COLLOQUIUM MATHEMATICUM

VOL. 104 2006 NO. 1

RESIDUE CLASS RINGS OF REAL-ANALYTIC AND ENTIRE FUNCTIONS

BY

MAREK GOLASIŃSKI (Toruń) and MELVIN HENRIKSEN (Clarement, CA)

Abstract. Let $\mathcal{A}(\mathbb{R})$ and $\mathcal{E}(\mathbb{R})$ denote respectively the ring of analytic and real entire functions in one variable. It is shown that if $\mathfrak{m}$ is a maximal ideal of $\mathcal{A}(\mathbb{R})$, then $\mathcal{A}(\mathbb{R})/\mathfrak{m}$ is isomorphic either to the reals or a real closed field that is an η_1-set, while if $\mathfrak{m}$ is a maximal ideal of $\mathcal{E}(\mathbb{R})$, then $\mathcal{E}(\mathbb{R})/\mathfrak{m}$ is isomorphic to one of the latter two fields or to the field of complex numbers. Moreover, we study the residue class rings of prime ideals of these rings and their Krull dimensions. Use is made of a classical characterization of algebraically closed fields due to E. Steinitz and techniques described in L. Gillman and M. Jerison's book on rings of continuous functions.

1. Introduction. Let $\mathbb{R}$ denote the field of reals and $\mathcal{A}(\mathbb{R})$ the ring of all analytic functions on $\mathbb{R}$. That is, $\mathcal{A}(\mathbb{R})$ consists of all real-valued functions f such that for each $x_0 \in \mathbb{R}$, there exists an open neighborhood V of x_0 such that for all $x \in V$, the value $f(x)$ is the sum of an absolutely convergent power series in powers of $x - x_0$. Let $\mathbb{K}$ be the field $\mathbb{R}$ of reals or the field $\mathbb{C}$ of complex numbers and $\mathcal{E}(\mathbb{K})$ the ring of entire functions over $\mathbb{K}$, i.e., the functions given by power series $\sum_{n=0}^{\infty} a_n x^n$ with $a_n \in \mathbb{K}$ for $n = 0, 1, \ldots$ and $\lim_{n\to\infty} |a_n|^{1/n} = 0$. Clearly, any $f \in \mathcal{E}(\mathbb{R})$ extends uniquely to an entire function over $\mathbb{C}$, whence there is an inclusion $\mathcal{E}(\mathbb{R}) \subseteq \mathcal{E}(\mathbb{C})$. As is well known, $\mathcal{A}(\mathbb{C})$ and $\mathcal{E}(\mathbb{C})$ coincide, while $\frac{1}{1+x^2} = \sum_{n=0}^{\infty}(-1)^n x^{2n}$ is in $\mathcal{A}(\mathbb{R}) \setminus \mathcal{E}(\mathbb{R})$.

After reviewing what is known about the ideal structure of $\mathcal{A}(\mathbb{R})$, a description of the maximal ideals of this ring and the corresponding residue class fields is given in Section 2. These residue class fields are either the real field $\mathbb{R}$ or non-archimedean totally ordered extensions of it. We show, in particular, that if CH (the continuum hypothesis) holds, then $\mathcal{A}(\mathbb{R})$ mod a maximal ideal must be isomorphic to one of two possible fields: the reals $\mathbb{R}$ or an H-field of cardinality 2^{ω}. In [8], the maximal ideals of $\mathcal{E}(\mathbb{C})$ were

2000 *Mathematics Subject Classification*: Primary 26E05, 30D20; Secondary 12D15, 13B30.

Key words and phrases: adequate ring, algebraically (real) closed field, analytic (entire) function, Bézout domain, filter, integral extension, Krull dimension, maximal (prime) ideal.

The second author acknowledges the hospitality and financial support of the Faculty of Mathematics and Computer Science of the Nicholas Copernicus University in Toruń during the time this work was revised and finished.

described, and it was shown that $\mathcal{E}(\mathbb{C})/\mathfrak{m}$ is always isomorphic as a ring to $\mathbb{C}$, even though for some maximal ideals $\mathfrak{m}$, the field $\mathcal{E}(\mathbb{C})/\mathfrak{m}$ is infinite-dimensional as an algebra over $\mathbb{R}$.

In Section 3, we apply techniques developed in [9] to carry out a similar program to study $\mathcal{E}(\mathbb{R})$. We make use of a characterization of algebraically closed fields proved by E. Steinitz in 1909 that is still not as well known as it should be. It states that an algebraically closed field is uniquely determined by its prime field and its degree of transcendency over it (see [17]). Thus, every algebraically closed field of degree of transcendency 2^ω over the field of rational numbers is isomorphic to $\mathbb{C}$. Use is also made of techniques described by Gillman and Jerison in [5].

Section 4 deals with prime ideals of $\mathcal{A}(\mathbb{R})$ and $\mathcal{E}(\mathbb{K})$. We apply the results of [12] to derive that the Krull dimension of both rings is at least 2^{ω_1}.

In what follows, all rings considered are assumed to be commutative and have an identity element. Such a ring is said to be *Bézout* if each of its finitely generated ideals is principal, and is called a *Bézout domain* if it is also an integral domain. The following proposition whose proof is given in [4] will be used in what follows.

PROPOSITION 1.1. *Let $\mathbb{K}$ be one of the fields $\mathbb{R}$ or $\mathbb{C}$. Whenever $\{x_n\}$ is any sequence of elements in $\mathbb{K}$ such that $\lim_{n\to\infty}|x_n| = \infty$ and $\{w_{nk}\}$ is any double sequence of elements of $\mathbb{K}$, there is an $f \in \mathcal{E}(\mathbb{K})$ such that $f^{(k)}(x_n) = w_{nk}$ for $n = 1, 2, \ldots$ and $k = 0, 1, 2, \ldots$, where $f^{(k)}$ denotes the kth derivative of f.*

In particular, because $\mathcal{E}(\mathbb{R}) \subseteq \mathcal{A}(\mathbb{R})$, there is also such an $f \in \mathcal{A}(\mathbb{R})$ for such sequences $\{x_n\}$ and $\{w_{nk}\}$ of real numbers.

What follows is the main theorem of [8].

THEOREM 1.2. *If $\mathfrak{m} \subseteq \mathcal{E}(\mathbb{C})$ is a maximal ideal, then $\mathcal{E}(\mathbb{C})/\mathfrak{m}$ is isomorphic as a ring to $\mathbb{C}$.*

The zeroset $Z_{\mathbb{C}}(f) = \{x \in \mathbb{C} : f(x) = 0\}$ of a function $f \in \mathcal{E}(\mathbb{K})$ is a closed discrete subset of $\mathbb{C}$ and hence is countable. Moreover, as is noted in [4], f is invertible if and only if $Z_{\mathbb{C}}(f) = \emptyset$.

