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EXT-ALGEBRAS AND DERIVED EQUIVALENCES

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Abstract. Using derived categories, we develop an alternative approach to defining Koszulness for positively graded algebras where the degree zero part is not necessarily semisimple.

The starting point for the work in this paper was to use derived categories to explain some of the results in [GRS]. In that paper the authors defined a notion of Koszulness for positively graded algebras where the degree zero part is not semisimple like it is in the classical Koszul case.

When generalising a theory it is always a question which features one would like to preserve. Some basic properties of classical Koszul algebras one as a minimum would like to keep are that each Koszul algebra has a dual Koszul algebra, that the Koszul dual of the Koszul dual is isomorphic to the algebra itself, and that there is a duality between certain module categories, the objects of which are called Koszul modules. The authors of [GRS] looked at the categories of Koszul modules in the classical Koszul case, and found some additional properties they wanted to keep in the generalised setting. They used the name "T-Koszul algebras" for their generalised version of classical Koszul algebras.

The T-Koszul algebras can also be viewed as a generalisation of tilting theory to the graded setting, because if one specialises to the case where the algebra is concentrated in degree zero (so basically we have an ungraded algebra), what you get is a finite-dimensional algebra together with a (Wakamatsu) cotilting module. In fact, the main purpose of [GRS] was to find a unified approach to both Koszul duality and the dualities arising from tilting theory.

While the approach in [GRS] is purely module (category) theoretic, in the present paper we look at the situation from the point of view of derived categories. In the classical Koszul case the duality on the level of Koszul modules can be explained as coming from an equivalence on the level of

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derived categories. We look at the specifics of this situation and make a generalised definition of Koszulness based on this. We feel that our definition is more natural and simpler than the one in [GRS]. We show that the T-Koszul algebras of [GRS] are also Koszul in our sense. With the help of our new definition we improve on some of the results and explain some of the phenomena observed in [GRS]. Our definition also specialises nicely to classical Koszul algebras and to Wakamatsu tilting theory.

In the present paper we heavily use ideas from [K] and also ideas gained in conversation with B. Keller.

We now describe the contents of the different sections.

In Section 1 we give some preliminary results about graded algebras and modules. In Sections 2 and 3 we recall some of the theory of Koszul algebras, both classical and T-Koszul. In Section 4 we explain how we look at graded algebras as (DG) categories, and what the concepts from the theory of DG categories mean for this special case.

In Section 5 we examine under what conditions two graded algebras have equivalent (unbounded) derived categories. The answer is used in Section 6, where we look at the special case that one of the algebras is the Ext-algebra of a module over the other algebra. We shall see that in the classical case of Koszul algebras, we get a derived equivalence on the whole derived categories only under very restrictive hypotheses. Therefore we look for ways to weaken the conditions and get equivalences on subcategories of the derived categories. We arrive at the notion of a graded self-orthogonal module.

Section 7 is the most important part of the paper. We give a general definition of Koszul algebras which we feel to be natural. We also show that in our setting we get generalised versions of the basic theorems for classical Koszul algebras. In Section 8 we prove that T-Koszul algebras are Koszul algebras in our sense. Like for the T-Koszul algebras, we show that our definitions specialise nicely to classical Koszul algebras and to Wakamatsu tilting theory.

1. Preliminaries. In this paper k always denotes a field and $\Lambda = \bigoplus_{n\geq 0} \Lambda_n$ denotes a positively graded k-algebra. The category of (left) ($\mathbb{Z}$)-graded modules with degree 0 maps is denoted by $\operatorname{Gr}\Lambda$. The graded shift $\langle 1\rangle$ acts as an autoequivalence on $\operatorname{Gr}\Lambda$. The full subcategory of finitely generated graded modules is denoted by $\operatorname{gr}\Lambda$. By $\operatorname{fgsyz}\Lambda$ we denote the full subcategory of $\operatorname{gr}\Lambda$ consisting of all graded modules which have a projective resolution such that all syzygies are finitely generated. Both $\operatorname{gr}\Lambda$ and $\operatorname{fgsyz}\Lambda$ are closed under $\langle 1\rangle$.

We record the following well known lemma, which can be found for instance in [NV, Corollary 2.4.4]. We shall often use it without explicit mention.

LEMMA 1.1. *Let* $\Lambda = \bigoplus_{n\geq 0} \Lambda_n$ *be a graded algebra and* M *and* N *be graded* Λ*-modules.*

(a) *If* M *is finitely generated*, *then*

$$\mathrm{Hom}_\Lambda(M,N) \simeq \bigoplus_{i\in\mathbb{Z}} \mathrm{Hom}_{\mathrm{Gr}\,\Lambda}(M, N\langle i\rangle).$$

(b) *If* $M \in \mathrm{fgsyz}\,\Lambda$, *then*

$$\mathrm{Ext}^n_\Lambda(M,N) \simeq \bigoplus_{i\in\mathbb{Z}} \mathrm{Ext}^n_{\mathrm{Gr}\,\Lambda}(M, N\langle i\rangle)$$

for all $n \geq 0$.

Suppose for the rest of the section that $\dim_k \Lambda_n < \infty$ for all $n \geq 0$. Denote by $l.f.\Lambda$ the full subcategory of $\mathrm{Gr}\,\Lambda$ consisting of all graded Λ-modules which are finite-dimensional in each degree. (Here $l.f.$ stands for *locally finite.*) The category $l.f.\Lambda$ is abelian and among its objects are the finitely generated and finitely cogenerated graded modules. In particular $l.f.\Lambda$ contains the finitely generated projective modules and the finitely cogenerated graded injective modules. The graded injective modules are not necessarily injective as ungraded modules [NV, Remark 2.3.3]. We have the following duality result, a reference for which is [M].

LEMMA 1.2. *Let* $\Lambda = \bigoplus_{n\geq 0} \Lambda_n$ *be a graded algebra* (*with* $\dim_k \Lambda_n < \infty$ *for all* $n \geq 0$). *There is a duality* $D = \mathrm{Hom}_k(-,k)\colon l.f.\Lambda \to l.f.\Lambda^{\mathrm{op}}$, *where the graded parts of the dual of a module* M *are given by* $(DM)_i = D(M_{-i})$.

The duality above restricts to a duality between finitely generated graded Λ-modules and the finitely cogenerated graded Λ^{op}-modules. The duality functor sends finitely generated (graded) projective modules to finitely cogenerated graded injective modules.

It also restricts to a duality between another interesting pair of subcategories. We say that a graded Λ-module M is *bounded below* if there is an integer n such that $M_{n'} = 0$ for all $n' \leq n$. Dually we say that a graded Λ-module M is *bounded above* if there is an integer n such that $M_{n'} = 0$ for all $n' \geq n$. The duality D restricts to a duality between the locally finite modules bounded below and the locally finite modules bounded above. A locally finite Λ-module M bounded below has a locally finite projective cover which is also bounded below. Obviously all submodules of modules bounded below are bounded below, therefore all syzygies of a locally finite Λ-module M bounded below are locally finite bounded below. (In fact the locally finite Λ-modules bounded below form an abelian category in which all objects have projective covers.) This gives the following useful result.

Proposition 1.3. *Let* $\Lambda = \bigoplus_{n\geq 0} \Lambda_n$ *be a graded algebra*, M *and* N *graded locally finite* Λ*-modules and let* D *denote the above duality. If* M *is bounded below or* N *is bounded above*, *then*

$$\operatorname{Ext}^n_{\operatorname{Gr}\Lambda}(M,N) \simeq \operatorname{Ext}^n_{\operatorname{Gr}\Lambda^{\mathrm{op}}}(DN,DM)$$

for all $n \geq 0$.

Combining the first two lemmas we get the following result concerning situations where the graded duality works well together with ungraded Hom and Ext spaces.

Proposition 1.4. *Let* $\Lambda = \bigoplus_{n\geq 0} \Lambda_n$ *be a graded algebra*, M *and* N *graded* Λ*-modules and let* D *denote the above duality.*

(a) *If* M *is finitely generated and* N *is finitely cogenerated*, *then*

$$\operatorname{Hom}_\Lambda(M,N) \simeq \operatorname{Hom}_{\Lambda^{\mathrm{op}}}(DN,DM).$$

(b) *If* $M \in \operatorname{fgsyz}\Lambda$ *and* N *has a graded injective resolution such that all cosyzygies are finitely cogenerated*, *then*

$$\operatorname{Ext}^n_\Lambda(M,N) \simeq \operatorname{Ext}^n_{\Lambda^{\mathrm{op}}}(DN,DM)$$

for all $n \geq 0$.

2. Classical Koszul algebras. In this section we recall some of the results about classical Koszul algebras. A standard reference is [BGS].

Let $\Lambda = \bigoplus_{n\geq 0} \Lambda_n$ be a graded algebra over a field k. A *linear projective resolution* of a graded Λ-module M is a projective resolution of M

$$\cdots \to P_n \to \cdots \to P_1 \to P_0 \to M$$

such that P_i is a projective module generated in degree i for all $i \geq 0$. We say that a finitely generated graded Λ-module M is a *Koszul module* if M has a linear projective resolution. The full subcategory of $\operatorname{gr}\Lambda$ consisting of Koszul modules is denoted by $\mathcal{K}(\Lambda)$. If Λ_0 has a linear projective resolution, we say that Λ is a *Koszul algebra*.

In [BGS, Proposition 2.1.3] we can find the following characterisation of Koszul algebras.

Proposition 2.1. *Let* $\Lambda = \bigoplus_{n\geq 0} \Lambda_n$ *be a graded algebra with* Λ_0 *semisimple. Then* Λ *is a Koszul algebra if and only if* $\operatorname{Ext}^n_{\operatorname{Gr}\Lambda}(\Lambda_0, \Lambda_0\langle i\rangle) \neq 0$ *implies* $i = n$.

We will only consider Koszul algebras where $\dim_k \Lambda_i < \infty$ for all $i \geq 0$.

We now list the most important basic theorems about Koszul algebras. If Λ is a Koszul algebra, we let $E(\Lambda)$ denote the graded algebra $E(\Lambda) = \bigoplus_{i\geq 0} \operatorname{Ext}^i_\Lambda(\Lambda_0,\Lambda_0)$. Since $\Lambda_0 \in \operatorname{fgsyz}\Lambda$, it follows from Lemma 1.1 and Proposition 2.1 that $E(\Lambda) \simeq \bigoplus_{i\geq 0} \operatorname{Ext}^i_{\operatorname{Gr}\Lambda}(\Lambda_0, \Lambda_0\langle i\rangle)$.

THEOREM 2.2. *Let Λ be a Koszul algebra. Then*

(a) *$E(\Lambda)$ is a Koszul algebra.*
(b) *$E(E(\Lambda)) \simeq \Lambda$ as graded algebras.*
(c) *The functor $\mathcal{E} = \bigoplus_{i\geq 0} \operatorname{Ext}^i_\Lambda(-,\Lambda_0)\colon \operatorname{Gr}\Lambda \to \operatorname{Gr} E(\Lambda)$ induces a duality $\mathcal{E}\colon \mathcal{K}(\Lambda) \to \mathcal{K}(E(\Lambda))$.*

There is also a duality (really an equivalence) on the level of derived categories. For this we need a covariant functor and we therefore define

$$\Gamma = E(\Lambda)^{\mathrm{op}} = \bigoplus_{i\geq 0} \operatorname{Ext}^i_\Lambda(\Lambda_0,\Lambda_0)^{\mathrm{op}}.$$

We call the algebra Γ the *Koszul dual* of Λ. We recall that an algebra is Koszul if and only if the opposite algebra is Koszul.

In general we cannot expect an equivalence (of triangulated categories) between the unbounded derived categories $\mathcal{D}\operatorname{Gr}\Lambda$ and $\mathcal{D}\operatorname{Gr}\Gamma$. We discuss this further in Section 6. The important duality result in [BGS, Theorem 2.12.1] is that there is a triangle equivalence between certain (big) subcategories of $\mathcal{D}\operatorname{Gr}\Lambda$ and $\mathcal{D}\operatorname{Gr}\Gamma$. In that paper we can also find the following result concerning bounded derived categories of finitely generated (graded) modules.

