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AN EXPLICIT CONSTRUCTION FOR THE HAPPEL FUNCTOR

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Abstract. An easy explicit construction is given for a full and faithful functor from the bounded derived category of modules over an associative algebra A to the stable category of the repetitive algebra of A. This construction simplifies the one given by Happel.

1. Introduction. For basic results on triangulated categories we refer to [7] or [2]. To fix notation, we denote by $\mathrm{C}^{\mathrm{b}}(\mathcal{A})$ the category of bounded differential complexes, by $\mathrm{K}^{\mathrm{b}}(\mathcal{A})$ the homotopy category and by $\mathrm{D}^{\mathrm{b}}(\mathcal{A})$ the derived category of an abelian category $\mathcal{A}$. For an algebra A (over a base field k), we denote by $\operatorname{mod} A$ the category of all finitely generated left modules, and by P_A (resp. I_A) the full subcategory given by the projectives (resp. injectives) in $\operatorname{mod} A$. Further, $\widehat{A}$ denotes the repetitive algebra, and $\underline{\operatorname{mod}}\,\widehat{A}$ the stable module category of $\operatorname{mod}\widehat{A}$.

THEOREM 1 (Happel). *If A is a finite-dimensional algebra, then there is a triangulated, full and faithful functor of triangulated categories $H : \mathrm{D}^{\mathrm{b}}(\operatorname{mod} A) \to \underline{\operatorname{mod}}\,\widehat{A}$, which is also dense if A is of finite global dimension.*

The definition of H in [2] is however rather involved and the proof technical. The proof was considerably shortened in [5], at the expense of explicitness. It was then noted in [3] that the following result from [6] could be used to provide a more direct proof.

PROPOSITION 2 (Rickard). *If Λ is a finite-dimensional selfinjective algebra then there is an equivalence $F : \underline{\operatorname{mod}}\,\Lambda \to \mathrm{D}^{\mathrm{b}}(\operatorname{mod}\Lambda)/\mathrm{K}^{\mathrm{b}}(\mathrm{P}_\Lambda)$ of triangulated categories.*

This result was found first for exterior algebras in [1]. A closer look at the proof reveals that all arguments hold in the case where Λ is the repetitive algebra $\widehat{A}$ for A a finite-dimensional algebra, or more generally a *Frobenius algebra*, that is, a locally bounded k-algebra whose projective and injective modules coincide (see [2]).

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The following is the main result of this paper; we stress at once that the functor $\widetilde{F}$ is given explicitly.

MAIN THEOREM.

(i) *If Λ is a Frobenius algebra then there exists a triangulated functor* $\widetilde{F}$: $\mathrm{D}^{\mathrm{b}}(\mathrm{mod}\,\Lambda) \to \underline{\mathrm{mod}}\,\Lambda$ *such that $F\widetilde{F}$ is isomorphic to the canonical projection* $\pi : \mathrm{D}^{\mathrm{b}}(\mathrm{mod}\,\Lambda) \to \mathrm{D}^{\mathrm{b}}(\mathrm{mod}\,\Lambda)/\mathrm{K}^{\mathrm{b}}(\mathrm{P}_\Lambda)$.

(ii) *If A is a finite-dimensional algebra and $\Lambda = \widehat{A}$, then the composition of $\widetilde{F}$ with the canonical inclusion* $\mathrm{D}^{\mathrm{b}}(\mathrm{mod}\,A) \to \mathrm{D}^{\mathrm{b}}(\mathrm{mod}\,\widehat{A})$ *is triangulated, full and faithful, and also dense in case A is of finite global dimension.*

Part (ii) was already mentioned in [3], using an unspecified quasi-inverse of F. We therefore briefly outline a direct proof.

Since the construction of $\widetilde{F}$ is explicit and can be carried out easily in examples, we start by describing it, although at first sight, it does not seem functorial, nor even well defined. We then give the proof that the construction actually works, and add some comments at the end.

