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A CLASS OF QUASITILTED RINGS THAT ARE NOT TILTED

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Abstract. Based on the work of D. Happel, I. Reiten and S. Smalø on quasitilted artin algebras, the first two authors recently introduced the notion of quasitilted rings. Various authors have presented examples of quasitilted artin algebras that are not tilted. Here we present a class of right quasitilted rings that not right tilted, and we show that they satisfy a condition that would force a quasitilted artin algebra to be tilted.

Inspired by the papers [9] and [8] on quasitilted artin algebras, in [6] the first two authors began an investigation of the class of rings R, called *right quasitilted* rings, admitting a split torsion theory $(\mathcal{X},\mathcal{Y})$ in Mod-R such that $R\in\mathcal{Y}$ and $\operatorname{proj}\dim\mathcal{Y}\le 1$. A quasitilted artin algebra is one admitting such a torsion theory $(\mathcal{X}_0,\mathcal{Y}_0)$ in mod-R, and examples of quasitilted artin algebras that are not tilted can be found in [10], for example. In [6] we present an example of a (non-noetherian) right quasitilted ring and state that, together with the third author, we would subsequently show that it is not tilted.

Here we shall verify our statement by presenting a rather large class of right quasitilted rings that are not right tilted. Also, recalling that according to [8] a quasitilted artin algebra is tilted if and only if the torsion free class $\mathcal{Y}_0$ in mod-R is cogenerated by a (cotilting) module in mod-R, we shall show that in each of our examples the torsion free class $\mathcal{Y}$ is cogenerated by a cotilting module.

Throughout we use the terminology and notation introduced in [6] and the standard results and terminology of [1], [5] and [11].

1. A class of quasitilted triangular matrix rings. In the following, let S be a non-semisimple hereditary prime two-sided Goldie ring with two-sided maximal quotient ring $Q=Q(S)$, and let

$$R=T(S)=\begin{bmatrix} Q & Q\\ 0 & S\end{bmatrix} \tag{1}$$

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denote the ring of upper triangular 2×2 matrices over Q with $2,2$-entries in S. (Our example in [6] had $S = \mathbb{Z}$ and $Q = \mathbb{Q}$.) We let

$$e = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad f = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

in R, and we note that if

$$J = \begin{bmatrix} 0 & Q \\ 0 & 0 \end{bmatrix},$$

then $J^2 = 0$ and

$$fRe = 0, \quad fR = fRf \cong S, \quad eRe \cong Q, \quad eRf = eJ = J \cong {}_Q Q_S.$$

We recall that Q is simple artinian, and $Q = E(S_S) = E({}_S S)$ is the two-sided injective envelope of S, and moreover

$$Q = \{sd^{-1} \mid d \in S \text{ is regular}, s \in S\} = \{d^{-1}s \mid d \in S \text{ is regular, } s \in S\}$$

and

$$\mathrm{End}(Q_S) = \mathrm{End}(Q_Q) \cong Q,$$

canonically. See [11, Chapter 2], for example.

Also, we note that eR/eJ is semisimple and J is nilpotent, so we see that

$$e = e_1 + \cdots + e_n$$

is a sum of orthogonal primitive idempotents with $e_1R \cong e_iR$ having unique maximal submodule e_iJ for all $1 \leq i \leq n$, and

$$eR = e_1R \oplus \cdots \oplus e_nR.$$

Using modifications of the arguments in Section 7 of [6] we shall show that each $R = T(S)$ as in (1) is right quasitilted.

