For this purpose we need some preliminaries. Let $x \in G_{\mathbf{n}} \setminus I_{\mathbf{k}}$. Then using $x \notin I_{\mathbf{k}}$, (2.7), (2.3), (2.6), and Lemma 2, one gets

$$(8.11)\quad |Z_{\mathbf{k},m}(\mathbf{x})| = |Y_{\mathbf{k}}(\mathbf{x})g_{\mathbf{k},m}(\mathbf{x})| = \Big|\sum_{\nu_1=0}^{N}\cdots\sum_{\nu_s=0}^{N} a_{\mathbf{k},\nu}x_1^{\nu_1}\cdots x_s^{\nu_s}g_{\mathbf{k},m}(\mathbf{x})\Big|$$
$$\le \sum_{\nu_1=0}^{N}\cdots\sum_{\nu_s=0}^{N}|y_{\mathbf{k},\nu,m}(\mathbf{x})| \le \sum_{\nu_1=0}^{N}\cdots\sum_{\nu_s=0}^{N}\frac{\varepsilon_{\mathbf{k}}}{8(N+1)^s} = \frac{\varepsilon_{\mathbf{k}}}{8}$$

for all $m \ge m_1$ and $\mathbf{x} \notin I_{\mathbf{k}}$. We recall that the degree N and m_1 depend on f, ε and $k_1, \dots, k_s$. It follows that

$$\int\limits_{G_{\mathbf{n}}\setminus I_{\mathbf{k}}} \omega(\mathbf{x})|Z_{\mathbf{k},m}(\mathbf{x})|\,d\mathbf{x} \le \int\limits_{G_{\mathbf{n}}\setminus I_{\mathbf{k}}} \omega(\mathbf{x})\,\frac{\varepsilon_{\mathbf{k}}}{8}\,d\mathbf{x} \le \frac{\varepsilon_{\mathbf{k}}}{8}\int\limits_{\mathbb{R}^s}\omega(\mathbf{x})\,d\mathbf{x} = \frac{\varepsilon_{\mathbf{k}}}{8}.$$

Together with (8.10), this shows that $J_{\mathbf{n},\mathbf{k}} < \varepsilon_{\mathbf{k}}/8 + \varepsilon_{\mathbf{k}}/8 = \varepsilon_{\mathbf{k}}/4$. Then from (8.5) we get

$$\begin{aligned}
I_2 &< \sum_{-\infty<n_1<\infty}\cdots\sum_{-\infty<n_s<\infty}\ \underset{(k_1,\dots,k_s)\ne(n_1,\dots,n_s)}{\sum_{k_1=n_1-1}^{n_1+1}\cdots\sum_{k_s=n_s-1}^{n_s+1}}\frac{\varepsilon_{\mathbf{k}}}{4}\\
&< \frac{1}{4}\sum_{-\infty<n_1<\infty}\cdots\sum_{-\infty<n_s<\infty}\ \sum_{k_1=n_1-1}^{n_1+1}\cdots\sum_{k_s=n_s-1}^{n_s+1}\frac{\varepsilon}{9^s\cdot 2^{|k_1|+\cdots+|k_s|}}\\
&= \frac{1}{4}\sum_{-\infty<n_1<\infty}\cdots\sum_{-\infty<n_s<\infty}\frac{\varepsilon}{3^s\cdot 2^{|n_1|+\cdots+|n_s|}} = \frac{\varepsilon}{4} \qquad \text{(by (2.2))}.
\end{aligned}$$

That gives

LEMMA 10. *We have*

$$\begin{aligned}
I_2 = &\sum_{-\infty<n_1<\infty}\cdots\sum_{-\infty<n_s<\infty}\int_{n_1}^{n_1+1}\cdots\int_{n_s}^{n_s+1}\omega(\mathbf{x})\\
&\times \sum_{\substack{-1\le i_1,\dots,i_s\le +1\\ i_1^2+\dots+i_s^2>0}} |Z_{n_1+i_1,\dots,n_s+i_s,m}(\mathbf{x})|\,d\mathbf{x} < \frac{\varepsilon}{4}.
\end{aligned}$$

c) *An upper bound for* I_3. The coordinates x_σ of any point $\mathbf{x} \in G_{\mathbf{n}}$ satisfy $n_\sigma \le x_\sigma \le n_\sigma+1$ for $\sigma = 1, \dots, s$. From the hypotheses $k_\sigma \ne n_\sigma-1, n_\sigma, n_\sigma+1$ $(\sigma = 1, \dots, s)$ for k_σ in I_3 we get

- when $k_\sigma \le n_\sigma - 2$: $x_\sigma \ge n_\sigma = (n_\sigma - 2) + 2 \ge k_\sigma + 2$;
- when $k_\sigma \ge n_\sigma + 2$: $x_\sigma \le n_\sigma + 1 = (n_\sigma + 2) - 1 \le k_\sigma - 1$.