2. The construction. Let Λ be a Frobenius algebra and X a complex of Λ-modules. For each integer n we may define a complex L_nX and a morphism $\lambda_{n,X} : X \to L_nX$ of complexes as follows:

$$\begin{tikzcd}
X \arrow[d, "\lambda_{n,X}"'] & \cdots \arrow[r] & X^{n-1} \arrow[r] \arrow[d, equal] & X^n \arrow[r] \arrow[d, "\varepsilon"'] & X^{n+1} \arrow[r] \arrow[d, "\varepsilon'"'] & X^{n+2} \arrow[r] \arrow[d, equal] & \cdots \\
L_nX & \cdots \arrow[r] & X^{n-1} \arrow[r] & I \arrow[r, "\alpha"] & C \arrow[r, "\beta"] & X^{n+2} \arrow[r] & \cdots
\end{tikzcd}$$

where the homomorphism $\varepsilon : X^n \to I$ is a fixed, arbitrarily chosen injective envelope and C is obtained as the push-out from d_X^n along ε. In case X^n is injective we choose ε as the identity. Note that the construction is not unique, it does depend on the choice of ε. Dually, we define a complex R_nX and a morphism $\varrho_{n,X} : R_nX \to X$, where $\varphi = \varrho_{n,X}^n : (R_nX)^n \to X^n$ is a fixed, arbitrarily chosen projective cover and $(R_nX)^{n-1}$ is obtained as the pull-back from d_X^{n-1} along φ.

If X is a bounded complex, say $X^i = 0$ for all $i < s$ and all $i > r$, where we suppose $s \le 0 \le r$, then we can apply this procedure several times to obtain a complex

$$\widetilde{X} = R_1R_2\cdots R_{r-1}R_r(L_{-1}L_{-2}\cdots L_{s+1}L_sX)$$

which has projective modules in positive degrees and injective modules in negative degrees. Note that $\widetilde{X}$ does not depend on the numbers r and s. Set $\widetilde{F}X = \widetilde{X}^0$, the Λ-module in degree zero, considered as an object in $\underline{\mathrm{mod}}\,\Lambda$.

If $f : X \to Y$ is a morphism of bounded complexes, there exists a morphism $L_nf : L_nX \to L_nY$ such that $L_nf \circ \lambda_{n,X} = \lambda_{n,Y} \circ f$, and hence a morphism $\widetilde{f} : \widetilde{X} \to \widetilde{Y}$. Again, the morphisms L_nf and $\widetilde{f}$ are not unique, and so the construction is not functorial.

Recall that the objects of $\mathrm{D^b}(\mathrm{mod}\,\Lambda)$ are the same as those of $\mathrm{K^b}(\mathrm{mod}\,\Lambda)$ but morphisms from X to Y are equivalence classes of pairs of morphisms $f' : Z_f \to X$ and $f'' : Z_f \to Y$, where f' is a *quasi-isomorphism*, that is, f' induces isomorphisms of all cohomology groups or equivalently, there exists a triangle $Z_f \xrightarrow{f'} X \to Z' \to$ in $\mathrm{K^b}(\mathrm{mod}\,\Lambda)$ with Z' acyclic. Two such pairs, (f', f'') and (g', g''), are equivalent if there exist quasi-isomorphisms $h : T \to Z_f$ and $h' : T \to Z_g$ such that $f'h = g'h'$ and $f''h = g''h'$.

As we will see, the construction carries over to morphisms in $\mathrm{K^b}(\mathrm{mod}\,\Lambda)$ in a straightforward way and sends quasi-isomorphisms to isomorphisms in $\underline{\mathrm{mod}}\,\Lambda$. Therefore, if $f = (f', f'') : X \to Y$ is a morphism in $\mathrm{D^b}(\mathrm{mod}\,\Lambda)$, we can set $\widetilde{f} = \widetilde{f''} \circ (\widetilde{f'})^{-1}$. The construction turns out to be independent of the choice of the representative.

Now, let A be a finite-dimensional algebra and let $\Lambda = \widehat{A}$ be the repetitive algebra of A. Then, for any complex $X \in \mathrm{D^b}(\mathrm{mod}\,A)$, considered as an object in $\mathrm{D^b}(\mathrm{mod}\,\widehat{A})$, we may apply the above construction and obtain an $\widehat{A}$-module $\widetilde{X}^0$, by taking the degree zero part of the complex $\widetilde{X}$.

3. Preparatory lemmas. We now start formalizing the construction explained in Section 2, paying extra care to functoriality. The main ingredient will be the following assertions. Their proofs are purely homological and straightforward.

We recall that for a complex X of Λ-modules in Section 2, we have constructed morphisms $\lambda_{n,X} : X \to L_nX$ and $\varrho_{n,X} : R_nX \to X$, which depend on the choice of an injective envelope $X^n \to I$ and a projective cover $P \to X^n$ respectively.

LEMMA 3.