2. Maximal ideals of $\mathcal{A}(\mathbb{R})$. We begin with a brief summary of the properties of $\mathcal{A}(\mathbb{R})$ given in [2]. Recall that an integral domain $\mathcal{R}$ with quotient field K is said to be *completely integrally closed* if for any $x \in K$, there exists a finitely generated $\mathcal{R}$-submodule $\mathcal{M}$ of K such that $\mathcal{R}[x] \subseteq \mathcal{M}$. By [2, Theorem 1.19], $\mathcal{A}(\mathbb{R})$ is a Bézout domain that is completely integrally closed as well. On the other hand, as is noted in [6], the sequence of functions $\{\sin(1/2^n)x : n = 1, 2, \ldots\}$ generates an infinite ascending chain of ideals that fails to stabilize, so $\mathcal{A}(\mathbb{R})$ is not a Noetherian ring.

Clearly, $\mathcal{A}(\mathbb{R})$ is a subring of the ring $\mathcal{C}(\mathbb{R})$ of continuous functions $f : \mathbb{R} \to \mathbb{R}$ with the usual pointwise operations.

Given a non-zero $f \in \mathcal{A}(\mathbb{R})$, its zeroset $Z_{\mathbb{R}}(f) = \{x \in \mathbb{R} : f(x) = 0\}$ is a closed discrete subset of $\mathbb{R}$, and hence is countable. The lemma that follows is needed below.

LEMMA 2.1. *A function $f \in \mathcal{A}(\mathbb{R})$ is invertible if and only if $Z_{\mathbb{R}}(f) = \emptyset$.*

Consequently, given a proper ideal $I \subsetneq \mathcal{A}(\mathbb{R})$, we have $Z_{\mathbb{R}}(f) \neq \emptyset$ for any $f \in I$. We recall that an ideal I is called *fixed* if $\bigcap_{f\in I} Z_{\mathbb{R}}(f) \neq \emptyset$. Otherwise I is called *free*. Suppose that $x_0 \in \bigcap_{f\in I} Z_{\mathbb{R}}(f)$ and I is a maximal ideal. Then I is principal ideal generated by the function $\mathrm{id}_{\mathbb{R}} - x_0$ and there is an isomorphism

$$\mathcal{A}(\mathbb{R})/I \xrightarrow{\cong} \mathbb{R}.$$

It follows that an element $f \in \mathcal{A}(\mathbb{R})$ is in a maximal ideal if and only if $Z_{\mathbb{R}}(f) \neq \emptyset$. Because $\mathcal{A}(\mathbb{R}) \subseteq \mathcal{C}(\mathbb{R})$, each maximal ideal of $\mathcal{A}(\mathbb{R})$ is contained in a maximal ideal of $\mathcal{C}(\mathbb{R})$. By a theorem of Gelfand and Kolmogorov, any maximal ideal of $\mathcal{C}(\mathbb{R})$ is determined by a point of the Stone–Čech compactification $\beta\mathbb{R}$ of the real line $\mathbb{R}$. More precisely, each maximal ideal is of the form

$$\mathfrak{m}^x = \{f \in \mathcal{C}(\mathbb{R}) : x \in \mathrm{cl}_{\beta\mathbb{R}}\, Z_{\mathbb{R}}(f)\}$$

for a unique $x \in \beta\mathbb{R}$ (see [5, 7.3]). Clearly $\mathfrak{m}^x$ is fixed if and only if $x \in \mathbb{R}$. Because the zeroset of a non-zero real-analytic function is closed and discrete, we need only consider a restricted subclass of subsets in $\beta\mathbb{R}$.

A point $x \in \beta\mathbb{R}\setminus\mathbb{R}$ in the $\beta\mathbb{R}$-closure of a closed discrete subspace of $\mathbb{R}$ is said to be *close to* $\mathbb{R}$. Otherwise, x is said to be *far from* $\mathbb{R}$. For an example of a far point, see [5, 4U]. Recall that a totally ordered set $(L, <)$ is called an *η_1-set* if whenever A and B are countable subsets of L such that $A < B$ (i.e., $a \in A$ and $b \in B$ imply $a < b$), then there is an $x \in L$ such that $A < x < B$, and a field such that any of its algebraic extensions is algebraically closed is said to be *real closed*. Such a field is totally ordered, its positive elements have square roots, and polynomial equations of odd degree have a root in that field (see [5, Chapter 13]).

We are now ready to describe the residue class fields of maximal ideals of $\mathcal{A}(\mathbb{R})$.

THEOREM 2.2. *If $\mathfrak{m}$ is a maximal ideal of $\mathcal{A}(\mathbb{R})$, then the residue class field $\mathcal{A}(\mathbb{R})/\mathfrak{m}$ has cardinality 2^ω and is*:

(1) *the field of reals if and only if $\mathfrak{m}$ is fixed*;

(2) *a real closed* (*totally ordered*) *η_1-field if and only if $\mathfrak{m}$ is free.*

Proof. (1) Obviously, if the maximal ideal $\mathfrak{m} \subseteq \mathcal{A}(\mathbb{R})$ is fixed then $\mathcal{A}(\mathbb{R})/\mathfrak{m}$ is the field of reals.

If $\mathcal{A}(\mathbb{R})/\mathfrak{m}$ is the field of reals then the ideal $\mathfrak{m}$ contains a polynomial function of positive degree. For otherwise, $\mathcal{A}(\mathbb{R})/\mathfrak{m}$ would contain a copy of the field $\mathbb{R}(X)$ of rational functions over $\mathbb{R}$, contrary to the fact that the only isomorphism of $\mathbb{R}$ is the identity. Because any real polynomial p of positive degree is a product of linear and irreducible quadratic factors, $\mathfrak{m}$ is a prime ideal and $Z_{\mathbb{R}}(p) \neq \emptyset$, the ideal $\mathfrak{m}$ must contain a linear factor. Hence, the maximal ideal $\mathfrak{m}$ is fixed.

(2) Clearly, the ideal $\mathfrak{m} \subseteq \mathcal{A}(\mathbb{R})$ is free provided $\mathcal{A}(\mathbb{R})/\mathfrak{m}$ is a real closed η_1-field.

