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ON THE SPECTRAL MULTIPLICITY OF A DIRECT SUM OF OPERATORS

BY

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Abstract. We calculate the spectral multiplicity of the direct sum $T\oplus A$ of a weighted shift operator T on a Banach space Y which is continuously embedded in l^p and a suitable bounded linear operator A on a Banach space X.

1. Introduction. Let $L(X)$ be the Banach algebra of all bounded linear operators on a Banach space X. A subspace $E\subset X$ is called a *cyclic subspace* of an operator $A\in L(X)$ if $\operatorname{span}\{A^nE : n\ge 0\}=X$, where span denotes closed linear hull. A vector $x\in X$ is called *cyclic* ($x\in \operatorname{Cyc}(A)$) if $\operatorname{span}\{A^nx : n\ge 0\}=X$. The *spectral multiplicity* $\mu(A)$ of $A\in L(X)$ is

$$\mu(A):=\inf\{\dim E : \operatorname{span}\{A^nE : n\ge 0\}=X\},$$

a nonnegative integer or ∞. Clearly, A is cyclic if and only if $\mu(A)=1$.

The spectral multiplicity is an important invariant of operators, and it plays a key role in operator theory and its applications. Clearly, the notion of cyclic subspace is important in connection with the general problem of existence of a nontrivial invariant subspace, because an operator $A\in L(X)$ has no nontrivial invariant subspace if and only if $x\in\operatorname{Cyc}(A)$ for every $x\in X\setminus\{0\}$. Cyclic vectors are important in weighted polynomial approximation theory. (More details can be found in [6].)

In this article we calculate the spectral multiplicity of the direct sum $T\oplus A$, where T is a weighted shift operator on a Banach space Y continuously embedded in l^p, and A a suitable bounded operator on a Banach space X (Section 2). Note that the main result of Section 2, Theorem 1, generalizes and strengthens some results of the author in [3, Theorem], [4, Theorem 3] and [5, Theorem 3].

First we introduce some notations and definitions. If $\{e_i\}_{i\ge0}$ is a sequence of vectors in a Banach space X, we say that $\{e_i\}_{i\ge0}$ is *uniformly minimal* if there exists a constant $\delta>0$ such that

$$d:=\inf_{i\ge0}\operatorname{dist}\{e_i/\|e_i\|,\ \operatorname{span}\{e_j : j\ne i\}\}>0.$$

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It is clear that if $\{e_i\}_{i\geq 0}$ is a basis in X, then $\{e_i\}_{i\geq 0}$ is uniformly minimal, and therefore for any $x = \sum_{n=0}^{\infty} x(n)e_n \in X$,

$$\|x(n)e_n\|_X \leq c\|x\|_X \tag{1}$$

for all $n \geq 0$ and some $c > 0$ (actually, one can take $c = 1/d$).

A *weighted shift operator* T in a Banach space X with basis $\{e_n\}_{n\geq 0}$ is defined by

$$Te_n = \lambda_n e_{n+1}, \qquad n \geq 0,$$

where $\{\lambda_n\}_{n\geq 0}$ is a bounded sequence of complex numbers. Obviously, $\mu(T) = 1$.

2. The spectral multiplicity of $T \oplus A$. Let $A \oplus B$ denote the direct sum of bounded linear operators A and B acting in Banach spaces X and Y, respectively,

$$(A \oplus B)(x \oplus y) = Ax \oplus By, \qquad x \oplus y \in X \oplus Y.$$

It is well known that

$$\max\{\mu(A), \mu(B)\} \leq \mu(A \oplus B) \leq \mu(A) + \mu(B). \tag{2}$$

One might try to characterize the extremes, that is: When $\mu(A \oplus B) = \max\{\mu(A), \mu(B)\}$? When $\mu(A \oplus B) = \mu(A) + \mu(B)$? It is well known that $\mu(A \oplus B) = \max\{\mu(A), \mu(B)\}$ if the spectra $\sigma(A)$ and $\sigma(B)$ are well separated, i.e., their polynomially convex hulls are disjoint (see, e.g., [7]).