LEMMA 1.1. *All direct sums of copies of eR and of eR/eJ are injective.*

Proof. First we shall show that e_1R is injective relative to both fR and e_1R, so [1, Propositions 16.10 and 16.13] apply. Note that $J = eRf_{fRf} \cong Q_S$. First suppose that $I = fIf \leq fR = fRf_{fRf} \cong S_S$ and $\gamma : I \to e_1R$. Then $\gamma(I) \leq e_1Rf = e_1J$ and the latter is injective over fRf, being a direct summand of $J = eRf$. Thus there is a map $\overline{\gamma} : fR \to e_1J \leq e_1R$ that extends γ. Next suppose that $I \leq e_1R$, $\gamma : I \to e_1R$, and $I \neq e_1R$. Then $I \subseteq e_1J = e_1Jf$ and so $\gamma(I) \leq e_1Rf = e_1J$, which is injective over fRf. Thus there is a map $\overline{\gamma} : e_1J \to e_1J$ that extends γ. Identifying $J = Q_S$ we may consider $\overline{\gamma} \in \mathrm{End}(Q_S) = Q$. Thus there is an $x \in Q$ such that $\overline{\gamma}(e_1j) = xe_1j = e_1xe_1j$ for all $e_1j \in e_1J$. Now $e_1xe_1 \in e_1Re_1 \cong \mathrm{End}(e_1R_R)$ and left multiplication by e_1xe_1 extends $\overline{\gamma}$, and hence γ.

If $I = fIf$, then $\mathrm{Hom}_R(I, eR/eJ) = 0$, so e_1R/e_1J is injective relative to $fR = fRf$. Suppose that $I \leq e_1R$ and $\gamma : I \to e_1R/e_1J$. Then either

$I \le e_1J = e_1Jf$ and $\gamma = 0$, or $I = e_1R$. Thus e_1R/e_1J is injective relative to e_1R, and, as before, e_1R/e_1J is injective.

Now we see that both $eR = e_1R \oplus \cdots \oplus e_nR$ and $eR/eJ \cong e_1R/e_1J \oplus \cdots \oplus e_nR/e_nJ$ are injective. Clearly ${}_{eRe}eR$ and ${}_{eRe}eR/eJ$ have the descending chain condition on submodules, and in particular on annihilators of subsets of R. Thus (see [7, p. 181]), R has the ascending chain condition on annihilators of subsets of eR and eR/eJ, so direct sums of copies of these modules are injective according to [7, Proposition 3, p. 184]. ∎

Let $\mathcal{C} = \{e_1R/K \mid 0 \ne K \le eR\}$ and let $(\mathcal{X}, \mathcal{Y})$ be the torsion theory generated by $\mathcal{C}$. Thus, letting

$$\mathcal{Y} = \{Y_R \mid \mathrm{Hom}_R(C, Y) = 0 \text{ for all } C \in \mathcal{C}\}$$

we have

$$\mathcal{X} = \{X_R \mid \mathrm{Hom}_R(X, Y) = 0 \text{ for all } Y \in \mathcal{Y}\}.$$

LEMMA 1.2. *$\mathcal{Y} = \{M \mid M \cong e_1R^{(\alpha)} \oplus N$ with $N = Nf\}$. In particular, $R \in \mathcal{Y}$ and* $\operatorname{proj\,dim} \mathcal{Y} \le 1$.

Proof. Note that if $x \in M$, then $xe_iR = ye_1R$ where $y = xe_iue_1$ with $e_1r \mapsto e_iue_1r$ an isomorphism $e_1R \to e_iR$. Let $Y \in \mathcal{Y}$. As $\mathrm{Hom}_R(e_1R/K, Y) = 0$ whenever $0 \ne K \le e_1R$, it follows for $x \in Y$ that $xe_1 \ne 0$ implies $xe_1R \cong e_1R$. Thus,

$$Y = \sum_A w_\alpha e_1R + \sum_L b_\lambda fR$$

with each $w_\alpha e_1R \cong e_1R$. Now let $H \subseteq A$ be maximal with $\{w_\alpha e_1R \mid \alpha \in H\}$ independent, so that $P = \bigoplus_H w_\alpha e_1R \cong e_1R^{(H)}$ is an (injective by Lemma 1.1) projective direct summand of $\sum_I w_\alpha e_1R$. But if some $w_\beta e_1R \not\subseteq P$ and $\sum_A w_\alpha e_1R = P \oplus L$, then the projection xe_1R of $w_\beta e_1R$ on L would have $xe_1 \ne 0$ and $xe_1R \ncong e_1R$. Thus $Y \cong e_1R^{(H)} \oplus N$ with $N = Nf$.

Suppose $M = e_1R^{(\alpha)} \oplus N$ with $N = Nf$. If $0 \ne \gamma \in \mathrm{Hom}_R(e_1R/K, M)$, then $\mathrm{Im}\,\gamma \subseteq e_1R^{(\alpha)}$ and $\mathrm{Im}\,\gamma \not\subseteq e_1J^{(\alpha)} = e_1J^{(\alpha)}f$, and so for some projection π_α, the composite $\pi_\alpha\gamma : e_1R/K \to e_1R$ is a split epimorphism. Thus $K = 0$ and $M \in \mathcal{Y}$.