If now $\mathfrak{m}$ is free, then $\mathfrak{m} \subseteq \mathfrak{m}^x$ for some x close to $\mathbb{R}$. Thus x is in the $\beta\mathbb{R}$-closure of some closed countable subset of $\mathbb{R}$. By Proposition 1.1, each coset of $\mathcal{C}(\mathbb{R})/\mathfrak{m}^x$ contains an element of $\mathcal{E}(\mathbb{R}) \subseteq \mathcal{A}(\mathbb{R})$. Consequently, $\mathcal{A}(\mathbb{R})/\mathfrak{m}$ and $\mathcal{C}(\mathbb{R})/\mathfrak{m}^x$ are isomorphic. It follows from [5, Chapter 13] that $\mathcal{C}(\mathbb{R})/\mathfrak{m}^x$ and hence $\mathcal{A}(\mathbb{R})/\mathfrak{m}$ is a real closed η_1-field, which cannot be the field of reals since the latter has a countable cofinal subset. That these fields have cardinality 2^ω is clear. ∎

The proof of the above also yields

Corollary 2.3. *If $\mathfrak{m}$ is a maximal ideal of $\mathcal{A}(\mathbb{R})$ that is contained in $\mathfrak{m}^x$ for some $x \in \beta\mathbb{R} \setminus \mathbb{R}$ that is close to $\mathbb{R}$, then $\mathcal{A}(\mathbb{R})/\mathfrak{m}$ is a real closed η_1-field.*

In [3], a real closed η_1-field is called an *H-field,* and it is shown that all H-fields of cardinality 2^ω are isomorphic if and only if the continuum hypothesis (CH) holds. Thus, if CH holds, then $\mathcal{A}(\mathbb{R})/\mathfrak{m}$ is isomorphic to one of only two possible fields. Note that every η_1-field is non-archimedean since it must contain elements larger than any integral multiple of the identity element.

We recall that given a set X, a family $\mathcal{F}$ of its subsets is called a *filter* on X if:

(1) $\emptyset \notin \mathcal{F}$;
(2) if $A, B \in \mathcal{F}$ then $A \cap B \in \mathcal{F}$;
(3) if $A \in \mathcal{F}$ and $A \subseteq B$ then $B \in \mathcal{F}$.

A collection of all subsetes of X containing a certain fixed non-empty subset $X_0 \subseteq X$ is a filter on X called a *principal* filter. A filter $\mathcal{F}$ on a set X is said to be an *ultrafilter* if it is maximal (with respect to inclusion) in the family of all filters on X.

Given a non-unit $f \in \mathcal{A}(\mathbb{R})$ and a filter $\mathcal{F}$ on $Z_{\mathbb{R}}(f)$, $I_{\mathcal{F}} = \{g \in \mathcal{E}(\mathbb{R}) : Z_{\mathbb{R}}(f) \cap Z_{\mathbb{R}}(g) \in \mathcal{F}\}$ is an ideal of $\mathcal{A}(\mathbb{R})$. On the other hand, any non-zero ideal $I \subseteq \mathcal{A}(\mathbb{R})$ yields a filter $\mathcal{F}_I = \{Z_{\mathbb{R}}(g) \cap Z_{\mathbb{R}}(f) : g \in I\}$ on the countable discrete subset $Z_{\mathbb{R}}(f)$ for a fixed non-zero function $f \in I$. Clearly, $\mathcal{F}_I$ is a non-principal ultrafilter if and only if I is a non-principal maximal ideal.

Now, for a maximal ideal $\mathfrak{m} \subseteq \mathcal{A}(\mathbb{R})$, fix an associated ultrafilter $\mathcal{F}$ as above. Then one can easily show that $f + \mathfrak{m} \geq 0$ if and only if $f + \mathfrak{m} = g + \mathfrak{m} \geq 0$ for some $g \in \mathcal{A}(\mathbb{R})$ which restricts to a non-negative function on some member of the filter $\mathcal{F}$. In particular, $f + \mathfrak{m}$ is a square for any non-negative function $f \in \mathcal{A}(\mathbb{R})$.

Recall (see, e.g., [1, Chapter 4]) that an ideal $I \subseteq \mathcal{R}$ of a commutative ring $\mathcal{R}$ is called *formally real* if $r_1^2 + \cdots + r_n^2 \in I$ implies $r_1, \ldots, r_n \in I$. Thus, Theorem 2.2 yields

COROLLARY 2.4. *Let* $\mathfrak{m} \subseteq \mathcal{A}(\mathbb{R})$ *be a maximal ideal. Then*:

(1) $\mathfrak{m}$ *is formally real*;
(2) *for any* $f \in \mathcal{A}(\mathbb{R})$, *there is* $g \in \mathcal{A}(\mathbb{R})$ *such that* $g^2 + f \in \mathfrak{m}$ *or* $g^2 - f \in \mathfrak{m}$.

A. Murillo has pointed out that [11, Corollary 1] says

REMARK 2.5. Any non-negative function $f \in \mathcal{A}(\mathbb{R})$ is a square. Thus, the Pythagoras number of $\mathcal{A}(\mathbb{R})$ is 1.

Let now S be the multiplicative system of $\mathcal{E}(\mathbb{R})$ determined by functions with empty real zerosets and $S^{-1}\mathcal{E}(\mathbb{R})$ the corresponding localization. Then, clearly, $S^{-1}\mathcal{E}(\mathbb{R}) \subseteq \mathcal{A}(\mathbb{R})$ and in view of Remark 2.5, this inclusion is proper because, e.g., $\sqrt{x^2+1} \in \mathcal{A}(\mathbb{R}) \setminus S^{-1}\mathcal{E}(\mathbb{R})$.

3. Properties of $\mathcal{E}(\mathbb{R})$. The first thorough study of the ideal structure of $\mathcal{E}(\mathbb{R})$ was made by O. Helmer in [6]. Indeed, he studied the ring of entire functions with everywhere convergent power series with coefficients in any subfield $\mathbb{K} \subseteq \mathbb{C}$. The most striking result in that paper is that $\mathcal{E}(\mathbb{K})$ is a Bézout domain. What is surprising is that for $\mathbb{K} = \mathbb{C}$ this was already proved in 1915 by J. M. H. Wedderburn [18]. Building on this latter result, M. Henriksen showed in [8] that the residue class ring of every maximal ideal $\mathfrak{m} \subseteq \mathcal{E}(\mathbb{C})$ is isomorphic to $\mathbb{C}$, even though the fact that $\mathcal{E}(\mathbb{C})$ contains all polynomials with complex coefficients shows that $\mathcal{E}(\mathbb{C})/\mathfrak{m}$ is infinite-dimensional as an algebra over $\mathbb{R}$. The proof relies on the theorem of E. Steinitz cited above [17].

A ring $\mathcal{R}$ is called an *elementary divisor ring* if whenever A is a matrix with entries from $\mathcal{R}$, there are invertible matrices P, Q of appropriate size such that PAQ is a diagonal matrix. A ring $\mathcal{R}$ is said to be *adequate* [7] if $\mathcal{R}$ is Bézout and for $a, b \in \mathcal{R}$ with $a \neq 0$, there exist $r, s \in \mathcal{R}$ such that $a = rs$, $(r, b) = (1)$, and if a non-unit s' divides s, then $(s', b) \neq (1)$. It is known that every elementary divisor ring is a Bézout ring and that a Bézout domain that is an adequate ring, is an elementary divisor ring as well. It is shown in [7, Theorem 4] that in an adequate domain every non-zero prime ideal is contained in a unique maximal ideal. For the proofs of other assertions in this paragraph, see [2, Theorems 3.18 and 3.19], and [14].