In this section we shall be interested in the equality $\mu(A \oplus B) = \mu(A) + \mu(B)$. In this connection we shall prove the following theorem which generalizes and strengthens some results in [3–5]. (More general results related to the equality $\mu(A \oplus B) = \mu(A) + \mu(B)$ can be found, for instance, in [2, 7–10]).

THEOREM 1. *Let Y be a Banach space with a basis $\{e_n\}_{n\geq 0}$ of unit vectors, which is continuously embedded in l^p for some p, $1 \leq p \leq \infty$. Let $\{\lambda_n\}_{n\geq 0}$ denote a bounded sequence of nonzero complex numbers $\lambda_n \in \mathbb{C}$, and let T be the corresponding weighted shift operator acting in Y, $Te_n = \lambda_n e_{n+1}$, $n \geq 0$. Let X be a separable Banach space and $A \in L(X)$. Suppose that:*

- (i) $\sum_{n,m\geq N} |w_{n+m}/w_n w_m| =: \Omega_N < \infty$ *for some* $N \geq 0$, *where* $w_n := \lambda_0\lambda_1 \cdots \lambda_{n-1}$, $w_0 := 1$.
- (ii) $\sum_{n=0}^{\infty} (\|A^n x\|_X / \|T^n e_0\|_Y)^q =: C_x < \infty$ *for all* $x \in X$, *where* $1/p + 1/q = 1$.
- (iii) $\|e_{n+m}\|_Y \leq c\|e_n\|_Y \|e_m\|_Y$ *for all* $n, m \geq 0$ *and for some* $c \geq 0$.

Then $\mu(T \oplus A) = \mu(T) + \mu(A) = 1 + \mu(A)$.

The proof of the theorem uses the following product:

$$f \,\widetilde{\circledast}\, g := \sum_{n,m\geq 0} \frac{w_{n+m}}{w_n w_m} f(n)g(m)e_{n+m}, \tag{3}$$

where $f = \sum_{n=0}^{\infty} f(n)e_n$, $g = \sum_{n=0}^{\infty} g(n)e_n \in Y$.

LEMMA 2. *$(Y, \widetilde{\circledast})$ is a Banach algebra with the unit $f = e_0$ and $\mathcal{M}(Y, \widetilde{\circledast}) = \{0\}$, i.e., its maximal ideal space $\mathcal{M}(Y, \widetilde{\circledast})$ consists of one homomorphism $f \mapsto f(0)$.*

Proof. First we prove that

$$\|f \,\widetilde{\circledast}\, g\| \leq C\|f\|\|g\| \tag{4}$$

for all $f, g \in Y$ and for some number $C > 0$, not depending on f, g.

Note that if $N = 0$, then (4) is immediate from (3), and so assume that $N \geq 1$. By setting $R_n(f) := \sum_{k\geq n} f(k)e_k$, and using (i) and the inequalities

$$\|R_n(T^k f)\| \leq \|T^k f\|, \qquad k = 0, 1, \ldots, N-1,$$
$$\|T^k\| \leq \sup\{|\lambda_n \lambda_{n+1} \cdots \lambda_{n+k-1}|\}_{n=1}^{\infty} =: \Lambda_k,$$