(a) *The morphisms $\lambda_{n,X}$ and $\varrho_{n,X}$ are quasi-isomorphisms* (*independently of the choice in the definition*).
(b) *For a morphism $f : X \to Y$ of complexes*, *different choices in the definition of L_nf* (*resp. R_nf*) *lead to homotopic morphisms.*
(c) *Suppose that $f : X \to Y$ is a morphism of complexes which is homotopic to zero. If Y^{n-1} is an injective Λ-module then L_nf is homotopic to zero*; *similarly*, *if X^{n+1} is a projective Λ-module then R_nf is homotopic to zero.*

For a complex X with $X^j = 0$ for all $j \le r$ and an integer $i > r$, define a complex $L_{<i}X = L_{i-1}L_{i-2} \cdots L_rX$ (notice the independence from

the choice of r) and extend the definition to morphisms in the obvious way. Denote by $\lambda_{<i,X}$ the composition of the following morphisms which are quasi-isomorphisms by Lemma 3(a):

$$X \xrightarrow{\lambda_{r,X}} L_rX \xrightarrow{\lambda_{r+1,L_rX}} L_{r+1}L_rX \to \cdots \to L_{<i-1}X \xrightarrow{\lambda_{i-1,L_{<i-1}X}} L_{<i}X.$$

The composition of $L_{<i}$ with the canonical projection $q : \mathrm{C^b}(\mathrm{mod}\,\Lambda) \to \mathrm{K^b}(\mathrm{mod}\,\Lambda)$ is functorial by Lemma 3(b) and factors through q by Lemma 3(c). We have thus constructed functors

$$\overline{L}_{<i} : \mathrm{K^b}(\mathrm{mod}\,\Lambda) \to \mathrm{K^b}(\mathrm{mod}\,\Lambda)$$

and quasi-isomorphisms $\overline{\lambda}_{<i,X} : X \to \overline{L}_{<i}X$, which form a morphism of functors $\overline{\lambda}_{<i} : \mathrm{id} \to \overline{L}_{<i}$. Similarly, we obtain a functor

$$\overline{R}_{>i} : \mathrm{K^b}(\mathrm{mod}\,\Lambda) \to \mathrm{K^b}(\mathrm{mod}\,\Lambda)$$

and a morphism of functors $\overline{\varrho}_{>i} : \overline{R}_{>i} \to \mathrm{id}$.

If $f : X \to Y$ is a quasi-isomorphism, then so are $\overline{L}_{<i}f$ and $\overline{R}_{>i}f$. Hence $\overline{L}_{<i}$ and $\overline{R}_{>i}$ induce functors

$$\widetilde{L}_{<i}, \widetilde{R}_{>i} : \mathrm{D^b}(\mathrm{mod}\,\Lambda) \to \mathrm{D^b}(\mathrm{mod}\,\Lambda),$$

which are equivalences. Clearly, we have isomorphisms of functors

$$\widetilde{\lambda}_{<i} : \mathrm{id} \to \widetilde{L}_{<i}, \quad \widetilde{\varrho}_{>i} : \widetilde{R}_{>i} \to \mathrm{id}.$$

The following result shows that the functors $\widetilde{L}_{<i}$ and $\widetilde{R}_{>i}$ commute (up to isomorphism of functors).

LEMMA 4. *With the above notations*,

(a) $\overline{L}_{<i}\overline{R}_{>i}\overline{\lambda}_{>i} : \overline{L}_{<i}\overline{R}_{>i} \to \overline{L}_{<i}\overline{R}_{>i}\overline{L}_{<i} = \overline{R}_{>i}\overline{L}_{<i}$ *is a morphism of functors* $\mathrm{K^b}(\mathrm{mod}\,\Lambda) \to \mathrm{K^b}(\mathrm{mod}\,\Lambda)$, *which evaluates to a quasi-isomorphism for each object.*

(b) $\widetilde{L}_{<i}\widetilde{R}_{>i}\widetilde{\lambda}_{>i} : \widetilde{L}_{<i}\widetilde{R}_{>i} \to \widetilde{L}_{<i}\widetilde{R}_{>i}\widetilde{L}_{<i} = \widetilde{R}_{>i}\widetilde{L}_{<i}$ *is an isomorphism of functors* $\mathrm{D^b}(\mathrm{mod}\,\Lambda) \to \mathrm{D^b}(\mathrm{mod}\,\Lambda)$.