As observed in [7], in the light of [10], the ring $\mathcal{E}(\mathbb{K})$ is adequate. Those arguments, with a minor modification, show that $\mathcal{A}(\mathbb{R})$ is also an adequate ring. Hence, $\mathcal{A}(\mathbb{R})$ and $\mathcal{E}(\mathbb{K})$, being adequate Bézout domains, are elementary divisor rings as well. Actually, that $\mathcal{E}(\mathbb{C})$ is an elementary division ring was shown first in [18].

We say that an ideal I of a ring $\mathcal{R}$ is *formally complex* if $r^2+1 \in I$ for some $r \in \mathcal{R}$. If $\mathcal{R}$ contains an element j such that $j^2 = -1$, then $(j - jr)^2 + 1 \in I$ provided that $r \in I$. So every ideal of $\mathcal{R}$ is formally complex. In particular, any ideal of a subring $\mathcal{R}$ of $\mathcal{E}(\mathbb{C})$ that contains the constant function $\mathbf{i}$ is formally complex.

If $f(x) = \sum_{k=0}^{\infty} a_k x^k \in \mathcal{E}(\mathbb{R})$, let $\widetilde{f}(z) = \sum_{k=0}^{\infty} a_k z^k$ denote its extension to a function in $\mathcal{E}(\mathbb{C})$. The lemmas below show that any maximal ideal $\mathfrak{m} \subseteq \mathcal{E}(\mathbb{R})$ is either formally real or formally complex.

LEMMA 3.1. *Let* $\mathfrak{m} \subseteq \mathcal{E}(\mathbb{R})$ *be a maximal ideal. Then the following properties are equivalent*:

(1) $Z_{\mathbb{R}}(f) \neq \emptyset$ *for each* $f \in \mathfrak{m}$;
(2) *for any maximal ideal* $\mathfrak{m}' \subseteq \mathcal{A}(\mathbb{R})$ *containing* $\mathfrak{m}$, *the inclusion map* $\mathcal{E}(\mathbb{R})/\mathfrak{m} \hookrightarrow \mathcal{A}(\mathbb{R})/\mathfrak{m}'$ *of residue class fields is onto*;
(3) $\mathfrak{m}$ *is formally real.*

Proof. Clearly, (1)⇒(2)⇒(3).

To show (3)⇒(1), let $\mathfrak{m} \subseteq \mathcal{E}(\mathbb{R})$ be a formally real maximal ideal and $f \in \mathfrak{m}$ with $Z_{\mathbb{R}}(f) = \emptyset$. Then the zeroset $Z_{\mathbb{C}}(f)$ is symmetric (about the real axis) and $Z_{\mathbb{C}}(f) = Z_1 \cup Z_2$, where Z_1 is the portion of $Z_{\mathbb{C}}(f)$ above and Z_2 the portion below the real axis. By Proposition 1.1, there is a function $f_1 = \widetilde{f}'_1 + i\widetilde{f}''_1 \in \mathcal{E}(\mathbb{C})$ such that $Z_{\mathbb{C}}(f_1) = Z_1$ for some $f'_1, f''_1 \in \mathcal{E}(\mathbb{R})$. Then $Z_{\mathbb{C}}(f_2) = Z_2$ for $f_2 = \widetilde{f}'_1 - i\widetilde{f}''_1$ and $f = f'((f'_1)^2 + (f''_1)^2)$ with some invertible function $f' \in \mathcal{E}(\mathbb{R})$.

Given a maximal ideal $\mathfrak{m}' \subseteq \mathcal{E}(\mathbb{C})$ containing $\mathfrak{m}$, we get $f_1 \in \mathfrak{m}'$ or $f_2 \in \mathfrak{m}'$. Then $(f'_1)^2 + (f''_1)^2 \in \mathfrak{m}$ and hence $f'_1, f''_1 \in \mathfrak{m}$. Consequently, the non-empty symmetric set $Z_{\mathbb{C}}(f'_1) \cap Z_{\mathbb{C}}(f''_1)$ is contained in $Z_{\mathbb{C}}(f_1) = Z_1$. This leads to a contradiction and the proof is complete. ∎

Furthermore, we have

LEMMA 3.2. *Let* $\mathfrak{m}$ *be a maximal ideal of* $\mathcal{E}(\mathbb{R})$. *Then the following properties are equivalent*:

(1) *there is an isomorphism* $\mathcal{E}(\mathbb{R})/\mathfrak{m} \xrightarrow{\cong} \mathbb{C}$ *of fields*;
(2) $\mathfrak{m}$ *is formally complex*;
(3) *there is* $f \in \mathfrak{m}$ *with* $Z_{\mathbb{R}}(f) = \emptyset$.

Proof. Clearly, (1)⇒(2)⇒(3).

To show (2)⇒(1), take any maximal ideal $\mathfrak{m}' \subseteq \mathcal{E}(\mathbb{C})$ containing $\mathfrak{m}$. Then (2) implies that the inclusion map $\mathcal{E}(\mathbb{R})/\mathfrak{m} \hookrightarrow \mathcal{E}(\mathbb{C})/\mathfrak{m}'$ of residue class fields is onto. Hence, the main result of [8] (see Theorem 1.2) yields the required implication.

Finally, we show (3)⇒(2). Given $f \in \mathfrak{m}$ with $Z_{\mathbb{R}}(f) = \emptyset$, take any maximal ideal $\mathfrak{m}' \subseteq \mathcal{E}(\mathbb{C})$ containing $\mathfrak{m}$. From the proof of Lemma 3.1, we easily derive that there are functions $f_1', f_1'' \in \mathcal{E}(\mathbb{R})$ such that $\widetilde{f}_1' + i\widetilde{f}_1'' \in \mathfrak{m}'$ or $\widetilde{f}_1' - i\widetilde{f}_1'' \in \mathfrak{m}'$. If $f_1', f_1'' \in \mathfrak{m}$ then the set $Z_{\mathbb{C}}(f_1') \cap Z_{\mathbb{C}}(f_1'') \neq \emptyset$ is symmetric and the inclusion $Z_{\mathbb{C}}(\widetilde{f}_1') \cap Z_{\mathbb{C}}(\widetilde{f}_1'') \subseteq Z_1$ leads to a contradiction. Hence, $f_1' \notin \mathfrak{m}$ or $f_1'' \notin \mathfrak{m}$ and so f_1' or f_1'' is invertible (mod $\mathfrak{m}$). Thus, $1 + i\widetilde{g}_1 \in \mathfrak{m}'$ or $\widetilde{g}_2 - i \in \mathfrak{m}'$ for some $g_1, g_2 \in \mathcal{E}(\mathbb{R})$. Consequently, $g_1^2 + 1 \in \mathfrak{m}$ or $g_2^2 + 1 \in \mathfrak{m}$, and this completes the proof. ∎

