A CLASS OF QUASITILTED RINGS THAT ARE NOT TILTED

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Abstract. Based on the work of D. Happel, I. Reiten and S. Smalø on quasitilted artin algebras, the first two authors recently introduced the notion of quasitilted rings. Various authors have presented examples of quasitilted artin algebras that are not tilted. Here we present a class of right quasitilted rings that are not tilted, and we show that they satisfy a condition that would force a quasitilted artin algebra to be tilted.

Inspired by the papers [9] and [8] on quasitilted artin algebras, in [6] the first two authors began an investigation of the class of rings \( R \), called right quasitilted rings, admitting a split torsion theory \((\mathcal{X}, \mathcal{Y})\) in \text{Mod-} R such that \( R \in \mathcal{Y} \) and \( \text{proj dim}\mathcal{Y} \leq 1 \). A quasitilted artin algebra is one admitting such a torsion theory \((\mathcal{X}_0, \mathcal{Y}_0)\) in \text{mod-} R, and examples of quasitilted artin algebras that are not tilted can be found in [10], for example. In [6] we present an example of a (non-noetherian) right quasitilted ring and state that, together with the third author, we would subsequently show that it is not tilted.

Here we shall verify our statement by presenting a rather large class of right quasitilted rings that are not right tilted. Also, recalling that according to [8] a quasitilted artin algebra is tilted if and only if the torsion free class \(\mathcal{Y}_0\) in \text{mod-} R is cogenerated by a (cotilting) module in \text{mod-} R, we shall show that in each of our examples the torsion free class \(\mathcal{Y}\) is cogenerated by a cotilting module.

Throughout we use the terminology and notation introduced in [6] and the standard results and terminology of [1], [5] and [11].

1. A class of quasitilted triangular matrix rings. In the following, let \( S \) be a non-semisimple hereditary prime two-sided Goldie ring with two-sided maximal quotient ring \( Q = Q(S) \), and let

\[
R = T(S) = \begin{bmatrix}
Q & Q \\
0 & S
\end{bmatrix}
\]

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denote the ring of upper triangular $2 \times 2$ matrices over $Q$ with $2, 2$-entries in $S$. (Our example in [6] had $S = \mathbb{Z}$ and $Q = \mathbb{Q}$.) We let
\[
e = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad f = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}
\]
in $R$, and we note that if
\[
J = \begin{bmatrix} 0 & Q \\ 0 & 0 \end{bmatrix},
\]
then $J^2 = 0$ and
\[
fR e = 0, \quad fR f \cong S, \quad eRe \cong Q, \quad eRf = eJ = J \cong qQ_S.
\]
We recall that $Q$ is simple artinian, and $Q = E(S_S) = E(S_S)$ is the two-sided injective envelope of $S$, and moreover
\[
Q = \{ sd^{-1} \mid d \in S \text{ is regular}, s \in S \} = \{ d^{-1} s \mid d \in S \text{ is regular}, s \in S \}
\]
and
\[
\text{End}(Q_S) = \text{End}(Q_Q) \cong Q,
\]
canonically. See [11, Chapter 2], for example.

Also, we note that $eR/eJ$ is semisimple and $J$ is nilpotent, so we see that
\[
e = e_1 + \cdots + e_n
\]
is a sum of orthogonal primitive idempotents with $e_1 R \cong e_1 R$ having unique maximal submodule $e_i J$ for all $1 \leq i \leq n$, and
\[
eR = e_1 R \oplus \cdots \oplus e_n R.
\]

Using modifications of the arguments in Section 7 of [6] we shall show that each $R = T(S)$ as in (1) is right quasiilted.