Clearly now $R \in \mathcal{Y}$, and $\operatorname{proj\,dim}(e_1R^{(\alpha)} \oplus Nf) \le 1$ since $e_1R^{(\alpha)}$ is projective and $\operatorname{proj\,dim} Nf \le 1$ as it is an $fR = fRf \cong S$-module. ∎

It only remains to show that $(\mathcal{X}, \mathcal{Y})$ splits, in order to prove

PROPOSITION 1.3. *The ring $R = T(S)$ is right quasitilted with torsion theory $(\mathcal{X}, \mathcal{Y})$.*

Proof. Let $X \in \mathcal{X}$. Since every direct sum of copies of e_1R/e_1J is injective by Lemma 1.1, we see, as in the proof of Lemma 1.2, that $X/XJ \cong e_1R/e_1J^{(\alpha)} \oplus N$ with $N = Nf$. But then $N \in \mathcal{X} \cap \mathcal{Y} = 0$. Thus, since J is nilpotent, there exist $t_\alpha \in X \setminus XJ$ such that $\sum t_\alpha e_1 R = X$, and so there is an exact sequence

$$0 \to K \to e_1 R^{(\alpha)} \to X \to 0.$$

If $Y \in \mathcal{Y}$, then we have an exact sequence

$$0 = \mathrm{Ext}^1_R(Y, e_1 R^{(\alpha)}) \to \mathrm{Ext}^1_R(Y, X) \to \mathrm{Ext}^2_R(Y, K) = 0,$$

where the first equality is by Lemma 1.1 and the second is because $\operatorname{proj\,dim} \mathcal{Y} \leq 1$. ∎

Following the artin algebra tradition, we say that a ring R is *right tilted* with torsion theory $(\mathcal{X}, \mathcal{Y})$ in Mod-R if there is a hereditary ring H with a tilting module V_H such that $R = \mathrm{End}(V_H)$ and $\mathcal{X} = \mathrm{Ker}(- \otimes_R V)$. (See [4] for noetherian examples of such rings.) In any case if V_H is a tilting module with $R = \mathrm{End}(V_H)$, then ${}_R V$ is a tilting module and so is finitely presented, so that $\mathrm{Ker}(\_ \otimes_R V)$ is closed under direct products.

To see that our ring $R = T(S)$ in (1) is not right tilted, we shall show that a split tilting torsion theory $(\mathcal{X}', \mathcal{Y}')$ in Mod-R with $R_R \in \mathcal{Y}'$ and $\operatorname{proj\,dim} \mathcal{Y}' \leq 1$ cannot have $\mathcal{X}'$ closed under direct products.

Now let

$$t = etf = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

so that $tS \leq J = eRf$ over both S and R.

Lemma 1.4. *There is a cardinal number α such that*

$$\operatorname{proj\,dim} (eR/tS)^\alpha = 2.$$

Proof. Suppose to the contrary that $\operatorname{proj\,dim} (eR/tS)^\alpha \leq 1$ for every cardinal α. Then in the exact sequences

$$0 \to tS^\alpha \to eR^\alpha \to (eR/tS)^\alpha \to 0,$$

$tS^\alpha \cong S^\alpha$ must be projective as a right R-module, and so as a right S-module. But according to Theorem 3.3 of [3] this would entail S being a right perfect ring. That is impossible since, being a non-artinian prime ring, the socle $\mathrm{Soc}_S S$ is zero (see, for example, [1, Exercise 14.11(2)]). ∎

Note that this last lemma shows that R is not right hereditary.