we have

$$\begin{aligned}
f \,\widetilde{\circledast}\, g &= \sum_{n,m\geq 0} \frac{w_{n+m}}{w_n w_m} f(n)g(m)e_{n+m} \\
&= f(0)\sum_{m=0}^{\infty} g(m)e_m + \frac{f(1)}{w_1}\sum_{m=0}^{\infty} g(m)\,\frac{w_{m+1}}{w_m}\,e_{m+1} \\
&\quad + \frac{f(2)}{w_2}\sum_{m=0}^{\infty} g(m)\,\frac{w_{m+2}}{w_m}\,e_{m+2} + \cdots \\
&\quad + \frac{f(N-1)}{w_{N-1}}\sum_{m=0}^{\infty} g(m)\,\frac{w_{m+N-1}}{w_m}\,e_{m+N-1} \\
&\quad + g(0)\sum_{n=N}^{\infty} f(n)e_n + \frac{g(1)}{w_1}\sum_{n=N}^{\infty} f(n)\,\frac{w_{n+1}}{w_n}\,e_{n+1} \\
&\quad + \frac{g(2)}{w_2}\sum_{n=N}^{\infty} f(n)\,\frac{w_{n+2}}{w_n}\,e_{n+2} + \cdots \\
&\quad + \frac{g(N-1)}{w_{N-1}}\sum_{n=N}^{\infty} f(n)\,\frac{w_{n+N-1}}{w_n}\,e_{n+N-1} \\
&\quad + \sum_{n=N}^{\infty}\sum_{m=N}^{\infty} \frac{w_{n+m}}{w_n\, w_m} f(n)g(m)e_{n+m}
\end{aligned}$$

$$= f(0)g + \frac{f(1)}{w_1}\,Tg + \frac{f(2)}{w_2}\,T^2\,g + \ldots + \frac{f(N-1)}{w_{N-1}}\,T^{N-1}g$$
$$+ g(0)R_N(f) + \frac{g(1)}{w_1}\,R_N(Tf) + \frac{g(2)}{w_2}\,R_N(T^2f) + \cdots$$
$$+ \frac{g(N-1)}{w_{N-1}}\,R_N(T^{N-1}f) + \sum_{n=N}^{\infty}\sum_{m=N}^{\infty}\frac{w_{n+m}}{w_n w_m}\,f(n)g(m)e_{n+m}.$$

From this, by using (iii), the equality $|f(i)| = \|f(i)e_i\|_Y$ and inequality (1) we obtain

$$\begin{aligned}\|f \mathbin{\widetilde{\circledast}} g\| &\le |f(0)|\,\|g\| + \frac{|f(1)|}{|w_1|}\|Tg\| + \cdots\\ &\quad + \frac{|f(N-1)|}{|w_{N-1}|}\,\|T^{N-1}g\| + |g(0)|\,\|R_N(f)\|\\ &\quad + \frac{|g(1)|}{|w_1|}\|R_N(Tf)\| + \cdots + \frac{|g(N-1)|}{|w_{N-1}|}\,\|R_N(T^{N-1}f)\|\\ &\quad + \sum_{n=N}^{\infty}\sum_{m=N}^{\infty}\left|\frac{w_{n+m}}{w_n w_m}\right|\|f(n)e_n\|\|g(m)e_m\|\\ &\le c\Bigg[\left(1 + \frac{\|T\|}{|w_1|} + \cdots + \frac{\|T^{N-1}\|}{|w_{N-1}|}\right) + \left(1 + \frac{\|T\|}{|w_1|} + \ldots + \frac{\|T^{N-1}\|}{|w_{N-1}|}\right)\\ &\quad + \sum_{n=N}^{\infty}\sum_{m=N}^{\infty}\left|\frac{w_{n+m}}{w_n w_m}\right|\Bigg]\|f\|\|g\|\\ &\le c\left[2\sum_{i=0}^{N-1}\frac{\Lambda_i}{|w_i|} + \Omega_N\right]\|f\|\|g\| =: C\|f\|\|g\|.\end{aligned}$$

By standard arguments (i.e., by passing to the equivalent norm in Y) we deduce from the last inequality that $(Y, \widetilde{\circledast})$ is a Banach algebra. Clearly, $f \mathbin{\widetilde{\circledast}} e_0 = f$ for each $f \in Y$.

