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THE NATURAL OPERATORS LIFTING HORIZONTAL 1-FORMS TO SOME VECTOR BUNDLE FUNCTORS ON FIBERED MANIFOLDS

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Abstract. Let $F : \mathcal{FM} \to \mathcal{VB}$ be a vector bundle functor. First we classify all natural operators $T_{\mathrm{proj}|\mathcal{FM}_{m,n}} \rightsquigarrow T^{(0,0)}(F_{|\mathcal{FM}_{m,n}})^*$ transforming projectable vector fields on Y to functions on the dual bundle $(FY)^*$ for any $\mathcal{FM}_{m,n}$-object Y. Next, under some assumption on F we study natural operators $T^*_{\mathrm{hor}|\mathcal{FM}_{m,n}} \rightsquigarrow T^*(F_{|\mathcal{FM}_{m,n}})^*$ lifting horizontal 1-forms on Y to 1-forms on $(FY)^*$ for any Y as above. As an application we classify natural operators $T^*_{\mathrm{hor}|\mathcal{FM}_{m,n}} \rightsquigarrow T^*(F_{|\mathcal{FM}_{m,n}})^*$ for some vector bundle functors F on fibered manifolds.

0. Introduction. In this paper we consider the following categories over manifolds: the category $\mathcal{M}f$ of manifolds and maps, the category $\mathcal{M}f_m$ of m-dimensional manifolds and embeddings, the category $\mathcal{FM}$ of fibered manifolds and fibered maps, the category $\mathcal{FM}_{m,n}$ of fibered manifolds with m-dimensional bases and n-dimensional fibers and fibered embeddings, and the category $\mathcal{VB}$ of all vector bundles and vector bundle maps.

The notions of bundle functors and natural operators can be found in the fundamental monograph [3].

In [5], given a vector bundle functor $F : \mathcal{M}f \to \mathcal{VB}$ we classified all natural operators $A : T_{|\mathcal{M}f_m} \rightsquigarrow T^{(0,0)}(F_{|\mathcal{M}f_m})^*$ transforming a vector field X on an m-manifold M into a function $A(X) : (FM)^* \to \mathbb{R}$ on the dual vector bundle $(FM)^*$ and proved that every natural operator $B : T^*_{|\mathcal{M}f_m} \rightsquigarrow T^*(F_{|\mathcal{M}f_m})^*$ transforming a 1-form ω on an m-manifold M into a 1-form $B(\omega)$ on $(FM)^*$ is of the form $B(\omega) = a\omega^V + \lambda$ for some uniquely determined canonical map $a : (FM)^* \to \mathbb{R}$ and some canonical 1-form λ on $(FM)^*$. These results were generalizations of [1, 4].

In the present paper we study similar problems for a vector bundle functor $F : \mathcal{FM} \to \mathcal{VB}$ on a fibered manifold instead of on a manifold. Modifying methods from [5], for natural numbers m and n we classify all natural operators $A : T_{\mathrm{proj}|\mathcal{FM}_{m,n}} \rightsquigarrow T^{(0,0)}(F_{|\mathcal{FM}_{m,n}})^*$ transform-

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ing a projectable vector field X on an (m,n)-dimensional fibered manifold Y into a function $A(X) : (FY)^* \to \mathbb{R}$ on the dual vector bundle $(FY)^*$ and prove (under some assumption on F) that every natural operator $B : T^*_{\mathrm{hor}|\mathcal{FM}_{m,n}} \rightsquigarrow T^*(F_{|\mathcal{FM}_{m,n}})^*$ transforming a horizontal 1-form ω on an (m,n)-dimensional fibered manifold Y into a 1-form $B(\omega)$ on $(FY)^*$ is of the form $B(\omega) = a\omega^V + \lambda$ for some uniquely determined canonical map $a : (FY)^* \to \mathbb{R}$ and some canonical 1-form λ on $(FY)^*$. As an application we describe all natural operators $B : T^*_{\mathrm{hor}|\mathcal{FM}_{m,n}} \rightsquigarrow T^*(F_{|\mathcal{FM}_{m,n}})^*$ for some vector bundle functors F on fibered manifolds.