Proof. This is an immediate consequence of the above. ∎

The equivalence

$$G = \widetilde{R}_{>0}\widetilde{L}_{<0} : \mathrm{D^b}(\mathrm{mod}\,\Lambda) \to \mathrm{D^b}(\mathrm{mod}\,\Lambda)$$

assigns to each complex X a complex GX, with injective modules in negative degrees and projective modules in positive degrees.

The following proposition shows that for Λ a Frobenius algebra, the construction $R_{>0}L_{<0}$, which is *not* functorial, is extended to a functorial one by composing with suitable functors, as shown in the diagram below.

PROPOSITION 5. *If Λ is Frobenius and $p : \mathrm{mod}\,\Lambda \to \underline{\mathrm{mod}}\,\Lambda$ the canonical projection, then $X \mapsto p(R_{>0}L_{<0}X)^0 =: F_1X$ defines a functor F_1 : $\mathrm{C^b}(\mathrm{mod}\,\Lambda) \to \underline{\mathrm{mod}}\,\Lambda$ which factors over the canonical projection q :*

$\mathrm{C^b}(\mathrm{mod}\,\Lambda) \to \mathrm{K^b}(\mathrm{mod}\,\Lambda)$. *So, we get a functor* $F_2 : \mathrm{K^b}(\mathrm{mod}\,\Lambda) \to \underline{\mathrm{mod}}\,\Lambda$ *with* $F_2 q = F_1$.

Proof. A direct verification shows that for a morphism $f : X \to Y$ in $\mathrm{C^b}(\mathrm{mod}\,\Lambda)$, $p(L_n f)^{n+1}$ is well defined, independently of the possible choices for $L_n f$. Consequently, $p(R_{>0}L_{<0}f)^0$ is well defined as a morphism in $\underline{\mathrm{mod}}\,\Lambda$, and therefore $F_1 : X \mapsto p(R_{>0}L_{<0}X)^0$ is functorial.

In the following commutative diagram, the broken arrow indicates a non-functorial construction, and the full arrows stand for functorial ones.

$$\begin{tikzcd}
\mathrm{C^b}(\mathrm{mod}\,\Lambda) \arrow[r, dashed, "R_{>0}L_{<0}"] \arrow[d, "q"'] \arrow[drr, "F_1"] & \mathrm{C^b}(\mathrm{mod}\,\Lambda) \arrow[r, "X \mapsto X^0"] & \mathrm{mod}\,\Lambda \arrow[d, "p"] \\
\mathrm{K^b}(\mathrm{mod}\,\Lambda) \arrow[rr, "F_2"'] & & \underline{\mathrm{mod}}\,\Lambda
\end{tikzcd}$$

Now, if f is homotopic to zero, then so is $R_{>0}L_{<0}f$ by Lemma 3, and hence for some homotopy h we have $(R_{>0}L_{<0}f)^0 = d_{\widetilde{Y}}^{-1}h^0 + h^1 d_{\widetilde{X}}^0$, a morphism which factors over a projective. Thus $p(R_{>0}L_{<0}f)^0 = 0$, and therefore the construction $F_2 : X \mapsto (R_{>0}L_{<0}X)^0$ is well defined in the homotopy category. ∎

For the proof of the following result, we denote by $Z^{\le 0}$ the *right truncation* of a complex Z, that is, the complex with $(Z^{\le 0})^i = 0$ for $i > 0$ and $(Z^{\le 0})^i = Z^i$ for $i \le 0$. Also, we denote by $Z^{=0}$ the *stalk complex* concentrated in degree 0, that is, $(Z^{=0})^i = 0$ for $i \ne 0$ and $(Z^{=0})^0 = Z^0$. Observe that there are canonical morphisms $Z \to Z^{\le 0}$ and $Z^{=0} \to Z^{\le 0}$.

The following lemma will be used in order to see that F_2 factors through the canonical projection $\pi' : \mathrm{K^b}(\mathrm{mod}\,\Lambda) \to \mathrm{D^b}(\mathrm{mod}\,\Lambda)$.