THEOREM 2.3 ([BGS, Theorem 2.12.6]). *Let Λ be a Koszul algebra and let $\Gamma = \bigoplus_{i\geq 0} \operatorname{Ext}^i_\Lambda(\Lambda_0,\Lambda_0)^{\mathrm{op}}$. Suppose Λ is artinian and Γ is noetherian. Then there is an equivalence of triangulated categories $\mathcal{D}^b \operatorname{gr}\Lambda \to \mathcal{D}^b \operatorname{gr}\Gamma$.*

As usual, inside the derived category modules are identified with stalk complexes. Restricting the equivalences on the level of derived categories to Koszul modules we recover Theorem 2.2(c), except that the variance is different. Instead of a duality we obtain an equivalence. What we get is that the category of *coKoszul modules* (the definition is dual to that of Koszul modules) over Λ is equivalent to the category of Koszul modules over Γ.

3. Generalised Koszul algebras. In this section we discuss a generalisation of Koszul algebras introduced by Green, Reiten and Solberg (see [GRS]). The main difference compared to the classical case is that we no longer assume Λ_0 is semisimple.

Let Δ be a finite-dimensional k-algebra. The category of (left) Δ-modules is denoted by $\operatorname{Mod}\Delta$. The full subcategory of finitely generated Δ-modules is denoted by $\operatorname{mod}\Delta$. If T is a finitely generated Δ-module we denote by $\operatorname{add} T$ the smallest full subcategory of $\operatorname{mod}\Delta$ containing T and closed under direct summands and finite direct sums. Let M be a module in $\operatorname{mod}\Delta$. A morphism $f\colon M \to M^T$ with M^T in $\operatorname{add} T$ is called a *left* $\operatorname{add} T$*-approximation* if for any morphism $g\colon M \to U$ with U in $\operatorname{add} T$, there is a morphism

$h\colon M^T \to U$ such that $g = hf$. The approximation is called *minimal* if for any morphism $j\colon M^T \to M^T$, if $jf = f$ then j is an isomorphism. Right $\operatorname{add} T$-approximations and minimal right $\operatorname{add} T$-approximations $M_T \to M$ are defined dually.

A module T in $\operatorname{mod} \Delta$ is called *self-orthogonal* if $\operatorname{Ext}^i_\Delta(T,T) = 0$ for all $i > 0$. A module T is called a *tilting module* if T is self-orthogonal, has finite projective dimension and there is an exact sequence

$$0 \to \Delta \to T_0 \xrightarrow{f_0} T_1 \xrightarrow{f_1} T_2 \to \cdots \to T_n \to 0$$

with T_i in $\operatorname{add} T$ for all $0 \le i \le n$. Let T be a self-orthogonal Δ-module. Define the left perpendicular category ${}^\perp T$ to be the full subcategory of $\operatorname{mod} \Delta$ consisting of modules C with $\operatorname{Ext}^i_\Delta(C,T) = 0$ for $i > 0$. We denote by $\mathcal{X}_T$ the full subcategory of ${}^\perp T$ consisting of modules C which have a coresolution

$$0 \to C \to T_0 \xrightarrow{f_0} T_1 \xrightarrow{f_1} T_2 \to \cdots$$

with T_i in $\operatorname{add} T$ and $\operatorname{Ext}^1_\Lambda(\operatorname{Ker} f_i, T) = 0$ for all $i \ge 0$. A module T over a finite-dimensional k-algebra Δ is called a *Wakamatsu cotilting module* if it is self-orthogonal and Δ is in $\mathcal{X}_T$. If T is a tilting module, then T is also a Wakamatsu cotilting module.

Let $\Lambda = \bigoplus_{n \ge 0} \Lambda_n$ be a graded k-algebra generated in degrees 0 and 1 and with $\dim_k \Lambda_n < \infty$ for all $n \ge 0$. We see that Λ_0 is finite-dimensional, so it is artinian. Let T be a Wakamatsu cotilting Λ_0-module. We view T as a Λ-module concentrated in degree 0. Let $\mathcal{G}_T(\Lambda)$ denote the category of all graded Λ-modules M, finitely generated with generators in degree 0, and where

$$\operatorname{Ext}^1_\Lambda(T,T)\operatorname{Ext}^i_\Lambda(M,T) = \operatorname{Ext}^{i+1}_\Lambda(M,T)$$

for all $i \ge 0$.

If M is a graded module, let $\lceil M_0 \rceil$ denote the graded submodule of M generated by M_0.

The *reject* of a graded module M with respect to T is defined by

$$\operatorname{rej}_T(M) = \bigcap \{\operatorname{Ker} f \mid f \in \operatorname{Hom}_{\operatorname{Gr}\Lambda}(M,T)\}.$$

In other words $\operatorname{rej}_T(M)$ is the unique smallest submodule of M such that the factor $M/\operatorname{rej}_T(M)$ is cogenerated by T. If M is generated in degree 0, then the module $M/\operatorname{rej}_T(M)$ is concentrated in degree 0. In that case since $M/\operatorname{rej}_T(M)$ is cogenerated by T, the module $M/\operatorname{rej}_T(M)$ has a minimal left $\operatorname{add} T$-approximation which is an inclusion. The cokernel of the $\operatorname{add} T$-approximation $0 \to M/\operatorname{rej}_T(M) \to M^T$ is called the *coreject* of M with respect to T and is denoted by $\operatorname{corej}_T(M)$.

Let $\mathcal{K}_T(\Lambda)$ be the largest full subcategory of $\mathcal{G}_T(\Lambda)$ closed under the operations $\operatorname{rej}_T(-)\langle -1\rangle$, $(-)/\operatorname{rej}_T(-)$, $\operatorname{corej}_T(-)$, $(\Omega_\Lambda(-)/\lceil \Omega_\Lambda(-)_0 \rceil)\langle -1\rangle$,

$\Lambda \otimes_{\Lambda_0} (-)_0$ and $\lceil \Omega_\Lambda(-)_0 \rceil$. The objects in $\mathcal{K}_T(\Lambda)$ are called *Koszul modules.* We say that Λ is a (*generalised*) *T-Koszul algebra* if Λ and T are in $\mathcal{K}_T(\Lambda)$.

It is quite easy to check that if $\Lambda = \bigoplus_{n\geq 0} \Lambda_n$ is a classical Koszul algebra, then it is also a T-Koszul algebra if we choose $T = \Lambda_0$. The other main class of examples is the algebras with $\Lambda = \Lambda_0$ and where T is a Wakamatsu cotilting module.

In [GRS, Lemma III.1.4] we find the following description of the reject and coreject of Koszul modules over a T-Koszul algebra.

Proposition 3.1. *Suppose Λ is a generalised T-Koszul algebra. Suppose $M \in \mathcal{K}_T(\Lambda)$. Then*

(a) $\operatorname{rej}_T(M) = M_{\geq 1}$.
(b) $M/\operatorname{rej}_T(M) = M_0$.
(c) $\operatorname{corej}_T(M)$ *is the cokernel of the left* $\operatorname{add} T$*-approximation* $M_0 \to M^T$ *and is cogenerated by T.*

The main result about T-Koszul algebras is that the basic theorems for Koszul algebras still hold in this generalised setting. If Λ is a T-Koszul algebra, we let $E(\Lambda)$ denote the graded algebra $E(\Lambda) = \bigoplus_{i\geq 0} \operatorname{Ext}^i_\Lambda(T,T)$. It follows from the definition of T-Koszul that $T \in \operatorname{fgsyz} \Lambda$, so $E(\Lambda) \simeq \bigoplus_{i\geq 0} \operatorname{Ext}^i_{\operatorname{Gr}\Lambda}(T, T\langle i\rangle)$.

Theorem 3.2 ([GRS, Theorems III.6.4 and III.6.5]). *Let Λ be a T-Koszul algebra. Then*

(a) *$E(\Lambda)$ is a T-Koszul algebra with respect to ${}_{E(\Lambda)}T$.*
(b) *$E(E(\Lambda)) \simeq \Lambda$ as graded algebras.*
(c) *The functor $\mathcal{E} = \bigoplus_{i\geq 0} \operatorname{Ext}^i_\Lambda(-,T)\colon \operatorname{Gr}\Lambda \to \operatorname{Gr} E(\Lambda)$ induces a duality $\mathcal{E}\colon \mathcal{K}_T(\Lambda) \to \mathcal{K}_T(E(\Lambda))$.*

In [GRS], the authors did not consider derived categories. In the present paper we explain their duality results as coming from an equivalence on the level of derived categories, similar to what happens in the classical Koszul case.

4. Graded algebras as DG categories. Our aim is to use the theory of DG categories [K] to prove results about derived categories for graded algebras. A *DG category* is a k-linear category where the morphism spaces are differential graded k-vector spaces. In this section our task is to find a way to view graded algebras as DG categories and see what the concepts from DG category theory mean in this special case. In fact we put every morphism in cohomological degree 0, so our categories can be viewed as ordinary k-linear categories. Still it is the DG category concepts that are important to us, since they will help us to deal with the derived categories.

We will only consider positively graded algebras, that is, algebras graded in the nonnegative integers. Other types of gradings can be codified in a similar fashion.

In general, if $\mathcal{A}$ is a small (meaning that the objects form a set) k-linear category, we define a (*left*) $\mathcal{A}$-*module* to be a k-linear functor $\mathcal{A} \to \operatorname{Mod} k$. The functor category of all $\mathcal{A}$-modules is denoted by $\operatorname{Mod}\mathcal{A}$. For each object A of $\mathcal{A}$, we have a special module $P_A = \operatorname{Hom}_{\mathcal{A}}(A, -)$. From the Yoneda Lemma we get

$$\operatorname{Hom}_{\operatorname{Mod}\mathcal{A}}(P_B, P_A) \simeq \operatorname{Hom}_{\mathcal{A}}(A, B)$$

for all pair of objects A, B of $\mathcal{A}$. In this way $\mathcal{A}^{\mathrm{op}}$ is equivalent to the full subcategory of $\operatorname{Mod}\mathcal{A}$ formed by the objects $\{P_A \mid A \in \mathcal{A}\}$.

If $\Lambda = \bigoplus_{n\geq 0} \Lambda_n$ is a graded k-algebra, we can codify the structure as a k-linear category $\mathcal{A}$ in the following way: As the set of objects we take the integers $\mathbb{Z}$. We use brackets around the integers to make the notation clearer. The space of morphisms between objects $\{m\}$ and $\{n\}$ is given by

$$\operatorname{Hom}_{\mathcal{A}}(\{m\}, \{n\}) = \Lambda_{n-m}.$$

The category $\operatorname{Mod}\mathcal{A}$ is then equivalent to $\operatorname{Gr}\Lambda$, the category of graded Λ-modules with degree 0 morphisms. Under this equivalence, the functor $P_{\{m\}} = \operatorname{Hom}_{\mathcal{A}}(\{m\}, -)$ corresponds to the graded Λ-module $\Lambda\langle m\rangle$ for each object $\{m\}$ of $\mathcal{A}$.

Another way to codify Λ as a k-linear category is sometimes useful. We assume that $\dim_k \Lambda_i < \infty$ for all $i \geq 0$. This means that Λ_0 itself is a finite-dimensional k-algebra. So there is a decomposition $1 = e_1 + \cdots + e_r$ of the identity into a sum of primitive orthogonal idempotents. Each primitive idempotent e_i corresponds to an indecomposable projective Λ-module P_i. We can define a category $\mathcal{A}'$ as follows:

$$\operatorname{ob}\mathcal{A}' = \{1, \dots, r\} \times \mathbb{Z}, \qquad \operatorname{Hom}_{\mathcal{A}'}(i\{m\}, j\{n\}) = e_j \Lambda_{n-m} e_i$$

for all $i, j \in \{1, \dots, r\}$, $m, n \in \mathbb{Z}$ with the obvious composition. Also here we see that the category $\operatorname{Mod}\mathcal{A}'$ is equivalent to $\operatorname{Gr}\Lambda$. With this description, the functor $P_{i\{m\}} = \operatorname{Hom}_{\mathcal{A}'}(i\{m\}, -)$ corresponds to the indecomposable graded projective Λ-module $P_i\langle m\rangle$ for each object $i\{m\}$ of $\mathcal{A}'$.

Now we look at what the concepts from the theory of DG categories [K] mean for graded algebras.

First of all, every ordinary category can be considered as a DG category concentrated in degree 0.

Let $\Lambda = \bigoplus_{n\geq 0} \Lambda_n$ be a graded algebra, and denote the corresponding category by $\mathcal{A}$. For the definitions in this section it does not matter which of the two methods above is used to codify Λ. A (left) DG $\mathcal{A}$-module is the same as a complex of graded (left) Λ-modules.