From now on the usual coordinates on $\mathbb{R}^{m,n}$, the trivial bundle $\mathbb{R}^m \times \mathbb{R}^n$ over $\mathbb{R}^m$, will be denoted by $x^1, \ldots, x^m, y^1, \ldots, y^n$.

All manifolds are assumed to be finite-dimensional and smooth, i.e. of class $\mathcal{C}^\infty$. Maps between manifolds are assumed to be smooth.

1. Natural operators $T_{\mathrm{proj}|\mathcal{FM}_{m,n}} \rightsquigarrow T^{(0,0)}(F_{|\mathcal{FM}_{m,n}})^*$. Let $F : \mathcal{FM} \to \mathcal{VB}$ be a vector bundle functor. Let m and n be natural numbers. In this section modifying methods from [5] we classify the natural operators $A : T_{\mathrm{proj}|\mathcal{FM}_{m,n}} \rightsquigarrow T^{(0,0)}(F_{|\mathcal{FM}_{m,n}})^*$ transforming a projectable vector field X on an (m,n)-dimensional fibered manifold Y into a function $A(X) : (FY)^* \to \mathbb{R}$ on the dual vector bundle $(FY)^*$.

We recall that a *projectable* vector field on a fibered manifold Y over M is a vector field X on Y such that there exists an underlying vector field $\underline{X}$ on M which is p-related with X, where $p : Y \to M$ is the bundle projection. The flow of a projectable vector field is formed by $\mathcal{FM}$-morphisms.

The following example is an extension of Example 1 in [5] to fibered manifolds.

EXAMPLE 1. Let $v \in F_0(\mathbb{R}^{1,0})$. Consider a projectable vector field X on an (m,n)-dimensional fibered manifold Y over M. We define $A^v(X) : (FY)^* \to \mathbb{R}$ by $A^v(X)_\eta = \langle \eta, F(\Phi^X_y)(v)\rangle$ for $\eta \in (F_yY)^*$, $y \in Y_x$, $x \in M$. Here $\Phi^X_y : (\varepsilon, \varepsilon) \to Y$ with $\Phi^X_y(t) = \mathrm{Exp}(tX)_y$ for $t \in (-\varepsilon, \varepsilon)$, $\varepsilon > 0$. We consider Φ^X_y as a fibered map $\mathbb{R}^{1,0} \to Y$ covering $\Phi^{\underline{X}}_x : (-\varepsilon, \varepsilon) \to M$, where $\Phi^{\underline{X}}_x(t) = \mathrm{Exp}(t\underline{X})_x$ for $t \in (-\varepsilon, \varepsilon)$. The correspondence $A^v : T_{\mathrm{proj}|\mathcal{FM}_{m,n}} \rightsquigarrow T^{(0,0)}(F_{|\mathcal{FM}_{m,n}})^*$ is a natural operator.

PROPOSITION 1. *Let $v_1, \ldots, v_L \in F_0\mathbb{R}^{1,0}$ be a basis of the vector space $F_0\mathbb{R}^{1,0}$. Every natural operator $A : T_{\mathrm{proj}|\mathcal{FM}_{m,n}} \rightsquigarrow T^{(0,0)}(F_{|\mathcal{FM}_{m,n}})^*$ is of the form*

$$A = H(A^{v_1}, \ldots, A^{v_L})$$

for some uniquely determined smooth map $H \in \mathcal{C}^\infty(\mathbb{R}^L)$.

Proof. We modify the proof of Proposition 1 in [5] as follows. Let $v_1^*, \ldots$ $\ldots, v_L^* \in (F_0\mathbb{R}^{1,0})^*$ be the dual basis. Let $q : \mathbb{R}^m \times \mathbb{R}^n \to \mathbb{R}$ be the projection onto the first factor. It is a fibered map $\mathbb{R}^{m,n} \to \mathbb{R}^{1,0}$ over the projection $\mathbb{R}^m \to \mathbb{R}$ onto the first factor. For A as above we define $H : \mathbb{R}^L \to \mathbb{R}$ by

$$H(t_1, \ldots, t_L) = A(\partial/\partial x^1)_{(F_0 q)^*(\sum_{s=1}^L t_s v_s^*)}.$$