To prove that $\mathcal{M}(Y, \widetilde{\circledast}) = \{0\}$, it suffices to show that an element $f \in Y$ is $\widetilde{\circledast}$-invertible if and only if $f(0) \neq 0$. In fact, let $f(0) \neq 0$. Let us prove that then $\mathcal{D}_f$, where $\mathcal{D}_f\, g := f \mathbin{\widetilde{\circledast}} g$, is an invertible operator in Y (boundedness of $\mathcal{D}_f$ is a consequence of (4)). Rewrite $\mathcal{D}_f$ in the form $\mathcal{D}_f = f(0)I + \mathcal{D}_{f-f(0)}$ and set

$$h := f - f(0) \quad \text{and} \quad F := \underbrace{h \mathbin{\widetilde{\circledast}} \cdots \mathbin{\widetilde{\circledast}} h}_{N+1}.$$

It is easy to verify that $F(0) = F(1) = \cdots = F(N) = 0$. Therefore for every $g \in Y$ and $M > N$ we obtain

$$\begin{aligned}
\mathcal{D}_h^{N+1} g &= F \,\widetilde{\circledast}\, g \\
&= g(0) R_{N+1}(f) + \frac{g(1)}{w_1} R_{N+1}(TF) + \frac{g(2)}{w_2} R_{N+1}(T^2 F) \\
&\quad + \cdots + \frac{g(N)}{w_N} R_{N+1}(T^N F) + \sum_{n \geq N+1} \sum_{m \geq N+1} \frac{w_{n+m}}{w_n w_m} F(n) g(m) e_{n+m} \\
&= \sum_{i=0}^{N} \frac{g(i)}{w_i} R_{N+1}(T^i F) + \sum_{n \geq N+1} \sum_{m \geq N+1} \frac{w_{n+m}}{w_n w_m} F(n) g(m) e_{n+m} \\
&= \sum_{i=0}^{N} \frac{g(i)}{w_i} R_{N+1}(T^i F) + \sum_{n=N+1}^{M} \sum_{m=N+1}^{M} \frac{w_{n+m}}{w_n w_m} F(n) g(m) e_{n+m} \\
&\quad + \sum_{n=N+1}^{M} \sum_{m=N+1}^{\infty} \frac{w_{n+m}}{w_n w_m} F(n) g(m) e_{n+m} \\
&\quad + \sum_{n=M+1}^{\infty} \sum_{m=N+1}^{\infty} \frac{w_{n+m}}{w_n w_m} F(n) g(m) e_{n+m}.
\end{aligned}$$

It is clear that the operator $\mathcal{K}_M$ defined by

$$\mathcal{K}_M g := \sum_{i=0}^{N} \frac{g(i)}{w_i} R_{N+1}(T^i F) + \sum_{n=N+1}^{M} \sum_{m=N+1}^{M} \frac{w_{n+m}}{w_n w_m} F(n) g(m) e_{n+m},$$

is a finite-rank (hence compact) operator. By considering (iii) and estimate (4) we obtain

$$\begin{aligned}
\|\mathcal{D}_h^{N+1} - \mathcal{K}_M\|_{L(Y)} &= \sup_{\|g\| \leq 1} \|\mathcal{D}_h^{N+1} g - \mathcal{K}_M g\| \\
&= \sup_{\|g\| \leq 1} \Bigg\| \sum_{n=N+1}^{M} \sum_{m=M+1}^{\infty} \frac{w_{n+m}}{w_n w_m} F(n) g(m) e_{n+m} \\
&\qquad + \sum_{n=M+1}^{\infty} \sum_{m=N+1}^{\infty} \frac{w_{n+m}}{w_n w_m} F(n) g(m) e_{n+m} \Bigg\| \\
&\leq C \|F\| \Bigg[\sum_{n=N+1}^{M} \sum_{m=M+1}^{\infty} \left| \frac{w_{n+m}}{w_n w_m} \right| + \sum_{n=M+1}^{\infty} \sum_{m=N+1}^{\infty} \left| \frac{w_{n+m}}{w_n w_m} \right| \Bigg] \to 0
\end{aligned}$$

as $M \to \infty$. Hence $\mathcal{K}_M \rightrightarrows \mathcal{D}_h^{N+1}$ as $M \to \infty$, which means that $\mathcal{D}_h^{N+1}$ is compact.