LEMMA 6. *There exists an isomorphism* $\xi : FF_2 \to \pi G \pi'$ *of functors* $\mathrm{K^b}(\mathrm{mod}\,\Lambda) \to \mathrm{D^b}(\mathrm{mod}\,\Lambda)/\mathrm{K^b}(\mathrm{P}_\Lambda)$, *where* $\pi' : \mathrm{K^b}(\mathrm{mod}\,\Lambda) \to \mathrm{D^b}(\mathrm{mod}\,\Lambda)$ *is the canonical projection.*

Proof. Consider a morphism $f : X \to Y$ in $\mathrm{K^b}(\mathrm{mod}\,\Lambda)$ and note that $G\pi'X$ and $G\pi'Y$ are objects with projective modules in all degrees except 0. But, for such an object Z, the canonical morphisms $Z \to Z^{\le 0}$ and $Z^{=0} \to Z^{\le 0}$ in $\mathrm{D^b}(\mathrm{mod}\,\Lambda)$ have mapping cones lying in $\mathrm{K^b}(\mathrm{P}_\Lambda)$. Therefore they become isomorphisms under the projection $\pi : \mathrm{D^b}(\mathrm{mod}\,\Lambda) \to \mathrm{D^b}(\mathrm{mod}\,\Lambda)/\mathrm{K^b}(\mathrm{P}_\Lambda)$. This shows that we have an isomorphism $\xi_X : \pi G\pi'X \to \pi(G\pi'X)^{=0}$ and $\pi(G\pi'f)^{=0}\xi_X = \xi_Y \pi(G\pi'f)$. The result now follows from the fact that the functor F is induced by the canonical inclusion $\mathrm{mod}\,\Lambda \to \mathrm{D^b}(\mathrm{mod}\,\Lambda)$ which sends a module to the stalk complex concentrated in degree 0, that is, $\pi(G\pi'X)^{=0} = FF_2X$ and $\pi(G\pi'f)^{=0} = FF_2 f$. ∎

Proposition 7. *For each quasi-isomorphism f in* $\mathrm{K}^{\mathrm{b}}(\operatorname{mod} \Lambda)$, *the morphism $F_2 f$ is an isomorphism in* $\underline{\operatorname{mod}}\, \Lambda$.

Proof. If $f : X \to Y$ is a quasi-isomorphism in $\mathrm{K}^{\mathrm{b}}(\operatorname{mod} \Lambda)$, then $\pi' f$ and $G\pi' f$ are isomorphisms in $\mathrm{D}^{\mathrm{b}}(\operatorname{mod} \Lambda)$. Therefore, by Lemma 6, $FF_2 f$ is an isomorphism and thus so is $F_2 f$, since F is an equivalence. ∎

It follows from Proposition 7 that F_2 factors over the canonical projection $\pi' : \mathrm{K}^{\mathrm{b}}(\operatorname{mod} \Lambda) \to \mathrm{D}^{\mathrm{b}}(\operatorname{mod} \Lambda)$ and so the factorization

$$\widetilde{F} : \mathrm{D}^{\mathrm{b}}(\operatorname{mod} \Lambda) \to \underline{\operatorname{mod}}\, \Lambda$$

is defined as follows: for an object X, we have $\widetilde{F} X = F_2 X$, and for a morphism $f : X \to Y$ represented by a pair (f', f'') of a quasi-isomorphism $f' : Z_f \to X$ and a morphism $f'' : Z_f \to Y$, we have

$$\widetilde{F} f = F_2 f'' \circ (F_2 f')^{-1} : F_2 X \to F_2 Y$$

(note that $F_2 f' : F_2 Z \to F_2 X$ is an isomorphism, according to Proposition 7, and that the definition is independent on the choice of representatives).

Lemma 8. *The isomorphism ξ, defined in Lemma* 6 *yields an isomorphism $\xi : F\widetilde{F} \to \pi G$ of functors* $\mathrm{D}^{\mathrm{b}}(\operatorname{mod} \Lambda) \to \mathrm{D}^{\mathrm{b}}(\operatorname{mod} \Lambda)/\mathrm{K}^{\mathrm{b}}(\mathrm{P}_\Lambda)$.

Proof. Since a morphism in $\mathrm{D}^{\mathrm{b}}(\operatorname{mod} \Lambda)$ is represented by a pair of a quasi-isomorphism and a morphism in $\mathrm{K}^{\mathrm{b}}(\operatorname{mod} \Lambda)$, the result follows easily from Lemma 6. ∎

4. Proof of the Main Theorem. Since $G \simeq \mathrm{id}_{\mathrm{D}^{\mathrm{b}}(\operatorname{mod} \Lambda)}$, it follows from Lemma 8 that $F\widetilde{F} : \mathrm{D}^{\mathrm{b}}(\operatorname{mod} \Lambda) \to \mathrm{D}^{\mathrm{b}}(\operatorname{mod} \Lambda)/\mathrm{K}^{\mathrm{b}}(\mathrm{P}_\Lambda)$ is isomorphic to the canonical projection π, which is a triangulated functor. Therefore, $\widetilde{F}$ is a triangulated functor, since F is a triangulated equivalence.