In the light of Theorem 2.2, and Lemmas 3.1 and 3.2, we summarize what we know about residue classes of maximal ideals of $\mathcal{E}(\mathbb{R})$ in:

THEOREM 3.3. *Let* $\mathfrak{m} \subseteq \mathcal{E}(\mathbb{R})$ *be a maximal ideal. Then*:

(1) *there is an isomorphism* $\mathcal{E}(\mathbb{R})/\mathfrak{m} \xrightarrow{\cong} \mathbb{C}$ *if and only if* $\mathfrak{m}$ *is formally complex*;
(2) *there is an isomorphism* $\mathcal{E}(\mathbb{R})/\mathfrak{m} \xrightarrow{\cong} \mathbb{R}$ *if and only if* $\mathfrak{m}$ *is fixed*;
(3) *the residue class field* $\mathcal{E}(\mathbb{R})/\mathfrak{m}$ *is real closed otherwise. Furthermore,* $\mathcal{E}(\mathbb{R})/\mathfrak{m}$ *is an* η_1*-field provided* $\mathfrak{m}$ *is a formally real maximal free ideal*, *and any two such fields are isomorphic if and only if the continuum hypothesis holds.*

We note that Theorem 3.3 leads to the following conclusion on real entire functions.

COROLLARY 3.4. *For a maximal ideal* $\mathfrak{m} \subseteq \mathcal{E}(\mathbb{R})$ *the following hold*:

(1) *if* $\mathfrak{m}$ *is formally real then for any function* $f \in \mathcal{E}(\mathbb{R})$ *there is* $g \in \mathcal{E}(\mathbb{R})$ *such that* $g^2 + f \in \mathfrak{m}$ *or* $g^2 - f \in \mathfrak{m}$;
(2) *if* $\mathfrak{m}$ *is formally complex then for any function* $f \in \mathcal{E}(\mathbb{R})$ *there is* $g \in \mathcal{E}(\mathbb{R})$ *such that* $g^2 - f \in \mathfrak{m}$.

Furthermore, formally real maximal ideals of $\mathcal{E}(\mathbb{R})$ might be characterized as follows.

Given a non-unit $f \in \mathcal{E}(\mathbb{R})$ and a filter $\mathcal{F}$ on $Z_{\mathbb{C}}(f)$, the set $I = \{g \in \mathcal{E}(\mathbb{R}) : Z_{\mathbb{C}}(f) \cap Z_{\mathbb{C}}(g) \in \mathcal{F}\}$ is an ideal of $\mathcal{E}(\mathbb{R})$. On the other hand, any non-zero ideal $I \subseteq \mathcal{E}(\mathbb{R})$ yields a filter $\mathcal{F} = \{Z_{\mathbb{C}}(g) \cap Z_{\mathbb{C}}(f) : g \in I\}$ on the countable discrete subset $Z_{\mathbb{C}}(f)$ for a fixed non-zero function $f \in I$. Clearly, $\mathcal{F}$ is an ultrafilter if and only if the ideal I is maximal.

For such a formally real maximal ideal $\mathfrak{m} \subseteq \mathcal{A}(\mathbb{R})$, fix an associated ultrafilter $\mathcal{F}$ as above. Then one can easily show that $f + \mathfrak{m} \geq 0$ if and only if $f + \mathfrak{m} = g + \mathfrak{m} \geq 0$ for some $g \in \mathcal{E}(\mathbb{R})$ which restricts to a non-negative function on some member of the filter $\mathcal{F}$. In particular, for any non-negative function $f \in \mathcal{E}(\mathbb{R})$ there is $g \in \mathcal{E}(\mathbb{R})$ such that $g^2 - f \in \mathfrak{m}$.

Let now $f \in \mathcal{E}(\mathbb{R})$ be a non-negative function. Because real zeros of f have even multiplicity, we get $f = gh^2$ for some $g, h \in \mathcal{E}(\mathbb{R})$ with $Z_{\mathbb{R}}(g) = \emptyset$, where h is an entire function such that $Z_{\mathbb{C}}(h) \subseteq \mathbb{R}$. Next, take the decomposition $g = g'((g_1')^2 + (g_2'')^2)$ presented in the proof of Lemma 3.1 for some $g', g_1', g_1'' \in \mathcal{E}(\mathbb{R})$ with g' invertible. Consequently, we recover the following result proved in [15].

COROLLARY 3.5. *Any non-negative function $f \in \mathcal{E}(\mathbb{R})$ is a sum of two squares,*

$$f = (f')^2 + (f'')^2$$

for some $f', f'' \in \mathcal{E}(\mathbb{R})$.

Clearly, not every non-negative real entire function is a square. Hence, the Pythagoras number of $\mathcal{E}(\mathbb{R})$ is 2.

Furthermore, because the zeroset $Z_{\mathbb{C}}(f)$ of any non-negative real polynomial function f of one variable is finite, for f' and f'' might be taken real polynomial functions of one variable as well. Hence, the polynomial f is a sum of squares of two such polynomials.

At the end of this section, given a maximal ideal $\mathfrak{m} \subseteq \mathcal{C}(\mathbb{R})$, consider the inclusion maps

$$\mathcal{E}(\mathbb{R})/(\mathfrak{m} \cap \mathcal{E}(\mathbb{R})) \hookrightarrow \mathcal{A}(\mathbb{R})/(\mathfrak{m} \cap \mathcal{A}(\mathbb{R})) \hookrightarrow \mathcal{C}(\mathbb{R})/\mathfrak{m}.$$

If $\mathfrak{m} \cap \mathcal{E}(\mathbb{R}) \neq (0)$ then by the methods presented in Section 2, those maps are onto. On the other hand, by [5, 4F, p. 61] there is a z-ultrafilter $\mathcal{F}$ on $\mathbb{R}$ containing only sets of infinite measure. Let $\mathfrak{m}_{\mathcal{F}} \subseteq \mathcal{C}(\mathbb{R})$ be the corresponding maximal ideal. Because the zeroset of any function in $\mathcal{E}(\mathbb{R})$ is discrete, we have $\mathfrak{m}_{\mathcal{F}} \cap \mathcal{E}(\mathbb{R}) = (0)$. Hence, the canonical map $\mathcal{E}(\mathbb{R}) \to \mathcal{C}(\mathbb{R})/\mathfrak{m}_{\mathcal{F}}$ is a monomorphism and implies the inclusion maps

$$\mathcal{E}(\mathbb{R})_{(0)} \hookrightarrow \mathcal{A}(\mathbb{R})_{(0)} \hookrightarrow \mathcal{C}(\mathbb{R})/\mathfrak{m}_{\mathcal{F}},$$

where $\mathcal{A}(\mathbb{R})_{(0)}$ and $\mathcal{E}(\mathbb{R})_{(0)}$ denote the quotient fields of $\mathcal{A}(\mathbb{R})$ and $\mathcal{E}(\mathbb{R})$, respectively.