Let $\Lambda = \bigoplus_{n\ge 0} \Lambda_n$ and $\Gamma = \bigoplus_{n\ge 0} \Gamma_n$ be graded k-algebras coded as DG categories $\mathcal{A}$ and $\mathcal{B}$ with morphisms concentrated in degree 0. An $\mathcal{A}$-$\mathcal{B}$-*bimodule* is a $\mathbb{Z}\times\mathbb{Z}$-graded space X where the left action by Λ uses the first component of the grading, the right action by Γ uses the second component and $(a_jx)b_i = a_j(xb_i) \in X_{m+j,n+i}$ for all $a_j \in \Lambda_j$, $b_i \in \Gamma_i$ and $x \in X_{mn}$. A *DG* $\mathcal{A}$-$\mathcal{B}$-*bimodule* is a complex of such $\mathcal{A}$-$\mathcal{B}$-bimodules.

If X is a DG $\mathcal{A}$-$\mathcal{B}$-bimodule, the differential in X respects the grading, so for each $(i,j) \in \mathbb{Z}\times\mathbb{Z}$ we have a complex

$$X_{ij}\colon\ \cdots \to X_{ij}^{-1} \to X_{ij}^{0} \to X_{ij}^{1} \to X_{ij}^{2} \to \cdots .$$

Similarly if M is a DG $\mathcal{B}$-module (a complex of graded Γ-modules), the differential respects the grading, so for each $l \in \mathbb{Z}$ we get a complex

$$M_l\colon\ \cdots \to M_l^{-1} \to M_l^{0} \to M_l^{1} \to M_l^{2} \to \cdots .$$

Now the tensor product of X and M is the DG $\mathcal{A}$-module (complex of graded Λ-modules) defined by

$$(X \otimes_{\mathcal{B}} M)_i = \bigoplus_{j+l=0} (X_{ij} \otimes_k M_l)/S$$

where S is the subspace generated by the elements of the form $xa\otimes m - x\otimes am$ with $x \in X_{i,-t}$, $a \in \Gamma_{t-s}$ and $m \in M_s$. The condition $j+l=0$ is the usual one for graded tensor products.

Let X be as above and let L and N be two DG $\mathcal{A}$-modules (complexes of graded Λ-modules). Then $\operatorname{Hom}^{\bullet}_{\operatorname{Gr}\Lambda}(L,N)$ is the complex with components

$$\operatorname{Hom}^{i}_{\Lambda}(L,N) = \prod_{n\in\mathbb{Z}} \operatorname{Hom}_{\operatorname{Gr}\Lambda}(L^n, N^{n+i})$$

and differential $df = d_N \circ f - (-1)^p f \circ d_L$ for $f \in \operatorname{Hom}^p_{\Lambda}(L,N)$. We denote by $\mathcal{H}\mathrm{om}(X,N)$ the DG $\mathcal{B}$-module with graded parts

$$(\mathcal{H}\mathrm{om}(X,N))_j = \operatorname{Hom}^{\bullet}_{\operatorname{Gr}\Lambda}(X_{*j}, N).$$

If $\mathcal{A}$ is a DG category we can define the ordinary category of DG $\mathcal{A}$-modules $\mathcal{CA}$ (morphisms are given by $\operatorname{Hom}_{\mathcal{CA}}(L,N) = Z^0 \operatorname{Hom}^{\bullet}_{\mathcal{A}}(L,N)$), the homotopy category $\mathcal{HA}$ and the derived category $\mathcal{DA}$. In our case this is the usual abelian category $\mathcal{C}(\operatorname{Gr}\Lambda)$ of (unbounded) complexes of graded Λ-modules with degree 0 morphisms, the homotopy category $\mathcal{K}(\operatorname{Gr}\Lambda)$ and the derived category $\mathcal{D}\operatorname{Gr}\Lambda$.

If X is a complex of projective Λ-Γ-bimodules, we get a pair of adjoint functors on the homotopy categories:

$$\mathcal{K}(\operatorname{Gr}\Lambda) \underset{X\otimes -}{\overset{\mathcal{H}\mathrm{om}(X,-)}{\rightleftarrows}} \mathcal{K}(\operatorname{Gr}\Gamma).$$

We also get a pair of adjoint derived functors:

$$\mathcal{D}\operatorname{Gr}\Lambda \underset{X\otimes^L_{\operatorname{Gr}\Gamma}-}{\overset{R\operatorname{Hom}_{\operatorname{Gr}\Lambda}(X,-)}{\rightleftarrows}} \mathcal{D}\operatorname{Gr}\Gamma.$$

5. Derived equivalence of graded algebras. In this section we discuss under what conditions two positively graded algebras are derived equivalent.

Let $\Lambda = \bigoplus_{n\geq 0}\Lambda_n$ and $\Gamma = \bigoplus_{n\geq 0}\Gamma_n$ be two graded k-algebras. We call Λ and Γ *derived equivalent* if $\mathcal{D}\operatorname{Gr}\Lambda$ and $\mathcal{D}\operatorname{Gr}\Gamma$ are equivalent as triangulated categories.

Let $\mathcal{T}$ be a triangulated category with infinite direct sums and let $\mathcal{X}$ be a set of objects in $\mathcal{T}$. Then $\mathcal{X}$ is called a *set of generators* (of $\mathcal{T}$) if $\mathcal{T}$ is the smallest full triangulated subcategory containing $\mathcal{X}$ and closed under arbitrary direct sums. The smallest full triangulated subcategory of $\mathcal{T}$ containing $\mathcal{X}$ and closed under finite direct sums is denoted by $\operatorname{tria}\mathcal{X}$. If s is an autoequivalence of $\mathcal{T}$, we define

$$\operatorname{tria}(\mathcal{X}, s) = \operatorname{tria}\{s^iU \mid i\in\mathbb{Z},\, U\in\mathcal{X}\}.$$

In other words $\operatorname{tria}(\mathcal{X}, s)$ is the smallest full triangulated subcategory of $\mathcal{T}$ containing $\mathcal{X}$, closed under finite direct sums and closed under s. We also define

$$\operatorname{trias}\mathcal{X} = \operatorname{tria}\{X' \mid X' \text{ is a direct summand of some } X\in\mathcal{X}\}.$$

An object U in $\mathcal{T}$ is called *compact* if the functor $\operatorname{Hom}_{\mathcal{T}}(U,-)$ commutes with infinite direct sums.

The following theorem is adapted from [K, Theorem 9.2]. Some small changes have been made because we consider derived categories of left modules instead of right modules.

THEOREM 5.1. *Let $\mathcal{A}$ and $\mathcal{B}$ be two small k-linear categories. The following are equivalent*:

(a) *There is a DG $\mathcal{A}$-$\mathcal{B}$-bimodule X such that $X\otimes^L_{\mathcal{B}} -\colon \mathcal{D}\mathcal{B}\to\mathcal{D}\mathcal{A}$ is an equivalence of triangulated categories.*
(b) *There is an equivalence of triangulated categories $\mathcal{D}\mathcal{B}\to\mathcal{D}\mathcal{A}$.*
(c) *$\mathcal{B}^{\mathrm{op}}$ is equivalent to a full subcategory $\mathcal{U}$ of $\mathcal{D}\mathcal{A}$ whose objects form a set of compact generators and satisfy* $\operatorname{Hom}_{\mathcal{D}\mathcal{A}}(U, V[n]) = 0$ *for all $U, V\in\mathcal{U}$ and $n\neq 0$.*

If $\mathcal{D}\mathcal{B}\to\mathcal{D}\mathcal{A}$ is a triangle equivalence, then the subcategory $\mathcal{U}$ of $\mathcal{D}\mathcal{A}$ is the image of the full subcategory $\{P_B \mid B\in\mathcal{B}\}$ of $\mathcal{D}\mathcal{B}$. (Recall that

$\{P_B \mid B \in \mathcal{B}\} \simeq \mathcal{B}^{\mathrm{op}}$.)

$$\begin{tikzcd} \{P_B \mid B \in \mathcal{B}\} \arrow[r, "\sim"] \arrow[d, hook] & \mathcal{U} \arrow[d, hook] \\ \mathcal{D}\mathcal{B} \arrow[r, "\sim"] & \mathcal{D}\mathcal{A} \end{tikzcd}$$

We will not here explain the general construction of the bimodule X from a given subcategory $\mathcal{U}$ (with the right properties) of $\mathcal{D}\mathcal{A}$, but again we refer to [K, 7.3]. In the case of rings (when $\mathcal{A}$ and $\mathcal{B}$ have only one object each), such a bimodule X is usually called a *twosided tilting complex.*

In this paper we will several times encounter subcategories $\mathcal{U}$ of $\mathcal{D}\mathcal{A}$ with the property that $\mathrm{Hom}_{\mathcal{D}\mathcal{A}}(U, V[n]) = 0$ for all $U, V \in \mathcal{U}$ and $n \neq 0$, but where the objects may not form a set of compact generators. In this situation we have a weaker result, as the following proposition shows.

Proposition 5.2. *Let $\mathcal{A}$ and $\mathcal{B}$ be two small k-linear categories. Suppose $\mathcal{B}^{\mathrm{op}}$ is equivalent to a full subcategory $\mathcal{U}$ of $\mathcal{D}\mathcal{A}$ with the property that* $\mathrm{Hom}_{\mathcal{A}}(U, V[n]) = 0$ *for all $U, V \in \mathcal{U}$ and $n \neq 0$. Then there is a DG $\mathcal{A}$-$\mathcal{B}$-bimodule X such that the functor $X \otimes^L_{\mathcal{B}} - \colon \mathcal{D}\mathcal{B} \to \mathcal{D}\mathcal{A}$ induces an equivalence of triangulated categories* $\mathrm{tria}\{P_B \mid B \in \mathcal{B}\} \to \mathrm{tria}\,\mathcal{U}$.

$$\begin{tikzcd} \mathrm{tria}\{P_B \mid B \in \mathcal{B}\} \arrow[r, "\sim"] \arrow[d, hook] & \mathrm{tria}\,\mathcal{U} \arrow[d, hook] \\ \mathcal{D}\mathcal{B} \arrow[r, "X \otimes^L_{\mathcal{B}} -"] & \mathcal{D}\mathcal{A} \end{tikzcd}$$

Again the details of the proof can be found in [K, 7.3]. The construction of the bimodule X is the same as in the previous theorem.

Now let us return to the case of graded algebras. Let $\Lambda = \bigoplus_{n \geq 0} \Lambda_n$ and $\Gamma = \bigoplus_{n \geq 0} \Gamma_n$ be two graded k-algebras. We view them as categories where the objects are indexed by the integers. Suppose there is a triangle equivalence $F \colon \mathcal{D}\,\mathrm{Gr}\,\Gamma \to \mathcal{D}\,\mathrm{Gr}\,\Lambda$. Then we have the following diagram:

$$\begin{tikzcd} \{\Gamma\langle n\rangle \mid n \in \mathbb{Z}\} \arrow[r, "\sim"] \arrow[d, hook] & \mathcal{U} \arrow[d, hook] \\ \mathcal{D}\,\mathrm{Gr}\,\Gamma \arrow[r, "\sim", "F"'] & \mathcal{D}\,\mathrm{Gr}\,\Lambda \end{tikzcd}$$

There are two special autoequivalences on $\mathcal{D}\,\mathrm{Gr}\,\Gamma$: the shift of complexes $[1]$, which gives the triangulated structure of the category, and the graded shift $\langle 1\rangle$. The graded shift $\langle 1\rangle$ also restricts to an autoequivalence on $\{\Gamma\langle n\rangle \mid n \in \mathbb{Z}\}$. By the definition of a triangle equivalence, the functor F has to respect the shift of complexes $[1]$ in the sense that $F[1] \simeq [1]F$. In contrast there is no such rule for the graded shift $\langle 1\rangle$. But if G is a quasi-inverse of F, then $s = F\langle 1\rangle G$ must be an autoequivalence on $\mathcal{D}\,\mathrm{Gr}\,\Lambda$ satisfying $sF \simeq F\langle 1\rangle$. Since the objects of $\{\Gamma\langle n\rangle \mid n \in \mathbb{Z}\}$ are given by repeated shifts and inverse shifts of the object Γ of $\mathcal{D}\,\mathrm{Gr}\,\Gamma$, up to isomorphism in $\mathcal{D}\,\mathrm{Gr}\,\Lambda$ the objects of $\mathcal{U}$ are $\{s^iT \mid i \in \mathbb{Z}\}$ for some object T of $\mathcal{D}\,\mathrm{Gr}\,\Lambda$.