On the other hand, if $g \in Y$ and $\mathcal{D}_f g = 0$, then it follows from (3) that

$$(\mathcal{D}_f g)(n) = \sum_{k=0}^{n} f(k) g(n-k) \frac{w_n}{w_k w_{n-k}} = 0, \qquad n = 0, 1, 2, \ldots$$

Since $f(0) \neq 0$, simple calculations show that $0 = g(0) = g(1) = \cdots$, that is, $g = 0$, which implies that $\ker(f(0)I + \mathcal{D}_h) = \ker \mathcal{D}_f = \{0\}$. Then by a well known theorem of S. M. Nikol'skiĭ (see [1]) we deduce that $\mathcal{D}_f$ is invertible in Y, that is, f is $\widetilde{\circledast}$-invertible in Y.

Conversely, if f is $\widetilde{\circledast}$-invertible then it follows from (3) that $f(0) \neq 0$, i.e., $f(0) \neq 0$. The lemma is proved.

LEMMA 3. *$f \in \operatorname{Cyc}(T)$ if and only if $f(0) \neq 0$.*

Proof. It follows from (3) that

$$\begin{aligned} w_1 z \,\widetilde{\circledast}\, g &= w_1 z \,\widetilde{\circledast} \sum_{n=0}^{\infty} g(n) e_n = \sum_{n=0}^{\infty} g(n) w_1 (e_1 \circledast e_n) \\ &= \sum_{n=0}^{\infty} g(n) w_1 \frac{w_{n+1}}{w_1 w_n} e_{n+1} = \sum_{n=0}^{\infty} g(n) \frac{w_{n+1}}{w_n} e_{n+1} \\ &= \sum_{n=0}^{\infty} g(n) \lambda_n e_{n+1} = \sum_{n=0}^{\infty} g(n) T e_n = T\Big(\sum_{n=0}^{\infty} g(n) e_n \Big) = Tg \end{aligned}$$

for all $g \in Y$, and in general,

$$T^n g = w_n e_n \,\widetilde{\circledast}\, g \tag{5}$$

for any $g \in Y$ and $n \geq 0$. Hence

$$\begin{aligned} E_f := \operatorname{span}\{T^n f : n \geq 0\} &= \operatorname{span}\{w_n e_n \,\widetilde{\circledast}\, f : n \geq 0\} \\ &= \operatorname{span}\{\mathcal{D}_f(w_n e_n) : n \geq 0\} = \operatorname{clos} \mathcal{D}_f Y. \end{aligned} \tag{6}$$

Therefore, if $f \in \operatorname{Cyc}(T)$ then $\operatorname{clos} \mathcal{D}_f Y = Y$, which implies the existence of a sequence $\{f_n\}_{n \geq 0} \in Y$ such that $f \,\widetilde{\circledast}\, f_n \to e_0$ in Y as $n \to \infty$. Consequently, $(f \,\widetilde{\circledast}\, f_n)(0) \to 1$ as $n \to \infty$, or $f(0) f_n(0) \to 1$ as $n \to \infty$, and therefore, $f(0) \neq 0$.

Conversely, if $f(0) \neq 0$, then according to the equality (6) and Lemma 2, we have $E_f = Y$, that is, $f \in \operatorname{Cyc}(T)$. ∎

Proof of Theorem 1. If $\mu(A) = \infty$, then by inequalities (2) the assertion of the theorem is obvious, and therefore we will assume that $\mu(A) = n < \infty$.