If now $\mathcal{M}(\mathbb{C}) = \mathcal{E}(\mathbb{C})_{(0)}$ denote the field of meromorphic functions on the complex plane $\mathbb{C}$ then the isomorphism $\mathcal{E}(\mathbb{R})(j) \xrightarrow{\cong} \mathcal{E}(\mathbb{C})$ yields $(\mathcal{E}(\mathbb{R})_{(0)})(j) \xrightarrow{\cong} \mathcal{M}(\mathbb{C})$, and consequently, we derive an inclusion map

$$\mathcal{M}(\mathbb{C}) \hookrightarrow (\mathcal{C}(\mathbb{R})/\mathfrak{m}_{\mathcal{F}})(j),$$

where $j^2 = -1$. By [5], the field $\mathcal{C}(\mathbb{R})/\mathfrak{m}_{\mathcal{F}}$ is real closed and the Steinitz Theorem [17] leads to an isomorphism $(\mathcal{C}(\mathbb{R})/\mathfrak{m}_{\mathcal{F}})(j) \xrightarrow{\cong} \mathbb{C}$. Hence, the field $\mathcal{C}(\mathbb{R})/\mathfrak{m}_{\mathcal{F}}$ coincides with the real closure of the formally real fields $\mathcal{A}(\mathbb{R})_{(0)}$ and $\mathcal{E}(\mathbb{R})_{(0)}$. Furthermore, in view of the Steinitz Theorem, the algebraic closures of the fields $\mathcal{A}(\mathbb{R})_{(0)}$, $\mathcal{E}(\mathbb{R})_{(0)}$ and $\mathcal{M}(\mathbb{C})$ coincide, being isomorphic to the field $\mathbb{C}$. The real-analytic function $\exp(1/(x^2+1))$ is surely not algebraic over the field $\mathcal{E}(\mathbb{R})_{(0)}$ so the inclusion $\mathcal{E}(\mathbb{R})_{(0)} \subseteq \mathcal{A}(\mathbb{R})_{(0)}$ is not an algebraic extension. Consequently, no isomorphism of the algebraic closures of $\mathcal{E}(\mathbb{R})_{(0)}$ and $\mathcal{A}(\mathbb{R})_{(0)}$ is over $\mathcal{E}(\mathbb{R})_{(0)}$.

Note that by Corollary 3.5 any element $f/g \in \mathcal{E}(\mathbb{R})_{(0)}$ is a sum of two squares provided that $f, g \in \mathcal{E}(\mathbb{R})$ are simultaneously positive or negative functions.

4. Krull dimension of $\mathcal{A}(\mathbb{R})$ and $\mathcal{E}(\mathbb{K})$. Recall that the Krull dimension K-dim $\mathcal{R}$ of a commutative ring $\mathcal{R}$ is the supremum of the lengths of chains of (proper) prime ideals.

In [16] Schilling claimed to have shown that K-dim $\mathcal{E}(\mathbb{C}) = 1$, but in 1952 Kaplansky showed that it is at least 2; then Henriksen proved [9] that it is at least 2^{ω_1} and also discussed the nature of the residue class rings $\mathcal{E}(\mathbb{C})/\mathfrak{p}$, where $\mathfrak{p}$ is a prime ideal of $\mathcal{E}(\mathbb{C})$.

For f in $\mathcal{A}(\mathbb{R})$ or $\mathcal{E}(\mathbb{R})$ that is neither 0 nor a unit, let $m(f)$ denote the maximum multiplicity of a zero of f, if it is finite, and let $m(f) = \infty$ otherwise. By Proposition 1.1, it is clear that any maximal ideal of these rings must contain an element f such that $m(f) = 1$. Though much of what follows could be found in [9], we have decided to present that once again below. It will be placed in a more general context, in which it is evident that it applies to $\mathcal{A}(\mathbb{R})$ or $\mathcal{E}(\mathbb{R})$ as well as to $\mathcal{E}(\mathbb{C})$.

Suppose $\mathcal{R}$ is an adequate domain that satisfies:

(1) if $\mathfrak{m}$ is a maximal ideal of $\mathcal{R}$ then its powers $\mathfrak{m}^n$ for $n = 1, 2, \ldots$ are distinct;
(2) if a non-maximal prime ideal $\mathfrak{p} \subseteq \mathcal{R}$ is contained in a maximal ideal $\mathfrak{m}$, then $\mathfrak{p} \subseteq \mathfrak{p}^\star = \bigcap_{n=1}^\infty \mathfrak{m}^n$.

We call such a ring *nearly analytic*.

Remark 4.1. (1) It follows easily that $\mathfrak{p}^\star$ is a prime ideal and hence it is the largest non-maximal prime ideal contained in the maximal ideal $\mathfrak{m}$.

(2) Using the fact that $\mathcal{A}(\mathbb{R})$ and $\mathcal{E}(\mathbb{R})$ are adequate domains, and Proposition 1.1, it is an exercise to show that each of these rings is nearly analytic. Note also that $\mathfrak{p}^\star = (0)$ if and only if the unique maximal ideal containing $\mathfrak{p}$ is fixed.

Recall that a commutative ring $\mathcal{R}$ such that for any non-zero $s, t \in \mathcal{R}$ either $s \mid t$ or $t \mid s$ is called a *valuation* ring.

Proposition 4.2. *If $\mathfrak{p}$ is a prime ideal of a nearly analytic ring $\mathcal{R}$ then $\mathcal{R}/\mathfrak{p}$ is a valuation ring.*

Proof. Because $\mathcal{R}$ is nearly analytic, we know that $\mathcal{R}/\mathfrak{p}$ is a Bézout domain with a unique maximal ideal $\mathfrak{m}/\mathfrak{p}$. By [13, Theorem 63], this shows that $\mathcal{R}/\mathfrak{p}$ is a valuation ring. ■

THEOREM 4.3. *If $\mathfrak{p}$ is a prime ideal of a nearly analytic ring $\mathcal{R}$, then $\mathcal{R}/\mathfrak{p}$ is Noetherian if and only if $\mathfrak{p} = \mathfrak{p}^\star$.*

Proof. Because $\mathcal{R}/\mathfrak{p}^\star$ is a valuation domain, by Proposition 4.3 for any $f \in \mathfrak{m} \setminus \mathfrak{p}^\star$ the coset $f + \mathfrak{p}^\star$ generates the only prime ideal $\mathfrak{m}/\mathfrak{p}^\star$ of $\mathcal{R}/\mathfrak{p}^\star$. Thus, $\mathcal{R}/\mathfrak{p}^\star$ is a Noetherian ring.