Suppose $T \in \mathcal{D}\,\mathrm{Gr}\,\Lambda$ is a complex and $s\colon \mathcal{D}\,\mathrm{Gr}\,\Lambda \to \mathcal{D}\,\mathrm{Gr}\,\Lambda$ is an equivalence. If the pair (T, s) satisfies $\mathrm{Hom}_{\mathcal{D}\,\mathrm{Gr}\,\Lambda}(s^iT, T[n]) = 0$ for $i < 0$ or $n \neq 0$, then motivated by the discussion above we define the graded algebra $E_\Lambda(T, s)$ as follows:

$$E_\Lambda(T, s)_i = \mathrm{Hom}_{\mathcal{D}\,\mathrm{Gr}\,\Lambda}(s^iT, T), \qquad f \cdot g = g \circ s^j(f)$$

for all $f \in E_\Lambda(T, s)_i$ and $g \in E_\Lambda(T, s)_j$. In the situation above we have $\Gamma \simeq E_\Lambda(T, s)$ as graded algebras.

We have a special description of the compact objects in $\mathcal{D}\,\mathrm{Gr}\,\Lambda$. A complex of Λ-modules is called *perfect* if it is a bounded complex of finitely generated projective Λ-modules. Similarly to the case of ungraded rings, a complex T is a compact object in $\mathcal{D}\,\mathrm{Gr}\,\Lambda$ if and only if it is quasi-isomorphic to a perfect complex [K, 5.3].

The following proposition explains which pairs (T, s) give rise to a derived equivalence.

PROPOSITION 5.3. *Let $\Lambda = \bigoplus_{n\geq 0} \Lambda_n$ and $\Gamma = \bigoplus_{n\geq 0} \Gamma_n$ be two graded k-algebras. There is an equivalence of triangulated categories $\mathcal{D}\,\mathrm{Gr}\,\Gamma \to \mathcal{D}\,\mathrm{Gr}\,\Lambda$ if and only if there exists a complex $T \in \mathcal{D}\,\mathrm{Gr}\,\Lambda$ and an autoequivalence $s\colon \mathcal{D}\,\mathrm{Gr}\,\Lambda \to \mathcal{D}\,\mathrm{Gr}\,\Lambda$ such that:*

(a) *T is quasi-isomorphic to a bounded complex of finitely generated projective graded Λ-modules.*
(b) *$\{s^iT \mid i \in \mathbb{Z}\}$ generates $\mathcal{D}\,\mathrm{Gr}\,\Lambda$.*
(c) *We have isomorphisms*

$$\mathrm{Hom}_{\mathcal{D}\,\mathrm{Gr}\,\Lambda}(s^iT, T[n]) \simeq \begin{cases} \Gamma_i, & n = 0, \\ 0, & n \neq 0, \end{cases}$$

for all $i \in \mathbb{Z}$, and via the isomorphisms $\mathrm{Hom}_{\mathcal{D}\,\mathrm{Gr}\,\Lambda}(s^iT, T) \simeq \Gamma_i$ we have

$$\Gamma \simeq E_\Lambda(T, s)$$

as graded algebras.

Proof. Theorem 5.1 interpreted in this situation gives that $\mathcal{D}\,\mathrm{Gr}\,\Gamma \simeq \mathcal{D}\,\mathrm{Gr}\,\Lambda$ if and only if $\{\Gamma\langle n\rangle \mid n \in \mathbb{Z}\} \simeq \mathcal{U}$, where the objects of $\mathcal{U}$ form a set of compact generators which satisfy $\mathrm{Hom}_{\mathcal{D}\mathcal{A}}(U, V[n]) = 0$ for all $U, V \in \mathcal{U}$ and $n \neq 0$. By previous comments we may assume $\mathcal{U} = \{s^iT \mid i \in \mathbb{Z}\}$ for some object T of $\mathcal{D}\,\mathrm{Gr}\,\Lambda$ and some autoequivalence $s\colon \mathcal{D}\,\mathrm{Gr}\,\Lambda \to \mathcal{D}\,\mathrm{Gr}\,\Lambda$. Since s is an autoequivalence, we see that T is compact if and only if s^iT is compact for all $i \in \mathbb{Z}$. ■

REMARK 5.4. Condition (b) can be replaced by the following:

(b′) $\Lambda\langle j\rangle \in \operatorname{trias}\{s^iT \mid i \in \mathbb{Z}\}$ for all $j \in \mathbb{Z}$.

Proof. In $\mathcal{D}\operatorname{Gr}\Gamma$, every object that is compact must be in $\operatorname{trias}\{\Gamma\langle i\rangle \mid i \in \mathbb{Z}\}$. So if we have an equivalence $\mathcal{D}\operatorname{Gr}\Gamma \to \mathcal{D}\operatorname{Gr}\Lambda$, then every compact object in $\mathcal{D}\operatorname{Gr}\Lambda$ must be in $\operatorname{trias}\{s^iT \mid i \in \mathbb{Z}\}$. Therefore $\Lambda\langle j\rangle \in \operatorname{trias}\{s^iT \mid i \in \mathbb{Z}\}$ for all $j \in \mathbb{Z}$.

If $\Lambda\langle j\rangle \in \operatorname{trias}\{s^iT \mid i \in \mathbb{Z}\}$ for all $j \in \mathbb{Z}$, then obviously $\{s^iT \mid i \in \mathbb{Z}\}$ generates $\mathcal{D}\operatorname{Gr}\Lambda$. ■

From Theorem 5.1 we also see that if we have an equivalence $\mathcal{D}\operatorname{Gr}\Gamma \to \mathcal{D}\operatorname{Gr}\Lambda$, then this equivalence is given by a functor $X \otimes^L_{\operatorname{Gr}\Lambda} - \colon \mathcal{D}\operatorname{Gr}\Gamma \to \mathcal{D}\operatorname{Gr}\Lambda$, where X is a complex of Λ-Γ-bimodules.

A pair (T, s) satisfying conditions (a)–(c) in Proposition 5.3 is called a *tilting pair*. If $\Lambda = \bigoplus_{n\ge 0}\Lambda_n$ is a graded algebra and (T, s) is a tilting pair for Λ, then $E_\Lambda(T, s)$ is derived equivalent to Λ.

We have used autoequivalences of the derived category $\mathcal{D}\operatorname{Gr}\Lambda$ as part of the description of derived equivalences. For this to be useful, we need some knowledge of the autoequivalences of $\mathcal{D}\operatorname{Gr}\Lambda$. It seems worthwhile to give a version of the proposition involving only the family $\{U_i \mid i \in \mathbb{Z}\}$ of objects where $U_i = s^iT$ without specifying the autoequivalence s.

We say that the elements in a set $\mathcal{X}$ of objects in $\mathcal{D}\operatorname{Gr}\Lambda$ are *pairwise orthogonal* if $\operatorname{Hom}_{\mathcal{D}\operatorname{Gr}\Lambda}(U, V[n]) = 0$ for all $U, V \in \mathcal{X}$ and $n \ne 0$.

PROPOSITION 5.5. *Let $\Lambda = \bigoplus_{n\ge 0}\Lambda_n$ and $\Gamma = \bigoplus_{n\ge 0}\Gamma_n$ be two graded k-algebras. There is an equivalence of triangulated categories $\mathcal{D}\operatorname{Gr}\Gamma \to \mathcal{D}\operatorname{Gr}\Lambda$ if and only if there exists a family $\{U_i \mid i \in \mathbb{Z}\}$ of pairwise orthogonal perfect complexes generating $\mathcal{D}\operatorname{Gr}\Lambda$ and isomorphisms*

$$\xi_{i,j}\colon \operatorname{Hom}_{\mathcal{D}\operatorname{Gr}B}(U_i, U_j) \simeq \Gamma_{i-j}$$

for all $i, j \in \mathbb{Z}$ such that

$$\gamma \cdot \gamma' = \xi_{t+n,t-m}[\xi^{-1}_{t+n,t}(\gamma') \circ \xi^{-1}_{t,t-m}(\gamma)] \in \Gamma_{m+n}$$

for all $\gamma \in \Gamma_m$, $\gamma' \in \Gamma_n$ and $t \in \mathbb{Z}$.

We call such a family $\{U_i \mid i \in \mathbb{Z}\}$ a *tilting family*. From a tilting family $\{U_i \mid i \in \mathbb{Z}\}$ we can construct a tilting pair (T, s) where $T = U_0$ and we define $s(U_i) = U_{i+1}$ and for a morphism $f \in \operatorname{Hom}_{\mathcal{D}\operatorname{Gr}B}(U_i, U_j)$ we set

$$s(f) = \xi^{-1}_{i+1,j+1}[\xi_{i,j}(f)] \in \operatorname{Hom}_{\mathcal{D}\operatorname{Gr}B}(U_{i+1}, U_{j+1}).$$

Since $\{U_i \mid i \in \mathbb{Z}\}$ is a set of generators, we can extend s to an autoequivalence on $\mathcal{D}\operatorname{Gr}\Lambda$.

We end this section by giving simple examples of derived equivalences between graded algebras.

EXAMPLE 5.6. Let $\Lambda = \bigoplus_{n\geq 0} \Lambda_n$ be a graded algebra, and let Γ be the graded algebra given by

$$\Gamma_{2i} = \Lambda_i, \qquad \Gamma_{2i+1} = 0,$$

for all $i \geq 0$. Then Γ is derived equivalent to the algebra $\Lambda \times \Lambda$. A tilting pair for $\Lambda \times \Lambda$ is $((\Lambda, 0), s)$, where s is the autoequivalence given by $s(M, N) = (N\langle 1\rangle, M)$. In fact, in this example the categories of graded modules $\operatorname{Gr}\Gamma$ and $\operatorname{Gr}(\Lambda \times \Lambda)$ are already equivalent.

EXAMPLE 5.7 (APR tilting). This is an example where the grading plays no special role, so effectively this example is covered by classical tilting theory for ungraded algebras. Let Λ be the path algebra given by the quiver

$$\begin{tikzcd} 1 \arrow[d, "\alpha"'] \arrow[dr, "\beta"] & & \\ 2 & 3 \arrow[l, "\gamma"'] \arrow[r, "\delta"] & 4 \end{tikzcd}$$

where $\deg(\alpha) = 1$ and $\deg(\beta) = \deg(\gamma) = \deg(\delta) = 0$. Let Δ_4 be the module with representation

$$\begin{tikzcd} 0 \arrow[d, "0"'] \arrow[dr, "0"] & & \\ k & k \arrow[l, "1"'] \arrow[r, "0"] & 0 \end{tikzcd}$$

Here we have $\Delta_4 = \operatorname{Tr} D(S_4)$, where S_4 is the simple module corresponding to vertex 4. For the definition of $\operatorname{Tr} D$, see [ARS].

Let (T, s) be the tilting pair where $T = P_1 \oplus P_2 \oplus P_3 \oplus \Delta_4$ and $s = \langle 1\rangle$. (Here P_i denotes the projective module corresponding to vertex i.) As expected $E_\Lambda(T, s)$ is the path algebra given by the quiver

$$\begin{tikzcd} 1^* \arrow[d, "\alpha^*"'] \arrow[dr, "\beta^*"] & & \\ 2^* & 3^* \arrow[l, "\gamma^*"'] & 4^* \arrow[l, "\delta^*"'] \end{tikzcd}$$

where $\deg(\alpha^*) = 1$ and $\deg(\beta^*) = \deg(\gamma^*) = \deg(\delta^*) = 0$.

6. Ext-algebras. In this section we first make use of the results in the previous section and describe derived equivalences involving Ext-algebras. Then we take a closer look at the classical Koszul algebras and see how they fit into this picture. In general there is not an equivalence on the level of unbounded derived categories between a Koszul algebra and its Koszul dual, but there is an equivalence between certain subcategories. We try to identify what is essential and special about the Koszul situation and try to find the right analogy in the case of graded algebras where the degree zero part is not semisimple.