This proves part (i) of the Main Theorem. For (ii), we assume A to be a finite-dimensional algebra. Clearly it is enough to prove that the composition

$$\Phi = \pi J : \mathrm{D}^{\mathrm{b}}(\operatorname{mod} A) \to \mathrm{D}^{\mathrm{b}}(\operatorname{mod} \widehat{A})/\mathrm{K}^{\mathrm{b}}(\mathrm{P}_{\widehat{A}})$$

of the canonical inclusion $J : \mathrm{D}^{\mathrm{b}}(\operatorname{mod} A) \to \mathrm{D}^{\mathrm{b}}(\operatorname{mod} \widehat{A})$ with the canonical projection π has the stated properties since F is an equivalence and $F\widetilde{F}J \simeq \pi J = \Phi$.

The homomorphism $\widehat{A} \to A$, $(a_i, \varphi_i)_i \mapsto a_0$, induces a functor $j : \operatorname{mod} A \to \operatorname{mod} \widehat{A}$, which is exact, full and faithful. Thus $J : \mathrm{D}^{\mathrm{b}}(\operatorname{mod} A) \to \mathrm{D}^{\mathrm{b}}(\operatorname{mod} \widehat{A})$ is full and triangulated.

Consider the following diagram, where $\widehat{i} : \operatorname{mod} \widehat{A} \to \mathrm{D}^{\mathrm{b}}(\operatorname{mod} \widehat{A})$ is the canonical embedding.

$$\begin{tikzcd}
\operatorname{mod} A \arrow[r, "i"] \arrow[d, "j"] & \mathrm{D^b}(\operatorname{mod} A) \arrow[d, "J"] \arrow[dd, bend left, "\Phi"] \\
\operatorname{mod} \widehat{A} \arrow[r, "\widehat{i}"] \arrow[d, "p"] & \mathrm{D^b}(\operatorname{mod} \widehat{A}) \arrow[d, "\pi"] \\
\underline{\operatorname{mod}}\, \widehat{A} \arrow[r, "F", "\sim"'] & \mathrm{D^b}(\operatorname{mod} \widehat{A})/\mathrm{K^b}(\mathrm{P}_{\widehat{A}})
\end{tikzcd}$$

The upper square is clearly commutative and the lower square commutes by the generalization of Proposition 2.

Clearly, $\Phi = \pi \circ J$ is triangulated and full, since π and J are. Suppose that $\Phi X \simeq 0$ for some non-zero object X of $\mathrm{D^b}(\operatorname{mod} A)$. Let n be minimal such that the cohomology group $H^n(X)$ is not trivial. Let m be such that $X^i = 0$ for all $i < m$. Let Y be the complex whose entries and differentials are constructed inductively by applying $L_m, L_{m+1}, \ldots$ to X. The complex Y is quasi-isomorphic to X (the quasi-isomorphism is also constructed inductively), its entries are injective $\widehat{A}$-modules, possibly infinitely many non-zero to the right, and Y has bounded cohomology. We now show by induction that $\operatorname{Ker} d_Y^i$ does not lie in $\mathrm{I}_{\widehat{A}}$, for any $i \geq n$.

We use the notation of the diagram in Section 2. Since $H^n(X) \neq 0$, we have $\operatorname{Ker} d_Y^n = \operatorname{Ker} \alpha \simeq \operatorname{Ker} d_X^n$, and $\operatorname{Ker} d_X^n$, being a non-zero A-module, does not lie in $\mathrm{I}_{\widehat{A}}$.

For the inductive step, suppose that $\operatorname{Ker} \beta \in \mathrm{I}_{\widehat{A}} = \mathrm{P}_{\widehat{A}}$. Then $\operatorname{Ker} \beta$ is a direct summand of C, say with retraction $p : C \to \operatorname{Ker} \beta$. Since

$$[\alpha\ \varepsilon'] : I \oplus X^{n+1} \to C$$

is surjective and $p \circ [\alpha\ \varepsilon']$ is split epi, and since no indecomposable direct summand of X^{n+1} lies in $\mathrm{I}_{\widehat{A}}$, it follows that $\operatorname{Ker} \beta$ is a direct summand of I, say $I = I' \oplus \operatorname{Ker} \beta$, and $p\alpha$ is split epi. Now $\operatorname{Im} \alpha \subseteq \operatorname{Ker} \beta$ implies that $\operatorname{Ker} \alpha = I' \in \mathrm{I}_{\widehat{A}}$, a contradiction.