Conversely, if $\mathfrak{p}^\star$ contains $\mathfrak{p}$ properly then consider the ideal $\langle f^k + \mathfrak{p} : k = 1, 2, \ldots \rangle$ of $\mathcal{R}/\mathfrak{p}$ for $f \in \mathfrak{m} \setminus \mathfrak{m}^2$. This cannot be finitely generated by property (1) of any nearly analytic ring. ■

By the above, the arguments given in the proof of [8, Theorem 8] and the results of [12], we can summarize properties of the rings $\mathcal{A}(\mathbb{R})$ and $\mathcal{E}(\mathbb{R})$, denoted by $\mathcal{R}$, as follows.

THEOREM 4.4. (1) *There exists a non-maximal, prime ideal $\mathfrak{p} \subseteq \mathcal{R}$;*

(2) *a necessary and sufficient condition for a prime ideal $\mathfrak{p} \subseteq \mathcal{R}$ to be non-maximal is that $m(f) = \infty$ for every $f \in \mathfrak{p}$;*
(3) *$\mathcal{R}/\mathfrak{p}$ is a valuation ring and its unique maximal ideal is principal for any non-zero prime ideal $\mathfrak{p} \subseteq \mathcal{R}$;*
(4) *the localization $\mathcal{R}_\mathfrak{p}$ is a valuation ring for any prime ideal $\mathfrak{p} \subseteq \mathcal{R}$;*
(5) *$\mathcal{R}$ is an elementary divisor domain in which every non-zero prime ideal is contained in a unique maximal ideal;*
(6) *for any maximal ideal $\mathfrak{m}$ such that $\mathfrak{p}^\star \neq (0)$, the ring $\mathcal{R}/\mathfrak{p}^\star$ is isomorphic to the ring $(\mathcal{R}/\mathfrak{m})[[X]]$ of formal power series over the field $\mathcal{R}/\mathfrak{m}$.*

By [1, Chapter 4], a prime ideal $\mathfrak{p} \subseteq \mathcal{E}(\mathbb{R})$ is formally real if and only if the quotient field of the residue class ring $\mathcal{E}(\mathbb{R})/\mathfrak{p}$ is formally real. Furthermore, given a prime non-maximal ideal $\mathfrak{p} \subseteq \mathcal{E}(\mathbb{R})$, in view of the integral extension $\mathcal{E}(\mathbb{R}) \subseteq \mathcal{E}(\mathbb{C})$, there is a non-maximal prime ideal $\mathfrak{p}' \subseteq \mathcal{E}(\mathbb{C})$ with $\mathfrak{p} = \mathcal{E}(\mathbb{R}) \cap \mathfrak{p}'$.

We show that the proofs of Lemmas 3.1 and 3.2, and Proposition 4.2, lead *mutatis mutandis* to a characterization of formally real and formally complex prime ideals of $\mathcal{E}(\mathbb{R})$. In particular, it follows that also any prime ideal of $\mathcal{E}(\mathbb{R})$ is either formally real or formally complex. As usual, if $\mathcal{R}$ is a ring, $\mathcal{R}[X]$ denotes the ring of polynomials with coefficients in $\mathcal{R}$. First, we state

PROPOSITION 4.5. *Let $\mathfrak{p}$ be a prime ideal of $\mathcal{E}(\mathbb{R})$. Then the following properties are equivalent:*

(1) *$\mathfrak{p}$ is formally complex;*

(2) *for any prime ideal* $\mathfrak{p}' \subseteq \mathcal{E}(\mathbb{C})$ *with* $\mathfrak{p} = \mathcal{E}(\mathbb{R}) \cap \mathfrak{p}'$, *the inclusion map* $\mathcal{E}(\mathbb{R})/\mathfrak{p} \hookrightarrow \mathcal{E}(\mathbb{C})/\mathfrak{p}'$ *of residue class rings is onto*;

(3) *there is* $f \in \mathfrak{p}$ *with* $Z_{\mathbb{R}}(f) = \emptyset$.

Proof. Clearly, the implications (1)⇒(2)⇒(3) are obvious.

(3)⇒(1): Let now $Z_{\mathbb{R}}(f) = \emptyset$. Then, by the proof of Lemma 3.1 there are $f_1', f_1'' \in \mathcal{E}(\mathbb{R})$ such that $(f_1')^2 + (f_1'')^2 \in \mathfrak{p}$ and $f_1', f_1'' \notin \mathfrak{p}$. In view of Proposition 4.2, $\mathcal{E}(\mathbb{R})$ is a valuation ring. Hence, $f_1' - gf_1'' \in \mathfrak{p}$ or $f_1'' - hf_1' \in \mathfrak{p}$ for some $g, h \in \mathcal{E}(\mathbb{R})$. Thus, $g^2 + 1 \in \mathfrak{p}$ or $h^2 + 1 \in \mathfrak{p}$ and the proof is complete. ∎

To characterize formally real prime ideals of $\mathcal{E}(\mathbb{R})$, we proceed by the following construction.

If $p(X) \in (\mathcal{E}(\mathbb{R})/\mathfrak{p})[X]$ then there are $f, g \in \mathcal{E}(\mathbb{R})$ and $q(X) \in (\mathcal{E}(\mathbb{R})/\mathfrak{p})[X]$ such that $p(X) = q(X)(X^2 + 1) + (f + \mathfrak{p}) + (g + \mathfrak{p})X$. The map

$$(\mathcal{E}(\mathbb{R})/\mathfrak{p})[X] \to \mathcal{E}(\mathbb{C})/\mathfrak{p}'$$

that sends $p(X)$ to $(\widetilde{f} + \widetilde{g}i) + \mathfrak{p}'$ may be regarded as a homomorphism onto $\mathcal{E}(\mathbb{C})/\mathfrak{p}'$ whose kernel contains the principal ideal $(X^2 + 1)$ of $(\mathcal{E}(\mathbb{R})/\mathfrak{p})[X]$. This yields a map

$$\eta : (\mathcal{E}(\mathbb{R})/\mathfrak{p})[X]/(X^2 + 1) \to \mathcal{E}(\mathbb{C})/\mathfrak{p}'.$$

PROPOSITION 4.6. *Let* $\mathfrak{p} \subseteq \mathcal{E}(\mathbb{R})$ *be a prime ideal. Then the following properties are equivalent*:

(1) $\mathfrak{p}$ *is formally real*;

(2) $Z_{\mathbb{R}}(f) \neq \emptyset$ *for each* $f \in \mathfrak{p}$;

(3) *for any prime ideal* $\mathfrak{p}' \subseteq \mathcal{E}(\mathbb{C})$ *with* $\mathfrak{p} = \mathcal{E}(\mathbb{R}) \cap \mathfrak{p}'$, *the map*

$$\eta : (\mathcal{E}(\mathbb{R})/\mathfrak{p})[X]/(X^2 + 1) \to \mathcal{E}(\mathbb{C})/\mathfrak{p}'$$

is a ring isomorphism.