Consider the situation where Λ is a graded algebra and Γ is the Ext-algebra, with the natural grading, of a Λ-module T. The next result describes under what conditions Λ and Γ are derived equivalent, with an equivalence sending Γ to T.

PROPOSITION 6.1. *Let $\Lambda = \bigoplus_{n\geq 0} \Lambda_n$ be a graded algebra and let T be a graded Λ-module. Let $\Gamma = \bigoplus_{n\geq 0} \operatorname{Ext}^n_\Lambda(T,T)^{\mathrm{op}}$. Then there is an equivalence of triangulated categories $\mathcal{D}\operatorname{Gr}\Gamma \to \mathcal{D}\operatorname{Gr}\Lambda$ sending Γ to T if and only if there exists an autoequivalence $s\colon \mathcal{D}\operatorname{Gr}\Gamma \to \mathcal{D}\operatorname{Gr}\Gamma$ such that*:

(a) $T \in \operatorname{fgsyz}\Lambda$.
(b) $\operatorname{pd}_\Lambda T < \infty$.
(c) $\Lambda\langle j\rangle \in \operatorname{trias}\{s^iT \mid i \in \mathbb{Z}\} \subseteq \mathcal{D}\operatorname{Gr}\Lambda$ *for all* $j \in \mathbb{Z}$.
(d) *We have isomorphisms*

$$\operatorname{Hom}_{\mathcal{D}\operatorname{Gr}\Lambda}(s^iT, T[n]) \simeq \begin{cases} \operatorname{Ext}^i_\Lambda(T,T), & n = 0,\\ 0, & n \neq 0,\end{cases}$$

for all $i \in \mathbb{Z}$, and via these isomorphisms, $\Gamma \simeq E_\Lambda(T,s)$ as graded algebras.

Proof. If we have an equivalence $\mathcal{D}\operatorname{Gr}\Gamma \to \mathcal{D}\operatorname{Gr}\Lambda$ and Γ is sent to T, then T must be compact. Since T is compact, it is isomorphic in the derived category to a bounded complex of finitely generated projective modules. This means that $\operatorname{pd} T < \infty$ and $T \in \operatorname{fgsyz}\Lambda$. The rest follows from Proposition 5.3. ∎

How does this apply to Koszul algebras? Obviously in this case we should choose $T = \Lambda_0$. We have $\Lambda_0 \in \operatorname{fgsyz}\Lambda$, so condition (a) in Proposition 6.1 is satisfied. The right way to choose s in Koszul-like situations is to let $s = \langle -1\rangle[-1]$. Obviously for any set $\mathcal{X}$ of objects in $\mathcal{D}\operatorname{Gr}\Lambda$ we have $\operatorname{tria}(\mathcal{X}, \langle -1\rangle[-1]) = \operatorname{tria}(\mathcal{X}, \langle 1\rangle)$. Since

$$\operatorname{Ext}^n_{\operatorname{Gr}\Lambda}(\Lambda_0, \Lambda_0\langle i\rangle) = \operatorname{Hom}_{\mathcal{D}\operatorname{Gr}\Lambda}(\Lambda_0, \Lambda_0\langle i\rangle[n]) = 0$$

when $n \neq i$, we have $\operatorname{Hom}_{\mathcal{D}\operatorname{Gr}\Lambda}(\Lambda_0\langle -i\rangle[-i], \Lambda_0[n-i]) = 0$ whenever $n-i \neq 0$. So by choosing $s = \langle -1\rangle[-1]$, condition (d) is satisfied.

But in general conditions (b) and (c) are not satisfied. For each $i \in \mathbb{Z}$, we have $\dim_k(\bigoplus_{n\in\mathbb{Z}} H^n(s^i\Lambda_0)) = \dim_k(H^i(s^i\Lambda_0)) = \dim_k \Lambda_0 < \infty$ for all $i \in \mathbb{Z}$. But this means that if $\dim_k \Lambda = \infty$ we cannot have $\Lambda \in \operatorname{trias}\{s^i(\Lambda_0) \mid i \in \mathbb{Z}\} \subseteq \mathcal{D}\operatorname{Gr}\Lambda$, so condition (c) in Proposition 6.1 is not met. If $\dim_k \Lambda < \infty$, then $\operatorname{gldim}\Lambda = \operatorname{pd}\Lambda_0$, and condition (b) will not be satisfied when $\operatorname{gldim}\Lambda = \infty$.

So the pair $(\Lambda_0, \langle -1\rangle[-1])$ will give rise to a derived equivalence (meaning that the unbounded derived categories are equivalent) between Λ and its Koszul dual algebra $\Gamma \simeq \bigoplus_{n\geq 0} \operatorname{Ext}^n_\Lambda(\Lambda_0, \Lambda_0)^{\mathrm{op}}$ if and only if Λ is a finite-dimensional algebra of finite global dimension. If Λ is an arbitrary Koszul algebra we can still use Proposition 5.2 and get equivalences on certain subcategories of the derived categories. Here it is better to use the second method to codify Λ and Γ as categories. For simplicity we assume that Λ (and therefore Γ) is basic. Let $1_\Gamma = e_1 + \cdots + e_r$ be a decomposition of the identity of Γ into a sum of primitive orthogonal idempotents. Let $\Gamma = Q_1 \oplus \cdots \oplus Q_r$ be the corresponding decomposition of Γ into a direct sum of indecomposable graded projective Γ-modules. We also have a corresponding decomposition $\Lambda_0 = S_1 \oplus \cdots \oplus S_r$ of Λ_0 into a direct sum of simple Λ-modules. We codify Γ as a category $\mathcal{B}$ having the set $\{1, \ldots, r\} \times \mathbb{Z}$ as objects. Then $\mathcal{B}^{\mathrm{op}}$ is equivalent to the full subcategory $\{S_i\langle -j\rangle[-j] \mid 1 \leq i \leq r,\, j \in \mathbb{Z}\}$ of $\mathcal{D}\operatorname{Gr}\Lambda$. If M is an object in the derived category of a graded algebra, we define $\operatorname{triagr} M$ to be the smallest full triangulated subcategory of the derived category containing all the summands of M and closed under graded shift. From Proposition 5.2 we get the following diagram:

$$\begin{tikzcd}
\operatorname{triagr}\Gamma \arrow[r, "\sim"] \arrow[d, hook] & \operatorname{triagr}\Lambda_0 \arrow[d, hook] \\
\mathcal{D}\operatorname{Gr}\Gamma \arrow[r, "X\otimes^L_{\operatorname{Gr}\Gamma}-"] & \mathcal{D}\operatorname{Gr}\Lambda
\end{tikzcd}$$

It is possible to extend this equivalence to larger subcategories; we discuss this in the next section.

When Λ is artinian and Γ is noetherian, then $\operatorname{triagr}\Lambda_0 \simeq \mathcal{D}^b \operatorname{gr}\Lambda$ and $\operatorname{triagr}\Gamma \simeq \mathcal{D}^b \operatorname{gr}\Gamma$. In this way we get Theorem 2.3.

Motivated by the (classical) Koszul case, we make the following definition in the case of $\Lambda = \bigoplus_{n\geq 0} \Lambda_n$ a graded k-algebra with $\dim_k \Lambda_n < \infty$ for all $n \geq 0$, but with Λ_0 not necessarily semisimple. Let T be a finitely generated Λ-module concentrated in degree 0. We say that T is a *graded self-orthogonal module* if $\operatorname{Ext}^i_{\operatorname{Gr}\Lambda}(T, T\langle j\rangle) = 0$ when $j \neq i$. If T is in $\operatorname{fgsyz}\Lambda$, then this is equivalent to saying that T satisfies condition (d) in Proposition 6.1 with $s = \langle -1\rangle[-1]$. In this case (when T is graded self-orthogonal and in $\operatorname{fgsyz}\Lambda$) we have

$$\bigoplus_{n\geq 0} \operatorname{Ext}^n_\Lambda(T, T)^{\mathrm{op}} \simeq \bigoplus_{n\geq 0} \operatorname{Ext}^n_{\operatorname{Gr}\Lambda}(T, T\langle n\rangle)^{\mathrm{op}}.$$

We do not assume that T is in $\operatorname{fgsyz}\Lambda$, so we are not certain that the above isomorphism holds. In the following we let always Γ denote the algebra defined by graded Ext. So if T is a graded self-orthogonal Λ-module, we let

$$\Gamma = \bigoplus_{n\geq 0} \operatorname{Ext}^n_{\operatorname{Gr}\Lambda}(T, T\langle n\rangle)^{\mathrm{op}}. \tag{6.1}$$

Our motivating example of a graded self-orthogonal module is the module Λ_0 in the case Λ is a (classical) Koszul algebra. In this case we saw that $\operatorname{triagr}\Gamma \simeq \operatorname{triagr}\Lambda_0$. The same holds for all graded self-orthogonal modules. Let $\Lambda = \bigoplus_{n\geq 0}\Lambda_n$ be a graded k-algebra, and let $T = T_0\oplus\cdots\oplus T_r$ be a graded self-orthogonal Λ-module. Let Γ be given by (6.1). Then $\operatorname{triagr}\Gamma \simeq \operatorname{triagr} T$. The equivalence is given by a functor $X \otimes^L_{\operatorname{Gr}\Gamma} -$, where X is a complex of Λ-Γ-bimodules. The functor $X\otimes^L_{\operatorname{Gr}\Gamma} -$ may induce an equivalence on larger subcategories of the derived categories. In the next section we will be interested in the case when $X\otimes^L_{\operatorname{Gr}\Gamma} -$ induces an equivalence whose image contains $D\Lambda$.

Since T can be viewed as a right Γ-module, we can use T (really the tilting family $\{T\langle -i\rangle[-i] \mid i\in\mathbb{Z}\}$) to define X. First define the complex of Λ-Γ-bimodules $\widetilde{X}$ by

$$\widetilde{X}^i_{jk} = \begin{cases} T & \text{when } i=-j=k,\\ 0 & \text{else.}\end{cases}$$

We let X be a projective Λ-Γ-bimodule resolution of $\widetilde{X}$. From now on we denote the functor $X\otimes^L_{\operatorname{Gr}\Gamma} -\colon \mathcal{D}\operatorname{Gr}\Gamma \to \mathcal{D}\operatorname{Gr}\Lambda$ by F. We denote its right adjoint functor $R\operatorname{Hom}_{\operatorname{Gr}\Lambda}(X,-)\colon \mathcal{D}\operatorname{Gr}\Lambda\to\mathcal{D}\operatorname{Gr}\Gamma$ by G.

7. CoKoszul modules and algebras. In this section, which is the most important of the paper, we define what we feel is the natural generalisation of the classical Koszul algebras. We find it more convenient to develop a coKoszul version of the theory. We compare our new definition with the T-Koszul algebras from Section 3 and see that our definition contains this class. We prove that the basic theorems for classical Koszul algebras also hold in our setting.

Let $\Lambda = \bigoplus_{n\geq 0}\Lambda_n$ be a graded algebra, and let T be a graded self-orthogonal module. We suppose that $\dim_k \Lambda_i < \infty$ for all $i\geq 0$. Let Γ be given by (6.1). We have seen that there is a functor $F = X\otimes^L_{\operatorname{Gr}\Gamma} -\colon \mathcal{D}\operatorname{Gr}\Gamma\to\mathcal{D}\operatorname{Gr}\Lambda$, where X is a complex of Λ-Γ-bimodules, which restricts to an equivalence $X\otimes^L_{\operatorname{Gr}\Gamma} -\colon \operatorname{triagr}\Gamma\simeq\operatorname{triagr}T$. The inverse of this equivalence is given by $G = R\operatorname{Hom}_{\operatorname{Gr}\Lambda}(X,-)\colon \operatorname{triagr}T\to\operatorname{triagr}\Gamma$. We have the diagram

$$\begin{tikzcd} \operatorname{triagr} T \arrow[r, "\sim"] \arrow[d, hook] & \operatorname{triagr}\Gamma \arrow[d, hook] \\ \mathcal{D}\operatorname{Gr}\Lambda \arrow[r, "G"] & \mathcal{D}\operatorname{Gr}\Gamma \end{tikzcd}$$

As we have seen, the complex X of bimodules can be constructed in such a way that for each $i\in\mathbb{Z}$, $X_{*,i}$ is quasi-isomorphic to $T\langle -i\rangle[-i]$ as a complex of Λ-modules. This means that the functors $R\operatorname{Hom}(X_{*,i},-)\colon \mathcal{D}\operatorname{Gr}\Lambda\to$

$\mathcal{D}(\operatorname{Mod} k)$ and $R\operatorname{Hom}(T\langle -i\rangle[-i], -)\colon \mathcal{D}\operatorname{Gr}\Lambda \to \mathcal{D}(\operatorname{Mod} k)$ are naturally isomorphic. From this we see that for all graded Λ-modules M we have

$$(H^j G(M))_i \simeq H^j R\operatorname{Hom}_{\mathcal{D}\operatorname{Gr}\Lambda}(T\langle -i\rangle[-i], M) \simeq \operatorname{Ext}^{i+j}_{\operatorname{Gr}\Lambda}(T, M\langle i\rangle).$$

We are interested in knowing which of the modules in $\mathcal{D}\operatorname{Gr}\Lambda$ (viewed as stalk complexes concentrated in degree 0) are sent by G to modules in $\mathcal{D}\operatorname{Gr}\Gamma$. From the above we see that the answer is exactly the modules M with the property that $\operatorname{Ext}^{i+j}_{\operatorname{Gr}\Lambda}(T, M\langle i\rangle) = 0$ for all $i \in \mathbb{Z}$ when $j \neq 0$.