We prove that $A = H(A^{v_1}, \ldots, A^{v_L})$. Since any projectable vector field X on an $\mathcal{FM}_{m,n}$-object Y such that the underlying vector field $\underline{X}$ is non-vanishing is locally $\partial/\partial x^1$ in some local fiber coordinates on Y, it is sufficient to show that

$$A(\partial/\partial x^1)_\eta = H(A^{v_1}(\partial/\partial x^1)_\eta, \ldots, A^{v_L}(\partial/\partial x^1)_\eta)$$

for any $\eta \in (F_0\mathbb{R}^{m,n})^*$. By the invariance of A and A^{v_s} with respect to $\mathcal{FM}_{m,n}$-morphisms $(x^1, t^{-1}x^2, \ldots, t^{-1}x^m, t^{-1}y^1, \ldots, t^{-1}y^n) : \mathbb{R}^m \times \mathbb{R}^n \to \mathbb{R}^m \times \mathbb{R}^n$ for $t \neq 0$ and next by letting $t \to 0$, we can assume that $\eta = (F_0 q)^*(\sum_{s=1}^L t_s v_s^*)$. Now, it remains to observe that $A^{v_s}(\partial/\partial x^1)_\eta = t_s$ for $s = 1, \ldots, L$.

The uniqueness of H is clear because $(A^{v_s}(\partial/\partial x^1))_{s=1}^L$ is a surjection onto $\mathbb{R}^L$. ∎

We have a functor $i : \mathcal{M}f \to \mathcal{FM}$, $i(M) = (\mathrm{id}_M : M \to M)$, $i(f) = f$, $M \in \mathrm{obj}(\mathcal{M}f)$, $f : M \to N$, which is an $\mathcal{M}f$-morphism.

Thus we have a vector bundle functor $F \circ i : \mathcal{M}f \to \mathcal{VB}$. So, by [2], we can choose a basis $v_1, \ldots, v_L \in F_0\mathbb{R}^{1,0} = (F \circ i)_0\mathbb{R}$ such that v_s is homogeneous of weight $n_s \in \mathbb{N} \cup \{0\}$, i.e. $F(\tau\,\mathrm{id})(v_s) = \tau^{n_s} v_s$ for any $\tau \in \mathbb{R}$.

(∗) By a permutation we assume that $v_1, \ldots, v_{k_1}$ are of weight 0, and $v_{k_1+1}, \ldots, v_{k_2}$ are of weight 1, and so on.

Then $A^{v_1}(X), \ldots, A^{v_{k_1}}(X)$ do not depend on X, i.e. $A^{v_1}, \ldots, A^{v_{k_1}}$ are natural functions on $(FY)^*$. Moreover $A^{v_{k_1+1}}(X), \ldots, A^{v_{k_2}}(X)$ depend linearly on X, i.e. $A^{v_{k_1+1}}, \ldots, A^{v_{k_2}}$ are linear operators.

The following corollaries are simple consequences of Proposition 1 and the homogeneous function theorem.

Corollary 1. *Every natural* (*canonical*) *function G on $(F_{|\mathcal{FM}_{m,n}})^*$ is of the form*

$$G = K(A^{v_1}, \ldots, A^{v_{k_1}})$$

for some uniquely determined $K \in \mathcal{C}^\infty(\mathbb{R}^{k_1})$. If $F \circ i$ has the point property, i.e. $F \circ i(\mathrm{pt}) = \mathrm{pt}$, then $G = \mathrm{const}$, where pt *denotes a one-point manifold.*

Corollary 2. *Let $A : T_{\mathrm{proj}|\mathcal{FM}_{m,n}} \rightsquigarrow T^{(0,0)}(F_{|\mathcal{FM}_{m,n}})^*$ be a natural linear operator. Then*

$$A = \sum_{s=k_1+1}^{k_2} K_s(A^{v_1}, \ldots, A^{v_{k_1}}) A^{v_s}$$

for some uniquely determined $K_s \in \mathcal{C}^\infty(\mathbb{R}^{k_1})$.

2. A decomposition proposition. Let F and $v_1, \ldots, v_L$ be as in Section 1 with the assumption $(*)$. Let $i : \mathcal{M}f \to \mathcal{FM}$ be the functor as in Section 1.

Let $p : Y \to M$ be a fibered manifold. A 1-form $\omega : TY \to \mathbb{R}$ on Y is called *horizontal* if $\omega|VY = 0$, where VY is the vertical bundle.