**Lemma 1.1.** All direct sums of copies of $eR$ and of $eR/eJ$ are injective.

**Proof.** First we shall show that $e_1 R$ is injective relative to both $fR$ and $e_1 R$, so [1, Propositions 16.10 and 16.13] apply. Note that $J = eRf_{fR} \cong Q_S$. First suppose that $I = fI f \leq fR = fRf_{fR} \cong S_S$ and $\gamma : I \rightarrow e_1 R$. Then $\gamma(I) \leq e_1 Rf = e_1 J$ and the latter is injective over $fRf$, being a direct summand of $J = eRf$. Thus there is a map $\overline{\gamma} : fR \rightarrow e_1 J \leq e_1 R$ that extends $\gamma$. Next suppose that $I \leq e_1 R$, $\gamma : I \rightarrow e_1 R$, and $I \neq e_1 R$. Then $I \subseteq e_1 J = e_1 Jf$ and so $\gamma(I) \leq e_1 Rf = e_1 J$, which is injective over $fRf$. Thus there is a map $\overline{\gamma} : e_1 J \rightarrow e_1 J$ that extends $\gamma$. Identifying $J = Q_S$ we may consider $\overline{\gamma} \in \text{End}(Q_S) = Q$. Thus there is an $x \in Q$ such that $\overline{\gamma}(e_1 j) = xe_1 j = e_1 xe_1 j$ for all $e_1 j \in e_1 J$. Now $e_1 xe_1 \in e_1 Re_1 \cong \text{End}(e_1 R)$ and left multiplication by $e_1 xe_1$ extends $\overline{\gamma}$, and hence $\gamma$.

If $I = fI f$, then $\text{Hom}_R(I, eR/eJ) = 0$, so $e_1 R/e_1 J$ is injective relative to $fR = fRf$. Suppose that $I \leq e_1 R$ and $\gamma : I \rightarrow e_1 R/e_1 J$. Then either
\[ I \leq e_1 J = e_1 J f \text{ and } \gamma = 0, \text{ or } I = e_1 R. \text{ Thus } e_1 R/e_1 J \text{ is injective relative to } e_1 R, \text{ and, as before, } e_1 R/e_1 J \text{ is injective.} \]

Now we see that both \( eR = e_1 R \oplus \cdots \oplus e_n R \) and \( eR/eJ \cong e_1 R/e_1 J \oplus \cdots \oplus e_n R/e_n J \) are injective. Clearly \( eRe R \) and \( eRe eR/eJ \) have the descending chain condition on submodules, and in particular on annihilators of subsets of \( R \). Thus (see [7, p. 181]), \( R \) has the ascending chain condition on annihilators of subsets of \( eR \) and \( eR/eJ \), so direct sums of copies of these modules are injective according to [7, Proposition 3, p. 184].

Let \( C = \{ e_1 R/K \mid 0 \neq K \leq eR \} \) and let \( (\mathcal{X}, \mathcal{Y}) \) be the torsion theory generated by \( C \). Thus, letting

\[ \mathcal{Y} = \{ Y_R \mid \text{Hom}_R(C, Y) = 0 \text{ for all } C \in C \}, \]

we have

\[ \mathcal{X} = \{ X_R \mid \text{Hom}_R(X, Y) = 0 \text{ for all } Y \in \mathcal{Y} \}. \]

**Lemma 1.2.** \( \mathcal{Y} = \{ M \mid M \cong e_1 R^{(\alpha)} \oplus N \text{ with } N = N f \}. \) In particular, \( R \in \mathcal{Y} \) and \( \text{proj dim } \mathcal{Y} \leq 1. \)

**Proof.** Note that if \( x \in M \), then \( xe_1 R = ye_1 R \) where \( y = xe_1 ue_1 \) with \( e_1 r \mapsto e_1 ue_1 r \) an isomorphism \( e_1 R \to e_1 R \). Let \( Y \in \mathcal{Y} \). As \( \text{Hom}_R(e_1 R/K, Y) = 0 \) whenever \( 0 \neq K \leq e_1 R \), it follows for \( x \in Y \) that \( xe_1 \neq 0 \) implies \( xe_1 R \cong e_1 R \). Thus,

\[ Y = \sum_{A} w_{\alpha} e_1 R + \sum_{L} b_{\lambda} f R \]

with each \( w_{\alpha} e_1 R \cong e_1 R \). Now let \( H \subseteq A \) be maximal with \( \{ w_{\alpha} e_1 R \mid \alpha \in H \} \) independent, so that \( P = \bigoplus_{H} w_{\alpha} e_1 R \cong e_1 R^{(H)} \) is an (injective by Lemma 1.1) projective direct summand of \( \sum_{A} w_{\alpha} e_1 R \). But if some \( w_{\beta} e_1 R \notin P \) and \( \sum_{A} w_{\alpha} e_1 R = P \oplus L \), then the projection \( xe_1 R \) of \( w_{\beta} e_1 R \) on \( L \) would have \( xe_1 \neq 0 \) and \( xe_1 R \neq e_1 R \). Thus \( Y \cong e_1 R^{(H)} \oplus N \) with \( N = N f \).