We now return to the problem of finding from our point of view the right generalisation of Koszul algebras. In the previous section we saw that in the more general case a graded self-orthogonal module T could be a natural substitute for the module Λ_0 from the classical Koszul case. But we need to put further conditions on Λ and T to get theorems of the same strength as in the classical Koszul case. One of the things we would like to have is that Γ contains enough information to reconstruct Λ. Then in some sense T cannot be too small. One way to ensure the reconstruction property is to demand that $D\Lambda$ should be in $\operatorname{triagr} T$. This is asking a bit too much though, because this is not always true even in the classical Koszul case. What we need is that $D\Lambda$ is in *some* subcategory of $\mathcal{D}\operatorname{Gr}\Lambda$ which is equivalent under G to some subcategory of $\mathcal{D}\operatorname{Gr}\Gamma$. This is the main idea; the details follow below.

DEFINITION 7.1. Let $\Lambda = \bigoplus_{i\geq 0}\Lambda_i$ be a graded algebra with $\dim_k \Lambda_i < \infty$ for all $i \geq 0$ and let T be a graded self-orthogonal Λ-module. We say that a graded Λ-module M is a *coKoszul* module (with respect to T) if:

(1) M is finitely cogenerated in degree 0.
(2) GM is a module, or equivalently $\operatorname{Ext}^i_{\operatorname{Gr}\Lambda}(T, M\langle j\rangle) = 0$ when $i \neq j$.
(3) GM is generated in degree 0.
(4) The counit map $\phi_M\colon FGM \to M$ is an isomorphism.

Condition (3) means that

$$\operatorname{Ext}^{i+1}_{\operatorname{Gr}\Lambda}(T, M\langle i+1\rangle) = \operatorname{Ext}^i_{\operatorname{Gr}\Lambda}(T, M\langle i\rangle)\cdot \operatorname{Ext}^1_{\operatorname{Gr}\Lambda}(T, T\langle 1\rangle)$$

for all $i \geq 0$. If M is a module such that conditions (1)–(3) hold, then GM is the graded Γ-module with graded parts

$$(GM)_i = \operatorname{Hom}_{\operatorname{Gr}\Lambda}(T, M)\cdot \operatorname{Ext}^i_{\operatorname{Gr}\Lambda}(T, T\langle i\rangle).$$

In that case condition (4) holds if and only if

$$\operatorname{Tor}_i^{\operatorname{Gr}\Gamma}(T\langle -j\rangle, GM) = 0 \qquad \text{when } i \neq j$$

and the natural maps $(\phi_M)_i\colon \operatorname{Tor}_i^{\operatorname{Gr}\Gamma}(T\langle -i\rangle, GM) \to M_{-i}$ are isomorphisms for all $i \geq 0$.

We denote by $\mathcal{C}_T(\Lambda)$ the full subcategory of $\operatorname{Gr}\Lambda$ consisting of the coKoszul Λ-modules.

PROPOSITION 7.2. *The category $\mathcal{C}_T(\Lambda)$ is closed under extensions and direct summands.*

Proof. Each of the four conditions can easily be checked to be closed under extensions and direct summands. ∎

The graded self-orthogonal module T is itself a coKoszul module. It is obvious that the module $D\Lambda$ satisfies the three first conditions. In keeping with the classical situation we make the following definition.

DEFINITION 7.3. We say that an algebra Λ is a *coKoszul algebra* (with respect to T) if $D\Lambda$ is a coKoszul module. If Λ is a coKoszul algebra, we say that Λ^{op} is a *Koszul algebra*.

In other words, a graded algebra is coKoszul if and only if T is a graded self-orthogonal module and the map $\phi_{D\Lambda}\colon FG(D\Lambda) \to D\Lambda$ is an isomorphism.

The next theorem shows that our definitions are good, since the basic theorems for classical Koszul algebras also hold in this new setting.

THEOREM 7.4. *Let Λ, T and Γ be as above. Suppose Λ is a coKoszul algebra. Then*

(a) Γ^{op} *is a coKoszul algebra with respect to* ${}_{\Gamma^{\mathrm{op}}}T$.
(b) $\Lambda \simeq \bigoplus_{i\geq 0} \operatorname{Ext}^i_{\operatorname{Gr}\Gamma^{\mathrm{op}}}(T, T\langle i\rangle)$ *as rings.*
(c) *There is a duality* $\mathcal{C}_T(\Lambda) \to \mathcal{C}_T(\Gamma^{\mathrm{op}})$.

Proof. (b) For each graded part with $i \geq 0$ we have

$$\begin{aligned}\Lambda_i &\simeq \operatorname{Hom}_{\mathcal{D}\operatorname{Gr}\Lambda}(D\Lambda, D\Lambda\langle -i\rangle) \simeq \operatorname{Hom}_{\mathcal{D}\operatorname{Gr}\Gamma}(DT, DT\langle i\rangle[i])\\ &\simeq \operatorname{Ext}^i_{\operatorname{Gr}\Gamma}(DT\langle -i\rangle, DT) \simeq \operatorname{Ext}^i_{\operatorname{Gr}\Gamma^{\mathrm{op}}}(T, T\langle i\rangle).\end{aligned}$$

The isomorphisms respect the multiplication. (In the derived categories multiplication is done by taking the necessary shifts and then composing the maps in the opposite order.)

(a) and (c). We have

$$\begin{aligned}\operatorname{Ext}^i_{\operatorname{Gr}\Gamma^{\mathrm{op}}}(T, T\langle j\rangle) &\simeq \operatorname{Ext}^i_{\operatorname{Gr}\Gamma}(DT\langle -j\rangle, DT) \simeq \operatorname{Hom}_{\mathcal{D}\operatorname{Gr}\Gamma}(DT, DT\langle j\rangle[i])\\ &\simeq \operatorname{Hom}_{\mathcal{D}\operatorname{Gr}\Lambda}(D\Lambda, D\Lambda\langle -j\rangle[i-j]) = 0 \quad \text{when } i \neq j,\end{aligned}$$

so T is self-orthogonal as a Γ^{op}-module.

Denote by $\widehat{G}$ the functor $R\operatorname{Hom}_{\operatorname{Gr}\Gamma^{\mathrm{op}}}(\widehat{X}, -)\colon \mathcal{D}\operatorname{Gr}\Gamma^{op} \to \mathcal{D}\operatorname{Gr}\Lambda^{\mathrm{op}}$ induced by T, where $\widehat{X}$ is a complex of bigraded Γ^{op}-Λ^{op}-bimodules quasi-isomorphic to T as a complex of left Γ^{op}-modules. Denote the right adjoint functor $\widehat{X} \otimes^L_{\operatorname{Gr}\Lambda^{\mathrm{op}}} -\colon \mathcal{D}\operatorname{Gr}\Lambda^{\mathrm{op}} \to \mathcal{D}\operatorname{Gr}\Gamma^{\mathrm{op}}$ by $\widehat{F}$.

If M is in $\mathcal{C}_T(\Lambda)$, then functorially in M we have

$$\operatorname{Tor}_i^{\operatorname{Gr}\Lambda^{\mathrm{op}}}(T\langle -i\rangle, DM) \simeq D\operatorname{Ext}^i_{\operatorname{Gr}\Lambda}(T\langle -i\rangle, M)$$

for all $i \geq 0$. (More generally, this formula holds for all locally finite modules M bounded above.) This induces a functorial isomorphism $\widehat{F}D(M) \simeq DG(M)$ for modules M in $\mathcal{C}_T(\Lambda)$. So $\widehat{F}D$ restricts to a duality $\widehat{F}D \colon \mathcal{C}_T(\Lambda) \to DG(\mathcal{C}_T(\Lambda))$. Therefore $\widehat{F} \colon D(\mathcal{C}_T(\Lambda)) \to DG(\mathcal{C}_T(\Lambda))$ is an equivalence with the adjoint functor $\widehat{G} \colon DG(\mathcal{C}_T(\Lambda)) \to D(\mathcal{C}_T(\Lambda))$ as an inverse. Since $\widehat{G}$ is fully faithful on $DG(\mathcal{C}_T(\Lambda))$, the counit map $\psi_M \colon \widehat{F}\widehat{G}M \to M$ is an isomorphism for all M in $DG(\mathcal{C}_T(\Lambda))$. This shows that the modules in $DG(\mathcal{C}_T(\Lambda))$ satisfy condition (4) for being a coKoszul Γ^{op}-module. The other three conditions are easily checked to be true, so $DG(\mathcal{C}_T(\Lambda)) \subseteq \mathcal{C}_T(\Gamma^{\mathrm{op}})$.

Since $D\Gamma^{\mathrm{op}} \simeq DG(T)$ is in $DG(\mathcal{C}_T(\Lambda))$, we deduce that $D\Gamma^{\mathrm{op}}$ is a coKoszul Γ^{op}-module. So Γ^{op} is a coKoszul algebra with respect to ${}_{\Gamma^{\mathrm{op}}}T$.

Changing the roles of Λ and Γ^{op} we get

$$\mathcal{C}_T(\Gamma^{\mathrm{op}}) = DGD\widehat{G}(\mathcal{C}_T(\Gamma^{\mathrm{op}})) \subseteq DG(\mathcal{C}_T(\Lambda)) \subseteq \mathcal{C}_T(\Gamma^{\mathrm{op}})$$

so $DG(\mathcal{C}_T(\Lambda)) = \mathcal{C}_T(\Gamma^{\mathrm{op}})$. Since DG is a duality onto its image, there is a duality $DG \colon \mathcal{C}_T(\Lambda) \to \mathcal{C}_T(\Gamma^{\mathrm{op}})$. ∎

In the next example the self-orthogonal module T does not give an equivalence between $\mathcal{D}\,\mathrm{Gr}\,\Lambda$ and $\mathcal{D}\,\mathrm{Gr}\,\Gamma$, but still we get an equivalence on the bounded derived categories.

Example 7.5. Let Λ be the path algebra given by the quiver

$$\alpha \circlearrowright 1 \xrightarrow{\beta} 2$$

where $\deg(\alpha) = 1$ and $\deg(\beta) = 0$ and with relations $\beta\alpha = 0 = \alpha^2$. This is a finite-dimensional k-algebra, so it is artinian. Let $T = S_1 \oplus I_2$. (Here I_2 denotes the injective module corresponding to vertex 2.) Then T is a graded self-orthogonal module. Since T is finitely generated and Λ is artinian, T is in $\mathrm{fgsyz}\,\Lambda$. Let

$$\Gamma = \bigoplus_{n \geq 0} \mathrm{Ext}^n_\Lambda(T, T)^{\mathrm{op}} \simeq \bigoplus_{n \geq 0} \mathrm{Ext}^n_{\mathrm{Gr}\,\Lambda}(T, T\langle n\rangle)^{\mathrm{op}}.$$

Then Γ is the path algebra given by the quiver

$$\alpha^* \circlearrowright 1^* \xrightarrow{\beta^*} 2^*$$

where $\deg(\alpha^*) = 1$ and $\deg(\beta^*) = 0$ and with the relation $\beta^*\alpha^* = 0$. This is a noetherian k-algebra with finite global dimension.