EXAMPLE 2. If $\omega : TY \to \mathbb{R}$ is a horizontal 1-form on a fibered manifold Y, we have its vertical lifting $B^V(\omega) = \omega \circ T\pi : T(FY)^* \to \mathbb{R}$ to $(FY)^*$, where $\pi : (FY)^* \to Y$ is the bundle projection. The correspondence $B^V : T^*_{\mathrm{hor}|\mathcal{FM}_{m,n}} \rightsquigarrow T^*(F_{|\mathcal{FM}_{m,n}})^*$ is a natural operator.

ASSUMPTION 1. From now on we assume that there exists a basis $w_1, \ldots$ $\ldots, w_K \in F_0\mathbb{R}^{m,n}$ such that w_s is homogeneous of weight $n_s \in \mathbb{N} \cup \{0\}$. This means that $F(\tau\, \mathrm{id}_{\mathbb{R}^m \times \mathbb{R}^n})(w_s) = \tau^{n_s} w_s$ for any $\tau \in \mathbb{R}$.

REMARK 1. It seems that every vector bundle functor $F : \mathcal{FM} \to \mathcal{VB}$ satisfies Assumption 1.

PROPOSITION 2 (Decomposition Proposition). *Consider a natural operator $B : T^*_{\mathrm{hor}|\mathcal{FM}_{m,n}} \rightsquigarrow T^*(F_{|\mathcal{FM}_{m,n}})^*$. Under Assumption* 1 *there exists a uniquely determined natural function a on $(F_{|\mathcal{FM}_{m,n}})^*$ such that*

$$B = aB^V + \lambda$$

for some canonical 1*-form λ on $(F_{|\mathcal{FM}_{m,n}})^*$.*

LEMMA 1. (a) *We have $(B(\omega) - B(0))|(V(F\mathbb{R}^{m,n})^*)_0 = 0$ for any horizontal* 1*-form ω on $\mathbb{R}^{m,n}$, where $(V(F\mathbb{R}^{m,n})^*)_0$ is the fiber over $0 \in \mathbb{R}^m \times \mathbb{R}^n$ of the vertical subbundle in $T(F\mathbb{R}^{m,n})^*$.*

(b) *If $F \circ i$ has the point property then $B(\omega)|(V(F\mathbb{R}^{m,n})^*)_0 = 0$ for any horizontal* 1*-form ω on $\mathbb{R}^{m,n}$.*

Proof. We modify the proof of Lemma 1 in [5] as follows.

(a) We use the invariance of $(B(\omega) - B(0))|(V(F\mathbb{R}^{m,n})^*)_0$ with respect to the homotheties $t^{-1}\, \mathrm{id}_{\mathbb{R}^m \times \mathbb{R}^n}$ for $t \neq 0$ and apply the homogeneous function theorem. We deduce that $(B(\omega) - B(0))|(V(F\mathbb{R}^{m,n})^*)_0$ is independent of ω.

(b) We observe that if $F \circ i$ has the point property then $(F_0\mathbb{R}^{m,n})^*$ has no non-zero homogeneous elements of weight 0. Next, we use the invariance of $B(\omega)|(V(F\mathbb{R}^{m,n})^*)_0$ with respect to the homotheties $t^{-1}\, \mathrm{id}_{\mathbb{R}^m \times \mathbb{R}^n}$ for $t \neq 0$ and let $t \to 0$. ∎

Proof of Proposition 2. We modify the proof of Proposition 2 in [5]. Replacing B by $B - B(0)$ we can assume $B(0) = 0$ and $B(\omega)|(V(F\mathbb{R}^{m,n})^*)_0 = 0$. Then B is determined by the values $\langle B(\omega)_\eta, F^*(\partial/\partial x^1)_\eta \rangle$ for all horizontal 1-forms $\omega = \sum_{i=1}^m \omega_i dx^i$ on $\mathbb{R}^{m,n}$ and $\eta \in (F_0\mathbb{R}^{m,n})^*$, with $F^*(\partial/\partial x^1)$ the complete lifting (flow prolongation) of $\partial/\partial x^1$ to $(F\mathbb{R}^{m,n})^*$. Using the invariance of B with respect to the homotheties $t^{-1}\, \mathrm{id}_{\mathbb{R}^m \times \mathbb{R}^n}$ for $t \neq 0$ we get the homogeneity condition

$$t\langle B(\omega)_\eta, F^*(\partial/\partial x^1)_\eta\rangle$$
$$= \langle B((t\,\mathrm{id}_{\mathbb{R}^m\times\mathbb{R}^n})^*\omega)_{F(t^{-1}\,\mathrm{id}_{\mathbb{R}^m\times\mathbb{R}^n})^*(\eta)}, F^*(\partial/\partial x^1)_{F(t^{-1}\,\mathrm{id}_{\mathbb{R}^m\times\mathbb{R}^n})^*(\eta)}\rangle.$$