Suppose that $\mu(T \oplus A) = \mu(A) = n$. Let $\{f_i \oplus x_i\}_{i=1}^n$ be a cyclic set for $T \oplus A$. Then $\{f_i\}_{i=1}^n$ is a cyclic set for T. Suppose that $f_k(0) \neq 0$ for $k = 1, \dots, l$ and $f_k(0) = 0$ for $k = l+1, \dots, n$. We set $g_k = f_k$, $y_k = x_k$ for $k = 1, \dots, l$, and $g_k = f_k - f_1$, $y_k = x_k - x_1$ for $k = l+1, \dots, n$. Then $\{g_i \oplus y_i\}_{i=1}^n$ is a cyclic set, and since $g_k(0) \neq 0$ for $k = 1, \dots, n$, by Lemmas 2 and 3 there exist $F_k \in Y$ such that $F_k \,\widetilde{\circledast}\, g_k = e_0$, $k = 1, \dots, n$. Set

$$\widetilde{x}_k = \sum_{m \geq 0} \frac{F_k(m)}{w_m} A^m y_k.$$

Then

$$g_k \oplus y_k = \sum_{m\geq 0} \frac{g_k(m)}{w_m}(T\oplus A)^m(e_0\oplus \widetilde{x}_k).$$

Therefore

$$g_k \oplus y_k \in \operatorname{span}\{(T\oplus A)^m(e_0\oplus\widetilde{x}_i) : m\geq 0,\, i=1,\dots,n\}$$

for $k=1,\dots,n$ and the set $\{e_0\oplus\widetilde{x}_i\}_{i=1}^n$ is cyclic for $T\oplus A$. We now set $\overline{x}_1=\widetilde{x}_1$, $\overline{x}_k=\widetilde{x}_1-\widetilde{x}_k$, $k=2,\dots,n$; this yields a new cyclic set $\{e_0\oplus\overline{x}_1, \mathbf{0}\oplus\overline{x}_2,\dots,\mathbf{0}\oplus\overline{x}_n\}$ for $T\oplus A$. Therefore, for any $x\in X$, there exists a family $\{P_{m,i} : 1\leq i\leq n,\, m\geq 1\}$ of polynomials such that

$$\lim_{m\to\infty} P_{m,1}(T)e_0 = 0 \quad \text{in } Y,$$

$$\lim_{m\to\infty}\sum_{i=1}^n P_{m,i}(A)\overline{x}_i = x \quad \text{in } X.$$

By using (5) we deduce that $\lim_{m\to\infty} q_{m,1}=0$ in Y, where

$$q_{m,1} := \sum_{k\geq 0} w_k P_{m,1}(k) e_k.$$

Then by using condition (ii) of the theorem, the equality $\|T^k e_0\| = |w_k|$ and the Hölder inequality we deduce that

$$\begin{aligned}
\|P_{m,1}(A)\overline{x}_1\| &= \Big\|\sum_{k\geq 0} P_{m,1}(k)A^k\overline{x}_1\Big\| \leq \sum_{k\geq 0}|P_{m,1}(k)|\,\|A^k\overline{x}_1\| \\
&= \sum_{k\geq 0}|w_k|\,|P_{m,1}(k)|\,\frac{\|A^k\overline{x}_1\|}{|w_k|} \\
&\leq \Big(\sum_{k\geq 0}|w_k|\,|P_{m,1}(k)|^p\Big)^{1/p}\bigg(\sum_{k\geq 0}\Big(\frac{\|A^k\overline{x}_1\|}{\|T^k e_0\|}\Big)^q\bigg)^{1/q} \\
&= C_{\overline{x}_1}^{1/q}\Big(\sum_{k\geq 0}|q_{m,1}(k)|^p\Big)^{1/p} = C_{\overline{x}_1}^{1/q}\|q_{m,1}\|_{l^p} \leq C_{\overline{x}_1}^{1/q}\widetilde{C}\|q_{m,1}\|_Y,
\end{aligned}$$

so that $\lim_{m\to\infty} P_{m,1}(A)\overline{x}_1 = 0$. Hence $\lim_{m\to\infty}\sum_{i=2}^n P_{m,i}(A)\overline{x}_i = x$. Since the vector x is arbitrary, the last relation means that $\{\overline{x}_i\}_{i=2}^n$ is a cyclic set for A, and hence $\mu(A)\leq n-1$. But this contradicts the assumption $\mu(A)=n$. ■

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