Since $\mathrm{pd}_\Lambda T = \infty$, the module T will not give rise to an unbounded derived equivalence between Λ and Γ. But still Proposition 5.2 gives $\mathrm{triagr}\,T \simeq \mathrm{triagr}\,\Gamma$. In this example we have $\mathrm{triagr}\,T \simeq \mathcal{D}^b\,\mathrm{gr}\,\Lambda$ and $\mathrm{triagr}\,\Gamma \simeq \mathcal{D}^b\,\mathrm{gr}\,\Gamma$. So there is an equivalence $\mathcal{D}^b\,\mathrm{gr}\,\Lambda \simeq \mathcal{D}^b\,\mathrm{gr}\,\Gamma$. Since $D\Lambda$ is in $\mathrm{triagr}\,T$, the algebra Λ is a coKoszul algebra according to our definition.

8. Comparison with T-Koszul algebras. Our aim in this section is to show that the T-Koszul algebras of Green, Reiten and Solberg are Koszul algebras (meaning that the opposite algebras are coKoszul) according to our definitions. We also show that our definitions specialise nicely to classical Koszul algebras and to Wakamatsu tilting theory.

We use the fact that the category $\mathcal{C}_T(\Lambda)$ is closed under certain homotopy colimits. Homotopy colimits are defined as follows. Let

$$X_0 \xrightarrow{g_1} X_1 \xrightarrow{g_2} X_2 \xrightarrow{g_3} \cdots$$

be a sequence of objects and morphisms in a triangulated category $\mathcal{D}$ having arbitrary coproducts. Then the *homotopy colimit* of the sequence, denoted $\operatorname{hocolim} X_i$, is by definition given, up to noncanonical isomorphism, by the triangle

$$\coprod_{i=0}^{\infty} X_i \xrightarrow{\Phi} \coprod_{i=0}^{\infty} X_i \to \operatorname{hocolim} X_i \to \coprod_{i=0}^{\infty} X_i[1]$$

where the map Φ is given by the infinite matrix

$$\begin{pmatrix} 1_{X_0} & 0 & 0 & 0 & \dots \\ -g_1 & 1_{X_1} & 0 & 0 & \dots \\ 0 & -g_2 & 1_{X_2} & 0 & \dots \\ 0 & 0 & -g_3 & 1_{X_3} & \dots \\ \vdots & \vdots & \vdots & \vdots & \end{pmatrix}.$$

Suppose the triangulated category $\mathcal{D}$ is the derived category of an abelian category $\mathcal{A}$ satisfying the axiom (AB5). We recall that the axiom (AB5) states that $\mathcal{A}$ has arbitrary coproducts, and that filtered colimits of exact sequences are exact. As commented in [BN], in such a situation we have $H^i(\operatorname{hocolim} X_j) \simeq \operatorname{colim} H^i(X_j)$ for all $i \in \mathbb{Z}$. The category $\operatorname{Gr}\Lambda$ satisfies (AB5).

PROPOSITION 8.1. *Let Λ^{op} be a DT-Koszul algebra as defined in Section 3. Then Λ is a coKoszul algebra with respect to T.*

Proof. First of all T is graded self-orthogonal by [GRS, V.5.12].

Let $\mathcal{K}_{DT}(\Lambda^{\mathrm{op}})$ be the category of Koszul modules in the T-Koszul algebra sense. Let $\mathcal{L}_T(\Lambda) = D(\mathcal{K}_{DT}(\Lambda^{\mathrm{op}}))$ be the corresponding dual category of graded Λ-modules. Since Λ^{op} is in $\mathcal{K}_{DT}(\Lambda^{\mathrm{op}})$, we see that $D\Lambda$ is in $\mathcal{L}_T(\Lambda)$. We want to show that $\mathcal{L}_T(\Lambda)$ is a subcategory of $\mathcal{C}_T(\Lambda)$ and therefore Λ is a coKoszul algebra.

By definition the modules in $\mathcal{L}_T(\Lambda)$ satisfy the first three conditions for being in $\mathcal{C}_T(\Lambda)$. We only have to show that they satisfy the fourth condition.

If M is an object of $\mathcal{C}_T(\Lambda)$ we know from dual facts about $\mathcal{K}_{DT}(\Lambda^{\mathrm{op}})$ that M_0 has a minimal right $\operatorname{add} T$-approximation $M_T \to M_0$. We define $\operatorname{cotr}_T(M)$ to be the kernel of this map. We know that $\mathcal{L}_T(\Lambda)$ is closed under the operations $(-)_0$, $(-)/(-)_0\langle 1\rangle$ and $\operatorname{cotr}_T(-)$.

Let $T^\perp$ denote the full subcategory of $\operatorname{mod} \Lambda_0$ (finitely generated Λ-modules concentrated in degree 0) whose objects are the modules M with $\operatorname{Ext}^i_{\Lambda_0}(T, M) = 0$ for $i > 0$. Let $\mathcal{Y}_T$ denote the full subcategory of $T^\perp$ whose objects are the modules M for which there exists a resolution in $\operatorname{add} T$, that is, an exact sequence

$$\cdots \to T^2 \xrightarrow{f_1} T^1 \xrightarrow{f_0} T^0 \to M \to 0$$

with T^i in $\operatorname{add} T$ and $\operatorname{Ext}^1_{\Lambda_0}(T, \operatorname{coker} f_i) = 0$ for all $i > 0$.

Before we continue we explain the strategy for the (rather technical) rest of the proof. We know that T satisfies condition (4). Therefore by the Five-Lemma all objects of $\operatorname{tria} T$ satisfy condition (4), even if they do not satisfy the other conditions for being in $\mathcal{C}_T(\Lambda)$. Our first step is to show that the objects of $\mathcal{L}_T(\Lambda) \cap \mathcal{Y}_T$ satisfy condition (4). These objects are not necessarily in $\operatorname{tria} T$, but they are certain homotopy colimits of such objects, and the given homotopy colimits preserve condition (4). Continuing we find that the objects of $\mathcal{T} = \operatorname{triagr}(\mathcal{L}_T(\Lambda) \cap \mathcal{Y}_T)$ satisfy condition (4), but again they do not have to be in $\mathcal{C}_T(\Lambda)$. The last step is to show that the objects of $\mathcal{L}_T(\Lambda)$ are homotopy colimits of objects of $\mathcal{T}$, and that the given homotopy colimits preserve condition (4).

First we prove that the category $\mathcal{L}_T(\Lambda) \cap \mathcal{Y}_T$ is a subcategory of $\mathcal{C}_T(\Lambda)$.

Let $M \in \mathcal{L}_T(\Lambda) \cap \mathcal{Y}_T$. Let

$$\begin{aligned}
E_0 &: 0 \to 0 \to 0 \to T^0 \to 0,\\
E_1 &: 0 \to 0 \to T^1 \to T^0 \to 0,\\
E_2 &: 0 \to T^2 \to T^1 \to T^0 \to 0,\\
&\qquad\vdots\\
E_j &: 0 \to T^j \to \cdots \to T^0 \to 0,\\
&\qquad\vdots
\end{aligned}$$

Then each $E_j \in \operatorname{triagr} T$ and $\operatorname{hocolim} E_j \simeq M$.

The object $G(M)$ has homology concentrated in degree 0 by assumption. For each $i \geq 0$ we have $\operatorname{cotr}(\operatorname{coker} f_i) \simeq \operatorname{coker} f_{i+1}$. Since $M = \operatorname{coker} f_0$ and $\mathcal{C}_T(\Lambda)$ is closed under the operation cotr, we get $\operatorname{coker} f_i \in \mathcal{C}_T(\Lambda)$ for all $i \geq 0$. Therefore $G(\operatorname{coker} f_i)$ has homology concentrated in degree 0 for all $i \geq 0$.

Consider the triangle

$$\operatorname{coker} f_{i+1}[i] \to E_i \to M \to \operatorname{coker} f_{i+1}[i+1]$$

where $i > 0$. Applying G we get a triangle

$$G(\operatorname{coker} f_{i+1}[i]) \to G(E_i) \to G(M) \to G(\operatorname{coker} f_{i+1}[i+1]).$$

We know that $G(M)$ has homology concentrated in degree 0 and that $G(\operatorname{coker} f_{i+1}[i])$ has nonzero homology only in degree $-i$. Using H^0 on this triangle we get an exact sequence

$$0 \to H^0G(E_i) \to H^0G(M) \to 0.$$

Using H^{-i} we get a sequence

$$0 \to H^{-i}G(\operatorname{coker} f_{i+1}[i]) \to H^{-i}G(E_i) \to 0.$$

We also see that $G(E_i)$ cannot have nonzero homology in any other degree.

For each $i \neq 0$ there exists a number n_i such that $H^iG(E_j) = 0$ for all $j \geq n_i$. So $H^i(\operatorname{hocolim} G(E_j)) \simeq \operatorname{colim} H^iG(E_j) = 0$ for all $i \in \mathbb{Z}$. We see that $\operatorname{hocolim} G(E_j)$ only has nonzero homology in degree 0. Since $H^0G(E_j) \simeq H^0G(M)$ for all $j > 0$, the system $\operatorname{colim} H^0G(E_j)$ is constant. Applying H^0G to the triangle

$$T^{j+1}[j] \to E_j \to E_{j+1} \to T^{j+1}[j+1]$$

we get an exact sequence

$$0 \to H^0G(E_j) \to H^0G(E_{j+1}) \to 0.$$

So $H^0(\operatorname{hocolim} G(E_j)) \simeq H^0G(M)$. Since 0 is the only degree where those objects have nonzero homology, we have

$$\operatorname{hocolim} G(E_j) \simeq G(\operatorname{hocolim} E_j) \simeq GM.$$

The functor F commutes with direct sums, so $FG(M) \simeq FG \operatorname{hocolim} E_j \simeq \operatorname{hocolim} FG(E_j)$. Since each E_j is in $\operatorname{triagr} T$, we have $\operatorname{hocolim} FG(E_j) \simeq \operatorname{hocolim} E_j \simeq M$. So for all $M \in \mathcal{L}_T(\Lambda) \cap \mathcal{Y}_T$ we have $FG(M) \simeq M$. This is just an abstract isomorphism which we have not proven to be functorial. We want to show that the natural map $\phi_M \colon FG(M) \to M$ coming from the counit $FG \mapsto \mathrm{id}$ is an isomorphism.

Consider again the triangle

$$\operatorname{coker} f_{i+1}[i] \to E_i \to M \to \operatorname{coker} f_{i+1}[i+1]$$

in $\mathcal{D}\operatorname{Gr}\Lambda$ with $i > 0$. Abstractly we have isomorphisms $FG(M) \simeq M$ and $FG(\operatorname{coker} f_{i+1}) \simeq \operatorname{coker} f_{i+1}$, since these objects are in $\mathcal{L}_T(\Lambda) \cap \mathcal{Y}_T$. The transformation $FG \mapsto \mathrm{id}$ gives the following diagram:

$$\begin{tikzcd}
FG(\operatorname{coker} f_{i+1}[i]) \arrow[r] \arrow[d] & FG(E_i) \arrow[r] \arrow[d, "\wr"'] & FG(M) \arrow[r] \arrow[d, "\phi_M"] & FG(\operatorname{coker} f_{i+1}[i+1]) \arrow[d] \\
\operatorname{coker} f_{i+1}[i] \arrow[r] & E_i \arrow[r] & M \arrow[r] & \operatorname{coker} f_{i+1}[i+1]
\end{tikzcd}$$

The outer terms do not have any nonzero homology in degree 0, so using H^0 we get a commutative square

$$\begin{tikzcd} H^0FG(E_i) \arrow[r, "\sim"] \arrow[d, "\wr"'] & H^0FG(M) \arrow[d, "H^0\phi_M"] \\ H^0E_i \arrow[r, "\sim"] & H^0M \end{tikzcd}$$

We see that $H^0\phi_M \colon H^0FG(M) \to H^0M$ is an isomorphism. Since $FG(M) \simeq M$ only has nonzero homology in degree 0, we find that $\phi_M \colon FG(M) \to M$ is an isomorphism in $\mathcal{D}\,\mathrm{Gr}\,\Lambda$.

We deduce that the modules in $\mathcal{L}_T(\Lambda) \cap \mathcal{Y}_T$ satisfy condition (4) and therefore $\mathcal{L}_T(\Lambda) \cap \mathcal{Y}_T$ is a subcategory of $\mathcal{C}_T(\Lambda)$.