This shows that Y belongs to $\mathrm{K}^{+,\mathrm{b}}(\mathrm{P}_{\widehat{A}})$, but also that Y cannot be isomorphic to an object $Z \in \mathrm{K^b}(\mathrm{P}_{\widehat{A}})$. Indeed, suppose that there exist two morphisms $f : Y \to Z$ and $g : Z \to Y$ with $gf \sim \mathrm{id}_Y$. Let r be maximal such that $Z^r \neq 0$. Observe that $\operatorname{Im} g^r \subseteq \operatorname{Ker} d_Y^r$ and hence for some homotopy h, we have

$$\mathrm{id}_{Y^r} = a + b, \quad \text{where} \quad a = g^r f^r + d_Y^{r-1} h^r,\ b = h^{r+1} d_Y^r.$$

Now $\operatorname{Im} a \subseteq \operatorname{Ker} d_Y^r \subseteq \operatorname{Ker} b$ and therefore a induces a split epi $Y^r \to \operatorname{Ker} d_Y^r$ in contradiction to the above.

This shows that $\pi(Y)$ is non-zero in contradiction to $\pi(Y) \simeq \Phi(X) \simeq 0$. Now, it is easy to see that Φ is faithful, following the same argument as

Rickard in the proof of [6, Theorem 2.1], stated in this paper as Proposition 2.

Suppose now that A is of finite global dimension. Then $\operatorname{mod} A$ is a generating subcategory in $\underline{\operatorname{mod}}\,\widehat{A}$ (see Proposition II.3.2 in [2]). Clearly $\operatorname{mod} A$ is a generating subcategory in $\mathrm{D}^{\mathrm{b}}(\operatorname{mod} A)$, hence the functor Φ sends a generating subcategory to a generating one and is thus an equivalence by Lemma II.3.4 of [2].

This completes the proof of part (ii) of the Main Theorem.

5. Comments. 1. The original construction of a functor $H : \mathrm{D}^{\mathrm{b}}(\operatorname{mod} A) \to \underline{\operatorname{mod}}\,\widehat{A}$ given in [2] can be expressed in our language as $X \mapsto \widetilde{R}_{>0}\widetilde{L}_{<s}X$, where $s > 0$ is any integer such that $X^i = 0$ for all $i \geq s$.

2. The construction outlined in Section 2 can be used, for A any finite-dimensional algebra, to give an explicit equivalence

$$L_{<\infty} : \mathrm{D}^{\mathrm{b}}(\operatorname{mod} A) \to \mathrm{K}^{+,b}(\mathrm{I}_A),$$

where $\mathrm{K}^{+,b}(\mathrm{I}_A)$ is the category of lower bounded complexes of injective A-modules with bounded cohomology and $L_{<\infty} = \cdots L_2 L_1 L_0 L_{<0}$.

3. Let A be the path algebra of the quiver Q modulo the ideal generated by all paths of length two, where

$$Q : \begin{tikzcd} & 1 \arrow[dl] \arrow[dr] & \\ 2 \arrow[dr] & & 3 \arrow[dl] \\ & 4 \arrow[uu] & \end{tikzcd}$$

Then the stalk complex $S_1[0]$ concentrated in degree zero with entry S_1, the simple at 1, is easily seen to be isomorphic in the quotient category $\mathrm{D}^{\mathrm{b}}(\operatorname{mod} A)/\mathrm{K}^{\mathrm{b}}(\mathrm{P}_A)$ to the stalk complex $(S_1 \oplus S_1)[-3]$. More generally $S_1[0] \simeq S_1^n[-3n]$, and therefore the endomorphism space of $S_1[0]$ is infinite-dimensional. This is an example showing, that for A not Gorenstein, the quotient $\mathrm{D}^{\mathrm{b}}(\operatorname{mod} A)/\mathrm{K}^{\mathrm{b}}(\mathrm{P}_A)$ cannot be equivalent to a subcategory of $\underline{\operatorname{mod}}\,B$ for any finite-dimensional algebra B, as was shown in [4] for A Gorenstein.

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