Then by the non-linear Peetre theorem [3], the homogeneous function theorem and $B(0)=0$ we deduce that $\langle B(\omega)_\eta, F^*(\partial/\partial x^1)_\eta\rangle$ is a linear combination of $\omega_1(0),\dots,\omega_m(0)$ with coefficients being smooth maps in homogeneous coordinates of η of weight 0. Then using the invariance of B with respect to $(x^1, t^{-1}x^2,\dots,t^{-1}x^m, t^{-1}y^1,\dots,t^{-1}y^n):\mathbb{R}^m\times\mathbb{R}^n\to\mathbb{R}^m\times\mathbb{R}^n$ for $t\neq 0$ and letting $t\to 0$ we end the proof. ∎

3. On canonical 1-forms on $(F_{|\mathcal{FM}_{m,n}})^*$. The injectivity in the following proposition is a consequence of Lemma 1(b).

PROPOSITION 3. *Every natural* (*canonical*) 1*-form λ on $(F_{|\mathcal{FM}_{m,n}})^*$ induces a natural linear operator $A^{(\lambda)}: T_{\mathrm{proj}|\mathcal{FM}_{m,n}} \rightsquigarrow T^{(0,0)}(F_{|\mathcal{FM}_{m,n}})^*$ such that $A^{(\lambda)}(X)_\eta = \langle\lambda_\eta, F^*(X)_\eta\rangle$ for $\eta\in(FY)^*$, $X\in\mathcal{X}_{\mathrm{proj}}(Y)$, where $F^*(X)$ is the complete lifting* (*flow operator*) *of X to $(FY)^*$. If $F\circ i$ has the point property, then* (*under Assumption* 1) *the correspondence $\lambda\mapsto A^{(\lambda)}$ is a linear injection.*

4. A corollary. Let $i:\mathcal{M}f\to\mathcal{FM}$ be the functor as in Section 1.

COROLLARY 3. *Assume that $F\circ i$ has the point property and there are no non-zero elements from $F_0\mathbb{R}^{1,0}$ of weight* 1. (*For example, let $F=F_1\otimes F_2:\mathcal{FM}\to\mathcal{VB}$ be the tensor product of two vector bundle functors $F_1,F_2:\mathcal{FM}\to\mathcal{VB}$ such that $F_1\circ i, F_2\circ i$ have the point property.*) *Then* (*under Assumption* 1) *every natural operator $B: T^*_{\mathrm{hor}|\mathcal{FM}_{m,n}}\rightsquigarrow T^*(F_{|\mathcal{FM}_{m,n}})^*$ is a constant multiple of the vertical lifting.*

Proof. Since there are no non-zero elements from $F_0\mathbb{R}^{1,0}$ of weight 1, we see that every canonical 1-form on $(F_{|\mathcal{FM}_{m,n}})^*$ is zero because of Corollary 2 and Proposition 3. Then Proposition 2 together with Corollary 1 ends the proof. ∎

5. Applications. From now on let r,s,q be natural numbers with $s\geq r\leq q$.

APPLICATION 1. The concept of r-jets can be generalized as follows (see [3]). Let $Y\to M$ and $Z\to N$ be fibered manifolds. We recall that two fibered maps $f,g:Y\to Z$ with base maps $\underline{f},\underline{g}:M\to N$ determine the same (r,s,q)-jet $j_y^{(r,s,q)}f = j_y^{(r,s,q)}g$ at $y\in Y_x$, $x\in M$, if $j_y^r f = j_y^r g$, $j_y^s(f|Y_x) = j_y^s(g|Y_x)$ and $j_x^q\underline{f} = j_x^q\underline{g}$. The space of all (r,s,q)-jets of Y into Z

is denoted by $J^{(r,s,q)}(Y,Z)$. The composition of fibered maps induces the composition of (r,s,q)-jets [3, p. 126].