Let $\mathcal{T}$ denote the smallest triangulated full subcategory of $\mathcal{D}\,\mathrm{Gr}\,\Lambda$ containing $\mathcal{L}_T(\Lambda) \cap \mathcal{Y}_T$ and closed under graded shifts. (In other words, $\mathcal{T} = \mathrm{triagr}(\mathcal{L}_T(\Lambda) \cap \mathcal{Y}_T)$.)

Now let $N = \bigoplus_{i \le 0} N_i$ be an arbitrary graded Λ-module in $\mathcal{L}_T(\Lambda)$. Since $\mathcal{L}_T(\Lambda)$ is closed under $(-)/(-)_0\langle 1\rangle$, we find that $N/N_{\ge -j}\langle j+1\rangle$ is in $\mathcal{L}_T(\Lambda)$ for all $j \ge 0$. Since $(N/N_{\ge -j}\langle j+1\rangle)_0 \simeq N_{-j-1}$, we have $N_i \in \mathcal{L}_T(\Lambda) \cap \mathcal{Y}_T$ for all $i \le 0$. Consider the triangle

$$N_0 \to N_{\ge -1} \to N_{-1} \to N_0[1].$$

Since $N_0 \in \mathcal{T}$ and $N_{-1} \in \mathcal{T}$, we must have $N_{\ge -1} \in \mathcal{T}$. If $N_{\ge -j} \in \mathcal{T}$, then the triangle

$$N_{\ge -j} \to N_{\ge -j-1} \to N_{-j-1} \to N_{\ge -j}[1]$$

shows that $N_{\ge -j-1} \in \mathcal{T}$. By induction, $N_{\ge -n} \in \mathcal{T}$ for all $n \ge 0$.

The module N has a filtration

$$N_0 \subset N_{\ge -1} \subset \cdots \subset N_{\ge -j} \subset \cdots \subset N.$$

Therefore $\operatorname{hocolim} N_{\ge -j} = N$.

Consider the triangle

$$(N/N_{\ge -j})[-1] \to N_{\ge -j} \to N \to N/N_{\ge -j}.$$

We use G and get a triangle

$$G(N/N_{\ge -j})[-1] \to G(N_{\ge -j}) \to G(N) \to G(N/N_{\ge -j}).$$

Since $N, N/N_{\ge -j}\langle j+1\rangle \in \mathcal{L}_T(\Lambda)$, it follows that $G(N)$ has nonzero homology only in degree 0 while $G(N/N_{\ge -j})[-1]$ has nonzero homology only in degree $-j$. We get isomorphisms

$$H^0(G(N_{\ge -j})) \simeq H^0G(N), \qquad H^{-j}(G(N/N_{\ge -j})[-1]) \simeq H^{-j}(G(N_{\ge -j})).$$

We also see that $G(N_{\geq -j})$ cannot have nonzero homology in any other degree. So $G(N_{\geq -j})$ has nonzero homology only in degrees 0 and $-j$.

We want to find the homotopy colimit of the sequence

$$G(N_0) \to G(N_{\geq -1}) \to \cdots \to G(N_{\geq -j}) \to \cdots .$$

It is clear that $\operatorname{colim} H^i G(N_{\geq -j}) = 0$ for all $i \neq 0$. To find out what happens in degree 0, we consider the triangle

$$N_{-j-1}\langle -j-1\rangle[-1] \to N_{\geq -j} \to N_{\geq -j-1} \to N_{-j-1}\langle -j-1\rangle.$$

Applying $H^0 G$ we get an exact sequence

$$0 \to H^0 G(N_{\geq -j}) \to H^0 G(N_{\geq -j-1}) \to 0.$$

So $\operatorname{colim} H^0 G(N_{\geq -j})$ is a constant system, and we have seen that its value is $H^0 G(N)$. Therefore $\operatorname{hocolim} G(N_{\geq -j}) \simeq G(\operatorname{hocolim}(N_{\geq -j}) \simeq G(N)$. We know that F commutes with sums so $FG(N) \simeq N$. As before we can show that the natural map $\phi_N \colon FG(N) \to N$ is an isomorphism. This is true for all $N \in \mathcal{L}_T(\Lambda)$ and this finishes the proof. ∎

We do not assume that our algebras are generated in degrees 0 and 1. Apart from that, we do not know if our definitions of (co)Koszul algebras are essentially different from those of Green, Reiten and Solberg. But on the level of modules they are different, as the following example illustrates.

Example 8.2. The following example was discussed in [GRS, Example I.4.1]. Let Λ be the path algebra given by the quiver

$$\begin{tikzcd}
1 & 2 \arrow[l, "x"'] & 3 \arrow[l, "y"'] \\
& 4 \arrow[ul, "z"] & 5 \arrow[l, "w"] \arrow[u, "a"]
\end{tikzcd}$$

and the relation $zw = xya$. The arrows x, y, z, w have degree 1, while a has degree 0. Let $T = D\Lambda_0$. Then T is a graded self-orthogonal Λ-module. In fact, T together with $\langle -1\rangle[-1]$ satisfies the conditions of Proposition 6.1 and therefore Λ and $\Gamma = \bigoplus_{n\geq 0} \operatorname{Ext}^n_\Lambda(T,T)^{\mathrm{op}}$ are derived equivalent.

Let M be the module cogenerated in degree 0 with representation

$$\begin{tikzcd}
k & k \arrow[l, "1"'] & 0 \arrow[l, "0"'] \\
& 0 \arrow[ul, "0"] & 0 \arrow[l, "0"] \arrow[u, "0"]
\end{tikzcd}$$

The module $\Omega^{-1}M$ is cogenerated in degrees -1 and -2 and therefore M is not a coKoszul module with the definitions in [GRS]. But since $G(M)$

is generated in degree 0 and M also satisfies the other conditions, it is a coKoszul module according to our definition.

A good thing about the T-Koszul algebras defined in [GRS] is that they specialise nicely to classical Koszul algebras and also to Wakamatsu tilting theory. Our definition also shares this good property, as we show next.

If Λ_0 is semisimple, then Λ_0 is a graded self-orthogonal Λ-module if and only if Λ is a (classical) Koszul algebra. In this case $D\Lambda$ is in $\operatorname{triagr}\Lambda_0$ if and only if $\dim_k \Lambda < \infty$, but $D\Lambda$ is always a coKoszul module. So an algebra with Λ_0 semisimple is coKoszul (with respect to Λ_0) if and only if it is a classical Koszul algebra. (Recall that classical Koszul is self-dual.) If a Λ-module M is a coKoszul module in our sense, then condition (3) says that

$$\operatorname{Ext}^i_{\operatorname{Gr}\Lambda}(\Lambda_0, M\langle j\rangle)) = 0 \quad \text{when } i \neq j.$$

This is equivalent to saying that M has a *co-linear injective resolution* (the definition is dual to that of linear projective resolution). If M has a co-linear injective resolution and Λ is a classical Koszul algebra, then the other conditions are also satisfied. Therefore our definitions coincide with the classical ones.

We next look at what happens in the case $\Lambda = \Lambda_0$. Then graded self-orthogonal just means self-orthogonal. If T is a self-orthogonal Λ-module, we let $\widetilde{G}$ denote the functor $\widetilde{G} = \operatorname{Hom}_\Lambda(T, -)\colon \operatorname{Mod}\Lambda \to \operatorname{Mod}\Gamma$, and let $\widetilde{F}$ denote the functor $\widetilde{F} = T \otimes_\Gamma -\colon \operatorname{Mod}\Gamma \to \operatorname{Mod}\Lambda$. If M is a finitely generated Λ-module, then an $\operatorname{add}T$*-presentation* of M is an exact sequence

$$T_1 \to T_0 \xrightarrow{f} M \to 0$$

with T_0 and T_1 in $\operatorname{add}T$ and f a minimal right $\operatorname{add}T$-approximation.

PROPOSITION 8.3. *Let Λ be a finite-dimensional algebra and let T be a finitely generated self-orthogonal module. Let M be a module in $T^\perp$. Then*

(a) $\widetilde{\phi}_M\colon \widetilde{F}\widetilde{G}(M) \to M$ *is an isomorphism if and only if M has an $\operatorname{add}T$-presentation $T_1 \to T_0 \to M \to 0$.*
(b) $\phi_M\colon FG(M) \to M$ *is an isomorphism if and only if $M \in \mathcal{Y}_T$.*

Proof. (a) We have

$$\widetilde{F}\widetilde{G}(M) = T \otimes_\Gamma \operatorname{Hom}_\Lambda(T, M) \simeq D\operatorname{Hom}_\Gamma(\operatorname{Hom}_{\Lambda^{\mathrm{op}}}(DM, DT), DT),$$

so $D\widetilde{F}\widetilde{G}(M) \simeq \operatorname{Hom}_\Gamma(\operatorname{Hom}_{\Lambda^{\mathrm{op}}}(DM, DT), DT)$. From results in [AS] it follows that $D(\widetilde{\phi}_M)\colon DM \to D\widetilde{F}\widetilde{G}(M)$ is an isomorphism if and only if there is an exact sequence

$$0 \to DM \xrightarrow{h} DT_0 \to DT_1$$

with DT_0 and DT_1 in $\operatorname{add} DT$ and h a minimal left $\operatorname{add} DT$-approximation. The statement follows by duality.

(b) If $\phi_M\colon FG(M) \to M$ is an isomorphism, then in particular $\widetilde{F}\widetilde{G}(M) \simeq M$. From part (a) it follows that M has an $\operatorname{add} T$-presentation

$$T_1 \to T_0 \xrightarrow{f} M \to 0.$$

The fact that f is a minimal right $\operatorname{add} T$-approximation implies that $\operatorname{Ker} f$ is in $T^{\perp}$. Using the Five-Lemma we can prove that $\phi_{\operatorname{Ker} f}\colon FG(\operatorname{Ker} f) \to \operatorname{Ker} f$ is an isomorphism, so $\operatorname{Ker} f$ also has an $\operatorname{add} T$-presentation. Continuing this way we deduce that $M \in \mathcal{Y}_T$.

If M is $\mathcal{Y}_T$, then by [GRS, III.3], DM is a DT-Koszul Λ^{op}-module. From the proof of Proposition 8.1 we conclude that M is a coKoszul module with our definitions and therefore $\phi_M\colon FG(M) \to M$ is an isomorphism. ■

If $\Lambda = \Lambda_0$ and T is a self-orthogonal Λ-module, what are the coKoszul modules? If M is a Λ-module, then conditions (1)–(3) are equivalent to saying that M is in $T^{\perp}$. From the proposition above it follows that M is coKoszul if and only if M is in $\mathcal{Y}_T$. The algebra Λ is coKoszul (with respect to T) if and only if $D\Lambda$ is in $\mathcal{Y}_T$. Since Wakamatsu cotilting is a self-dual concept, this is true if and only if T is a Wakamatsu cotilting module. If T is a Wakamatsu cotilting module, then the duality we get from Theorem 7.4 is already well known. Also here $D\Lambda$ can be a coKoszul module without being in $\operatorname{triagr} T$. This happens for instance when $\operatorname{pd} T < \infty$ and $\operatorname{pd} D\Lambda = \infty$.

In the situation above let $\mathcal{Z}_T$ denote the full subcategory of $T^{\perp}$ consisting of objects M with the property that $\widetilde{\phi}_M\colon \widetilde{F}\widetilde{G}(M) \to M$ is an isomorphism. We have $\mathcal{Y}_T \subseteq \mathcal{Z}_T \subseteq T^{\perp}$. These subcategories are related to the following conjecture.

Conjecture 1 (Wakamatsu Tilting Conjecture). *Let T be a Wakamatsu cotilting module with finite projective dimension. Then T is a tilting module.*

This conjecture is quite strong and implies for instance the Generalised Nakayama Conjecture [AR]. It is possible to reformulate the Wakamatsu Tilting Conjecture in the following way.

Conjecture 2 (Wakamatsu Tilting Conjecture, alternative version). *Let T be a Wakamatsu cotilting module with finite projective dimension. Then $\mathcal{Y}_T = T^{\perp}$.*

Also in the case when $\operatorname{pd} T = \infty$, so T cannot be a tilting module, we do not know any examples where $\mathcal{Y}_T \neq T^{\perp}$. For more information on the Wakamatsu Tilting Conjecture, see [BR, IV.3].

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