Proposition 1.5. *The ring $R = T(S)$ is not right tilted.*

Proof. Since it is indecomposable, $e_1 R / e_1 tS$, and hence eR/tS, belongs to either $\mathcal{X}'$ or $\mathcal{Y}'$. Now $\mathcal{Y}'$ is closed under products and $\operatorname{proj\,dim} \mathcal{Y}' \leq 1$, so the latter is impossible by Lemma 1.4. Thus to see that $\mathcal{X}'$ cannot be closed under direct products we need only show that $eR \in \mathrm{Cogen}(eR/tS)$. Clearly $\mathrm{Rej}_{eR/tS}(eR) = \bigcap \{\mathrm{Ker}\,\gamma \mid \gamma : eR \to eR/tS\} \subseteq tS$, so it will suffice to show that any tn with $n \in S$ belongs to the kernel of some $\gamma : eR \to eR/tS$.

Now $tnS \cong nS$ is S-isomorphic to a direct summand of S, so tnS is R-isomorphic to a direct summand of tS. Thus there is an R-homomorphism $g : tnS \to tS$ which, since eR is injective, extends to a map $h : eR \to eR$. But then $tn \in \operatorname{Ker} \pi h$, where $\pi : eR \to eR/tS$ is the natural epimorphism. ■

2. Cotilting cogenerator for $\mathcal{Y}$. Let

$$R = T(S) = \begin{bmatrix} Q & Q \\ 0 & S \end{bmatrix}$$

be the ring (1) of Section 1 with quasitilting torsion theory $(\mathcal{X}, \mathcal{Y})$ where $\mathcal{Y} = \{M \mid M \cong e_1R^{(\alpha)} \oplus N \text{ with } N = Nf\}$. We shall show that (in contrast to the artin algebra case [8]) even though R is not right tilted, $\mathcal{Y} = \operatorname{Cogen} U$ for a certain cotilting module U_R.

If C_0 is any injective cogenerator in Mod-S, then letting $C_R = [0 \ \ C_0]$ we see that $C = Cf$ is injective over fRf and cogenerates every $N = Nf$ in Mod-R.

Proposition 2.1. *Let $R = T(S)$ and let $C = Cf$ be an R-module such that Cf is an injective cogenerator in* Mod-fRf, *and let $U = e_1R \oplus C$. Then $\mathcal{Y} = \operatorname{Cogen} U$, and U is a cotilting module in the sense that* $\operatorname{Cogen} U = \operatorname{Ker} \operatorname{Ext}^1_R(-, U)$.

Proof. Clearly C cogenerates every $N = Nf$ in Mod-R, and so by Lemma 1.2, $\mathcal{Y} = \operatorname{Cogen} U$. Also, by Lemma 1.2, $\mathcal{Y} \subseteq \operatorname{Ker} \operatorname{Ext}^1_R(-, U)$. Indeed, since $e_1R^{(\alpha)}$ is projective and e_1R and C_{fRf} are injective,

$$\operatorname{Ext}^1_R(e_1R^{(\alpha)} \oplus Nf, e_1R \oplus C) = \operatorname{Ext}^1_R(Nf, C) \cong \operatorname{Ext}^1_{fRf}(Nf, C) = 0.$$

Finally, if $M = X \oplus Y$ with $X \in \mathcal{X}$, and $Y \in \mathcal{Y}$, then since $X \in \operatorname{Gen}(e_1R)$ there is an exact sequence

$$0 \to K \to e_1R^{(\beta)} \xrightarrow{\pi} X \to 0$$

with $e_1R^{(\beta)} \xrightarrow{\pi} X \to 0$ a projective cover, and $0 \neq K \subseteq e_1J^{(\beta)}$. Thus $K = Kf \in \operatorname{Cogen} C$ and we have an exact sequence

$$0 = \operatorname{Hom}_R(e_1R^{(\beta)}, C) \to \operatorname{Hom}_R(K, C) \to \operatorname{Ext}^1_R(X, C)$$

showing that $M \in \mathcal{Y}$ whenever $\operatorname{Ext}^1_R(M, U) = 0$. ■

If R is a hereditary prime noetherian (HNP) ring, it follows from [2, Lemma 1] and [1, Exercise 14.11(2)] that the injective S-module Q/S is a cogenerator. Thus J/tS is injective over fRf and cogenerates every $N = Nf$ in Mod-R, so that when R is an HNP ring we may choose $C = J/tS$ in Proposition 2.1.

Finally one is led to wonder which right quasitilted rings are right tilted. For example, is "$\mathcal{X}$ closed under direct products and $\mathcal{Y}$ cogenerated by a cotilting module" enough?

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