Suppose \( M = e_1 R^{(\alpha)} \oplus N \) with \( N = N f \). If \( 0 \neq \gamma \in \text{Hom}_R(e_1 R/K, M) \), then \( \text{Im } \gamma \subseteq e_1 R^{(\alpha)} \) and \( \text{Im } \gamma \notin e_1 J^{(\alpha)} = e_1 J^{(\alpha)} f \), and so for some projection \( \pi_\alpha \), the composite \( \pi_\alpha \gamma : e_1 R/K \to e_1 R \) is a split epimorphism. Thus \( K = 0 \) and \( M \in \mathcal{Y} \).

Clearly now \( R \in \mathcal{Y} \), and \( \text{proj dim } (e_1 R^{(\alpha)} \oplus N f) \leq 1 \) since \( e_1 R^{(\alpha)} \) is projective and \( \text{proj dim } N f \leq 1 \) as it is an \( f R = f R f \cong S \)-module.

It only remains to show that \( (\mathcal{X}, \mathcal{Y}) \) splits, in order to prove

**Proposition 1.3.** The ring \( R = T(S) \) is right quasitilted with torsion theory \( (\mathcal{X}, \mathcal{Y}) \).

**Proof.** Let \( X \in \mathcal{X} \). Since every direct sum of copies of \( e_1 R/e_1 J \) is injective by Lemma 1.1, we see, as in the proof of Lemma 1.2, that \( X/XJ \cong e_1 R/e_1 J^{(\alpha)} \oplus N \) with \( N = N f \). But then \( N \in \mathcal{X} \cap \mathcal{Y} = 0 \). Thus, since \( J \) is
nilpotent, there exist \( t_\alpha \in X \setminus XJ \) such that \( \sum t_\alpha e_1 R = X \), and so there is an exact sequence

\[
0 \to K \to e_1 R(\alpha) \to X \to 0.
\]

If \( Y \in \mathcal{Y} \), then we have an exact sequence

\[
0 = \text{Ext}^1_R(Y, e_1 R(\alpha)) \to \text{Ext}^1_R(Y, X) \to \text{Ext}^2_R(Y, K) = 0,
\]

where the first equality is by Lemma 1.1 and the second is because \( \text{proj dim} \mathcal{Y} \leq 1 \).

Following the artin algebra tradition, we say that a ring \( R \) is right tilted with torsion theory \( (\mathcal{X}, \mathcal{Y}) \) in \( \text{Mod-}R \) if there is a hereditary ring \( H \) with a tilting module \( V_H \) such that \( R = \text{End}(V_H) \) and \( \mathcal{X} = \text{Ker}(- \otimes_R V) \). (See [4] for noetherian examples of such rings.) In any case if \( V_H \) is a tilting module with \( R = \text{End}(V_H) \), then \( RV \) is a tilting module and so is finitely presented, so that \( \text{Ker}(\_ \otimes_R V) \) is closed under direct products.

To see that our ring \( R = T(S) \) in (1) is not right tilted, we shall show that a split tilting torsion theory \( (\mathcal{X}', \mathcal{Y}') \) in \( \text{Mod-}R \) with \( R_R \in \mathcal{Y}' \) and \( \text{proj dim} \mathcal{Y}' \leq 1 \) cannot have \( \mathcal{X}' \) closed under direct products.

Now let

\[
t = etf = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}
\]

so that \( tS \leq J = eRf \) over both \( S \) and \( R \).

**Lemma 1.4.** There is a cardinal number \( \alpha \) such that

\[
\text{proj dim} (eR/tS)^{\alpha} = 2.
\]

**Proof.** Suppose to the contrary that \( \text{proj dim} (eR/tS)^{\alpha} \leq 1 \) for every cardinal \( \alpha \). Then in the exact sequences

\[
0 \to tS^{\alpha} \to eR^{\alpha} \to (eR/tS)^{\alpha} \to 0,
\]

\( tS^{\alpha} \cong S^{\alpha} \) must be projective as a right \( R \)-module, and so as a right \( S \)-module. But according to Theorem 3.3 of [3] this would entail \( S \) being a right perfect ring. That is impossible since, being a non-artinian prime ring, the socle \( \text{Soc}_S S \) is zero (see, for example, [1, Exercise 14.11(2)]).

Note that this last lemma shows that \( R \) is not right hereditary.