The vector r-tangent bundle functor $T^{(r)} = (J^r(\cdot,\mathbb{R})_0)^* : \mathcal{M}f \to \mathcal{VB}$ can be generalized as follows. Let $\mathbb{R}^{1,1} = \mathbb{R}\times\mathbb{R}$ be the trivial bundle over $\mathbb{R}$. The space $J^{(r,s,q)}(Y,\mathbb{R}^{1,1})_0$, $0 \in \mathbb{R}^2$, has an induced structure of a vector bundle over Y. Every fibered map $f : Y \to Z$, $f(y) = z$, induces a linear map $\lambda(j^{(r,s,q)}_y f) : J^{(r,s,q)}_z(Z,\mathbb{R}^{1,1})_0 \to J^{(r,s,q)}_y(Y,\mathbb{R}^{1,1})_0$ by means of the jet composition. If we denote by $T^{(r,s,q)}Y$ the dual vector bundle of $J^{(r,s,q)}(Y,\mathbb{R}^{1,1})_0$ and define $T^{(r,s,q)}f : T^{(r,s,q)}Y \to T^{(r,s,q)}Z$ by using the dual maps to $\lambda(j^{(r,s,q)}_y f)$, we obtain a vector bundle functor $T^{(r,s,q)} : \mathcal{FM} \to \mathcal{VB}$.

Example 3. We have canonical 1-forms $\lambda^{(r,s,q)}_\alpha : TJ^{(r,s,q)}(Y,\mathbb{R}^{1,1})_0 \to \mathbb{R}$ on $J^{(r,s,q)}(Y,\mathbb{R}^{1,1})_0$ for $\alpha = 1,2$ defined by $\lambda^{(r,s,q)}_\alpha(v) = d\gamma_\alpha(T\pi(v))$ for $v \in T_w J^{(r,s,q)}(Y,\mathbb{R}^{1,1})_0$, $w = j^{(r,s,q)}_y(\gamma_1,\gamma_2)$, $y \in Y$, where $\pi : J^{(r,s,q)}(Y,\mathbb{R}^{1,1})_0 \to Y$ is the bundle projection.

Corollary 4. *Every natural operator*

$$B : T^*_{\mathrm{hor}|\mathcal{FM}_{m,n}} \rightsquigarrow T^*(J^{(r,s,q)}(\cdot,\mathbb{R}^{1,1})_0)$$

is a linear combination of the vertical lifting B^V *and the canonical* 1*-forms* $\lambda^{(r,s,q)}_1$ *and* $\lambda^{(r,s,q)}_2$ *with real coefficients.*

Proof. The vector bundle functor $T^{(r,s,q)}$ satisfies Assumption 1. Moreover, $T^{(r,s,q)} \circ i$ has the point property and the subspace of elements from $T^{(r,s,q)}_0\mathbb{R}^{1,0}$ of weight 1 is 2-dimensional. Then by Proposition 3 together with Corollaries 1 and 2, the space of canonical 1-forms on $J^{(r,s,q)}(\cdot,\mathbb{R}^{1,1})_0$ is at most 2-dimensional. Now, Proposition 2 ends the proof. ■

Application 2. Let r,s be integers such that $s \geq r \geq 0$. The concept of r-jets can also be generalized as follows (see [3]). Let $Y \to M$ be a fibered manifold and Q be a manifold. We recall that two maps $f,g : Y \to Q$ determine the same (r,s)-jet $j^{(r,s)}_y f = j^{(r,s)}_y g$ at $y \in Y_x$, $x \in M$, if $j^r_y f = j^r_y g$ and $j^s_y(f|Y_x) = j^s_y(g|Y_x)$. The space of all (r,s)-jets of Y into Q is denoted by $J^{(r,s)}(Y,Q)$.