Furthermore, any prime ideal of $\mathcal{A}(\mathbb{R})$ *is formally real.*

Proof. First, we show that (1)⇔(2) and (2)⇔(3).

(1)⇒(2): Suppose that $Z_{\mathbb{R}}(f) = \emptyset$ for some $f \in \mathfrak{p}$. Then, by Proposition 4.5, there is $g \in \mathcal{E}(\mathbb{R})$ with $g^2 + 1 \in \mathfrak{p}$, contrary to (1).

(2)⇒(1): Let $f_1^2 + \cdots + f_n^2 \in \mathfrak{p}$ for some $f_1, \ldots, f_n \in \mathcal{E}(\mathbb{R})$ and write g for the greatest common divisor of $f_1, \ldots, f_n$. Then $f_1 = gf_1', \ldots, f_n = gf_n'$ for some $f_1', \ldots, f_n' \in \mathcal{E}(\mathbb{R})$. Hence, $g \in \mathfrak{p}$ or $(f_1')^2 + \cdots + (f_n')^2 \in \mathfrak{p}$. Because $Z_{\mathbb{R}}((f_1')^2 + \cdots + (f_n')^2) = \emptyset$, we get $g \in \mathfrak{p}$ and consequently $f_1, \ldots, f_n \in \mathfrak{p}$.

(2)⇒(3): If $\widetilde{f} + \widetilde{g}i \in \mathfrak{p}'$ for some $f, g \in \mathcal{E}(\mathbb{R})$ then $f^2 + g^2 \in \mathfrak{p}$. Then as above the greatest common divisor of f and g is in $\mathfrak{p}$ and so also $f, g \in \mathfrak{p}$. Hence, the induced map $(\mathcal{E}(\mathbb{R})/\mathfrak{p})[X]/(X^2 + 1) \to \mathcal{E}(\mathbb{C})/\mathfrak{p}'$ is a ring isomorphism.

(3)⇒(2): Suppose that $Z_{\mathbb{R}}(f) = \emptyset$ for some $f \in \mathfrak{p}$. Then, by Proposition 4.5, $g^2 + 1 \in \mathfrak{p}$ for some $g \in \mathcal{E}(\mathbb{R})$. Consequently, $\widetilde{g} + i \in \mathfrak{p}'$ or $\widetilde{g} - i \in \mathfrak{p}'$ and hence $g + X + (X^2 + 1)$ or $g - X + (X^2 + 1)$ is in the kernel of the map $\eta : (\mathcal{E}(\mathbb{R})/\mathfrak{p})[X]/(X^2+1) \to \mathcal{E}(\mathbb{C})/\mathfrak{p}'$. This contradiction completes the proof of the last implication.

Let now $\mathfrak{p} \subseteq \mathcal{A}(\mathbb{R})$ be a prime ideal and $f_1^2 + \cdots + f_n^2 \in \mathfrak{p}$ for some $f_1, \ldots, f_n \in \mathcal{A}(\mathbb{R})$. Because the ring $\mathcal{A}(\mathbb{R})$ is a Bézout domain [2, Theorem 1.19], the proof of (2)⇒(1) above also shows that $f_1, \ldots, f_n \in \mathfrak{p}$. ∎

Methods of [9] and [12] applied to the ring $\mathcal{R}$ lead to the following generalization of [9, Theorem 5].

THEOREM 4.7. *If* $\mathfrak{p}_1 \subsetneq \mathfrak{p}_2 \subsetneq \mathfrak{p}_3$ *are prime ideals of* $\mathcal{R}$ *then there exists a chain of* 2^{ω_1} *prime ideals between* $\mathfrak{p}_1$ *and* $\mathfrak{p}_3$. *In particular*, *the Krull dimension* $\text{K-dim}\,\mathcal{R} \geq 2^{\omega_1}$.

REFERENCES

[1] J. Bochnak, M. Coste and M.-F. Roy, *Real Algebraic Geometry*, Ergeb. Math. Grenzgeb. 36, Springer, Berlin, 1998.

[2] J. M. Brewer, J. W. Bunce and F. S. van Vleck, *Linear Systems over Commutative Rings*, Lecture Notes in Pure and Appl. Math. 104, Dekker, New York, 1986.

[3] A. Dow, *On ultrapowers of Boolean algebras*, Topology Proc. 9 (1984), 269–291.

[4] R. H. J. Germay, *Extension d'un théorème de E. Picard relatif aux produits indéfinis de facteurs primaires*, Bull. Roy. Sci. Liège 17 (1948), 138–143.

[5] L. Gillman and M. Jerison, *Rings of Continuous Functions*, D. van Nostrand, Princeton, NJ, 1960.

[6] O. Helmer, *Divisibility properties of integral functions*, Duke Math. J. 6 (1940), 345–356.

[7] —, *The elementary divisor theorem for certain rings without chain conditions*, Bull. Amer. Math. Soc. 49 (1943), 225–236.

[8] M. Henriksen, *On the ideal structure of the ring of entire functions*, Pacific J. Math. 2 (1952), 179–184.

[9] —, *On the prime ideals of the ring of entire functions*, ibid. 3 (1953), 711–720.

[10] —, *Some remarks on elementary divisor rings*, *II*, Michigan Math. J. 3 (1955/56), 159–163.

[11] P. Jaworski, *Positive definite analytic functions and vector bundles*, Bull. Acad. Polon. Sci. Sér. Sci. Math. 30 (1982), 501–506.

[12] C. U. Jensen, *Some curiosities of rings of analytic functions*, J. Pure Appl. Algebra 38 (1985), 277–283.

[13] I. Kaplansky, *Commutative Rings*, Univ. of Chicago Press, 1974.

[14] M. Larsen, W. Lewis and T. Shores, *Elementary divisor rings and finitely generated modules*, Trans. Amer. Math. Soc. 187 (1974), 231–247.

[15] L. A. Rubel, *Sums of squares of real entire functions of one complex variable*, in: Proceedings of the Conference on Complex Analysis (Tianjin, 1992), Conf. Proc. Lecture Notes Anal. I, Internat. Press, Cambridge, MA, 1994, 180–187.

[16] O. F. G. Schilling, *Ideal theory on open Riemann surfaces*, Bull. Amer. Math. Soc. 52 (1946), 945–963.

[17] E. Steinitz, *Algebraische Theorie der Körper*, de Gruyter, Berlin, 1930.

[18] J. M. H. Wedderburn, *On matrices whose coefficients are functions of a single variable*, Trans. Amer. Math. Soc. 16 (1915), 328–332.

Faculty of Mathematics and Computer Science
Nicolaus Copernicus University
Chopina 12/18
87-100 Toruń, Poland
E-mail: marek@mat.uni.torun.pl

Department of Mathematics
Harvey Mudd College
Clarement, CA 91711, U.S.A.
E-mail: henriksen@hmc.edu

Received 26 January 2005;
revised 10 June 2005 (4561)