Consequently, $\mathbf{x} \notin I_{\mathbf{k}}$ in each case, which follows immediately from (2.10). Again we may apply (8.11), (2.2), and (1.1):

$$
\begin{aligned}
I_3 &= \sum_{-\infty<n_1<\infty} \cdots \sum_{-\infty<n_s<\infty} \int_{n_1}^{n_1+1} \cdots \int_{n_s}^{n_s+1} \omega(\mathbf{x}) \\
&\quad \times \sum_{\substack{-\infty<k_1<\infty \\ k_\sigma \neq n_\sigma-1, n_\sigma, n_\sigma+1 \\ (\sigma=1,\dots,s)}} \cdots \sum_{-\infty<k_s<\infty} |Z_{\mathbf{k},m}(\mathbf{x})| \, d\mathbf{x} \\
&\le \sum_{-\infty<n_1<\infty} \cdots \sum_{-\infty<n_s<\infty} \int_{n_1}^{n_1+1} \cdots \int_{n_s}^{n_s+1} \omega(\mathbf{x}) \\
&\quad \times \sum_{-\infty<k_1<\infty} \cdots \sum_{-\infty<k_s<\infty} \frac{\varepsilon_{\mathbf{k}}}{8} \, d\mathbf{x} \\
&\le \frac{\varepsilon}{8} \sum_{-\infty<n_1<\infty} \cdots \sum_{-\infty<n_s<\infty} \int_{n_1}^{n_1+1} \cdots \int_{n_s}^{n_s+1} \omega(\mathbf{x}) \, d\mathbf{x} \\
&= \frac{\varepsilon}{8} \int_{\mathbb{R}^s} \omega(\mathbf{x}) \, d\mathbf{x} = \frac{\varepsilon}{8}.
\end{aligned}
$$

LEMMA 11. *We have*

$$
\begin{aligned}
I_3 &= \sum_{-\infty<n_1<\infty} \cdots \sum_{-\infty<n_s<\infty} \int_{n_1}^{n_1+1} \cdots \int_{n_s}^{n_s+1} \omega(\mathbf{x}) \\
&\quad \times \sum_{\substack{-\infty<k_1<\infty \\ k_\sigma \neq n_\sigma-1, n_\sigma, n_\sigma+1 \\ (\sigma=1,\dots,s)}} \cdots \sum_{-\infty<k_s<\infty} |Z_{\mathbf{k},m}(\mathbf{x})| \, d\mathbf{x} \le \frac{\varepsilon}{8}.
\end{aligned}
$$

Collecting together the results from Lemmas 9–11, one finally has

$$\|f - H\|_\omega = I_1 + I_2 + I_3 \le \varepsilon/8 + \varepsilon/4 + \varepsilon/8 = \varepsilon/2 < \varepsilon.$$

This completes the proof of Theorem 1.

REFERENCES

[1] M. Boshernitzan, *Universal formulae and universal differential equations*, Ann. of Math. 124 (1986), 273–291.

[2] K. Briggs, *Another universal differential equation*, ArXiv: math.CA/02 111 42; http://members.lycos.co.uk/keithmbriggs/

[3] R. C. Buck, *The solutions of a smooth PDE can be dense in $C[I]$*, J. Differential Equations 41 (1981), 239–244.

[4] R. J. Duffin, *Rubel's universal differential equation*, Proc. Nat. Acad. Sci. U.S.A. 78 (1981), 4661–4662.

[5] C. Elsner, *On a universal differential equation for the analytic terms of C^∞-superpositions on the real line*, J. Math. Anal. Appl. 295 (2004), 315–330.

[6] —, *A universal differential equation of degree six*, ibid. 244 (2000), 533–541.

[7] C. Elsner, *On a universal functional equation of degree six*, Comm. Appl. Nonlinear Anal. 9 (2002), no. 3, 23–29.

[8] —, *On a linear transcendence measure for the solutions of a universal differential equation at algebraic points*, J. Math. Anal. Appl. 279 (2003), 684–699.

[9] —, *On a method to transform algebraic differential equations into universal equations*, Abh. Math. Sem. Univ. Hamburg 74 (2004), 33–48.

[10] S. G. Krantz and H. R. Parks, *A Primer of Real Analytic Functions*, Birkhäuser, Basel, 1992.

[11] M. A. Neumark, *Normierte Algebren*, Harri Deutsch, Thun–Frankfurt am Main, 1990.

[12] M. B. Pour-El, *Abstract computability and its relation to the general purpose analog computer* (*some connections between logic*, *differential equations*, *and analog computers*), Trans. Amer. Math. Soc. 199 (1974), 1–28.

[13] L. A. Rubel, *A universal differential equation*, Bull. Amer. Math. Soc. 4 (1981), 345–349.

[14] —, *Uniform approximation by rational functions which all satisfy the same algebraic differential equation*, J. Approx. Theory 84 (1996), 123–128.

[15] C. E. Shannon, *Mathematical theory of the differential analyzer*, J. Math. Phys. 20 (1941), 337–354.

Institut für Mathematik
Universität Hannover
Welfengarten 1
D-30167 Hannover, Germany
E-mail: elsner@math.uni-hannover.de

Received 30 July 2004;
revised 2 June 2005 (4476)