The vector r-tangent bundle functor $T^{(r)} = (J^r(\cdot,\mathbb{R})_0)^* : \mathcal{M}f \to \mathcal{VB}$ can be generalized as follows. The space $J^{(r,s)}(Y,\mathbb{R})_0$, $0 \in \mathbb{R}$, has an induced structure of a vector bundle over Y. Every fibered map $f : Y \to Z$, $f(y) = z$, induces a linear map $\lambda(j^{(r,s)}_y f) : J^{(r,s)}_z(Z,\mathbb{R})_0 \to J^{(r,s)}_y(Y,\mathbb{R})_0$ by means of the jet composition. If we denote by $T^{(r,s)}Y$ the dual vector bundle of $J^{(r,s)}(Y,\mathbb{R})_0$ and define $T^{(r,s)}f : T^{(r,s)}Y \to T^{(r,s)}Z$ by using the dual maps to $\lambda(j^{(r,s)}_y f)$, we obtain a vector bundle functor $T^{(r,s)} : \mathcal{FM} \to \mathcal{VB}$.

Example 4. Assume additionally $r \geq 1$. We have a canonical 1-form $\lambda^{(r,s)} : TJ^{(r,s)}(Y,\mathbb{R})_0 \to \mathbb{R}$ on $J^{(r,s)}(Y,\mathbb{R})_0$ defined by $\lambda^{(r,s)}(v) = d\gamma(T\pi(v))$

for $v \in T_w J^{(r,s)}(Y,\mathbb{R})_0$, $w = j_y^{(r,s)}(\gamma)$, $y \in Y$, where $\pi : J^{(r,s)}(Y,\mathbb{R})_0 \to Y$ is the bundle projection.

Corollary 5. *Let r, s be as above. Every natural operator*

$$B : T^*_{\mathrm{hor}|\mathcal{FM}_{m,n}} \rightsquigarrow T^*(J^{(r,s)}(\cdot,\mathbb{R})_0)$$

is a linear combination of the vertical lifting B^V and the canonical 1-form $\lambda^{(r,s)}$ with real coefficients. If $r = 0$, then the $\lambda^{(0,s)}$ do not occur.

Proof. Note that the subspace of elements from $T_0^{(r,s)}\mathbb{R}^{1,0}$ of weight 1 is 1-dimensional, and use the same arguments as in the proof of Corollary 4. ∎

Application 3. For any fibered manifold Y we have the vertical bundle VY of Y and for every $\mathcal{FM}$-map $f : Y \to Z$ we have the induced map $Vf : VY \to VZ$. The functor $V : \mathcal{FM} \to \mathcal{VB}$ is a vector bundle functor. Let $V^* = (V_{|\mathcal{FM}_{m,n}})^*$ be the dual bundle functor.

Corollary 6. *Every natural operator $B : T^*_{\mathrm{hor}|\mathcal{FM}_{m,n}} \rightsquigarrow T^*V^*$ is a constant multiple of the vertical lifting B^V.*

Proof. We observe that $V \cong T^{(0,1)}$ and apply Corollary 5. Given a $\mathcal{FM}$-object $p : Y \to M$, an isomorphism $i : VY \to T^{(0,1)}Y$ is given by $i(v)(j_y^{(0,1)}\gamma) = d_y(\gamma|Y_{p(y)})(v)$. ∎

Application 4. For any fibered manifold Y we have a vector bundle

$$J^r T^*_{\mathrm{hor}} Y = \{ j^r_y \omega \mid \omega \text{ is a horizontal 1-form on } Y,\ y \in Y\}$$

over Y. Let $(J^r T^*_{\mathrm{hor}})^* Y = (J^r T^*_{\mathrm{hor}} Y)^*$ be the dual bundle. Every $\mathcal{FM}$-map $f : Y \to Z$ induces a vector bundle map $(J^r T^*_{\mathrm{hor}})^* f : (J^r T^*_{\mathrm{hor}})^* Y \to (J^r T^*_{\mathrm{hor}})^* Z$ covering f such that $\langle (J^r T^*_{\mathrm{hor}})^* f(\eta), j^r_{f(y)}\omega\rangle = \langle \eta, j^r_y(f^*\omega)\rangle$ for $\eta \in (J^r T^*_{\mathrm{hor}})^*_y Y$, $j^r_{f(y)}\omega \in (J^r T^*_{\mathrm{hor}})_{f(y)} Z$, $y \in Y$. The functor $(J^r T^*_{\mathrm{hor}})^* : \mathcal{FM} \to \mathcal{VB}$ is a vector bundle functor.