**Proposition 1.5.** The ring \( R = T(S) \) is not right tilted.

**Proof.** Since it is indecomposable, \( e_1 R/e_1 tS \), and hence \( eR/tS \), belongs to either \( \mathcal{X}' \) or \( \mathcal{Y}' \). Now \( \mathcal{Y}' \) is closed under products and \( \text{proj dim} \mathcal{Y}' \leq 1 \), so the latter is impossible by Lemma 1.4. Thus to see that \( \mathcal{X}' \) cannot be closed under direct products we need only show that \( eR \in \text{Cogen}(eR/tS) \). Clearly \( \text{Rej}_{eR/tS}(eR) = \bigcap \{ \text{Ker} \gamma \mid \gamma : eR \to eR/tS \} \subseteq tS \), so it will suffice to show that any \( tn \) with \( n \in S \) belongs to the kernel of some \( \gamma : eR \to eR/tS \).
Now $tnS \cong nS$ is $S$-isomorphic to a direct summand of $S$, so $tnS$ is $R$-isomorphic to a direct summand of $tS$. Thus there is an $R$-homomorphism $g : tnS \to tS$ which, since $eR$ is injective, extends to a map $h : eR \to eR$. But then $tn \in \text{Ker } \pi h$, where $\pi : eR \to eR/tS$ is the natural epimorphism. ■

2. Cotilting cogenerator for $\mathcal{Y}$. Let

$$R = T(S) = \begin{bmatrix} Q & Q \\ 0 & S \end{bmatrix}$$

be the ring (1) of Section 1 with quasitilting torsion theory $(\mathcal{X}, \mathcal{Y})$ where $\mathcal{Y} = \{ M \mid M \cong e_1R(\alpha) \oplus N \text{ with } N = Nf \}$. We shall show that (in contrast to the artin algebra case [8]) even though $R$ is not right tilted, $\mathcal{Y} = \text{Cogen } U$ for a certain cotilting module $U_R$.

If $C_0$ is any injective cogenerator in Mod-$S$, then letting $C_R = [0 \ C_0]$ we see that $C = Cf$ is injective over $fRf$ and cogenerates every $N = Nf$ in Mod-$R$.

**Proposition 2.1.** Let $R = T(S)$ and let $C = Cf$ be an $R$-module such that $Cf$ is an injective cogenerator in Mod-$fRf$, and let $U = e_1R \oplus C$. Then $\mathcal{Y} = \text{Cogen } U$, and $U$ is a cotilting module in the sense that $\text{Cogen } U = \text{Ker Ext}^1_R(\cdot, U)$.

**Proof.** Clearly $C$ cogenerates every $N = Nf$ in Mod-$R$, and so by Lemma 1.2, $\mathcal{Y} = \text{Cogen } U$. Also, by Lemma 1.2, $\mathcal{Y} \subseteq \text{Ker Ext}^1_R(\cdot, U)$. Indeed, since $e_1R(\alpha)$ is projective and $e_1R$ and $C_{fRf}$ are injective,

$$\text{Ext}^1_R(e_1R(\alpha) \oplus Nf, e_1R \oplus C) = \text{Ext}^1_R(Nf, C) \cong \text{Ext}^1_{fRf}(Nf, C) = 0.$$

Finally, if $M = X \oplus Y$ with $X \in \mathcal{X}$, and $Y \in \mathcal{Y}$, then since $X \in \text{Gen}(e_1R)$ there is an exact sequence

$$0 \to K \to e_1R(\beta) \to X \to 0$$

with $e_1R(\beta) \to X \to 0$ a projective cover, and $0 \neq K \subseteq e_1J(\beta)$. Thus $K = Kf \in \text{Cogen } C$ and we have an exact sequence

$$0 = \text{Hom}_R(e_1R(\beta), C) \to \text{Hom}_R(K, C) \to \text{Ext}^1_R(X, C)$$

showing that $M \in \mathcal{Y}$ whenever $\text{Ext}^1_R(M, U) = 0$. ■

If $R$ is a hereditary prime noetherian (HNP) ring, it follows from [2, Lemma 1] and [1, Exercise 14.11(2)] that the injective $S$-module $Q/S$ is a cogenerator. Thus $J/tS$ is injective over $fRf$ and cogenerates every $N = Nf$ in Mod-$R$, so that when $R$ is an HNP ring we may choose $C = J/tS$ in Proposition 2.1.
Finally one is led to wonder which right quasitilted rings are right tilted. For example, is \( \mathcal{X} \) closed under direct products and \( \mathcal{Y} \) cogenerated by a cotilting module enough?

**REFERENCES**


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