Given an $\mathcal{FM}_{m,n}$-object Y we have a canonical 1-form θ^r on $J^r T^*_{\mathrm{hor}} Y$ such that

$$\langle \theta^r_w, v\rangle = \langle \omega_y, T\pi(v)\rangle$$

for $v \in T_w(J^r T^*_{\mathrm{hor}} Y)$, $w = j^r_y \omega$, $y \in Y$, $\omega \in \Omega^1_{\mathrm{hor}}(Y)$, where $\pi : J^r T^*_{\mathrm{hor}} Y \to Y$ is the bundle projection.

Corollary 7. *Every natural operator $B : T^*_{\mathrm{hor}|\mathcal{FM}_{m,n}} \rightsquigarrow T^*(J^r T^*_{\mathrm{hor}})$ is a linear combination of the vertical lifting B^V and θ^r with real coefficients.*

Proof. We observe that the subspace of elements from $(J^r T^*_{\mathrm{hor}})^*_0 \mathbb{R}^{1,0}$ of weight 1 is 1-dimensional. ∎

Application 5. We can generalize Application 4 as follows. For any fibered manifold Y we have a vector bundle

$$J^r(\wedge^k T^*_{\mathrm{hor}})Y = \{ j^r_y \omega \mid \omega \text{ is a horizontal } k\text{-form on } Y,\ y \in Y\}$$

over Y. Let $(J^r(\wedge^k T^*_{\mathrm{hor}}))^*Y = (J^r(\wedge^k T^*_{\mathrm{hor}})Y)^*$ be the dual bundle. Every $\mathcal{FM}$-map $f : Y \to Z$ induces a vector bundle map $(J^r(\wedge^k T^*_{\mathrm{hor}}))^* f : (J^r(\wedge^k T^*_{\mathrm{hor}}))^* Y \to (J^r(\wedge^k T^*_{\mathrm{hor}}))^* Z$ covering f such that

$$\langle (J^r(\wedge^k T^*_{\mathrm{hor}}))^* f(\eta), j^r_{f(y)}\omega\rangle = \langle \eta, j^r_y(f^*\omega)\rangle$$

for $\eta \in (J^r(\wedge^k T^*_{\mathrm{hor}}))^*_y Y$, $j^r_{f(y)}\omega \in (J^r(\wedge^k T^*_{\mathrm{hor}}))_{f(y)} Z$, $y \in Y$. Then $(J^r(\wedge^k T^*_{\mathrm{hor}}))^* : \mathcal{FM} \to \mathcal{VB}$ is a vector bundle functor.

Corollary 8. *Let $k \ge 2$. Every natural operator $B : T^*_{\mathrm{hor}|\mathcal{FM}_{m,n}} \rightsquigarrow T^*(J^r(\wedge^k T^*_{\mathrm{hor}}))$ is a constant multiple of the vertical lifting B^V.*

Proof. We observe that the subspace of elements from $(J^r(\wedge^k T^*_{\mathrm{hor}}))^*_0 \mathbb{R}^{1,0}$ of weight 1 is 0-dimensional. ■

Similar facts hold for

$$J^r(\otimes^k T^*_{\mathrm{hor}})Y = \{j^r_y\tau \mid \tau \text{ is a horizontal tensor field of type } (0,k) \text{ on } Y,\ y \in Y\},$$

$$J^r(\odot^k T^*_{\mathrm{hor}})Y = \{j^r_y\tau \mid \tau \text{ is a horizontal symmetric tensor field of type } (0,k) \text{ on } Y,\ y \in Y\}$$

in place of $J^r(\wedge^k T^*_{\mathrm{hor}})Y$.

Application 6. We can also generalize Application 4 as follows. Let r and s be two integers with $s \ge r \ge 0$. For any fibered manifold Y we have a vector bundle

$$J^{(r,s)}T^*_{\mathrm{hor}}Y = \{j^{(r,s)}_y\omega \mid \omega \text{ is a horizontal 1-form on } Y,\ y \in Y\}$$

over Y. Let $(J^{(r,s)}T^*_{\mathrm{hor}})^*Y = (J^{(r,s)}T^*_{\mathrm{hor}}Y)^*$ be the dual bundle. Every $\mathcal{FM}$-map $f : Y \to Z$ induces a vector bundle map $(J^{(r,s)}T^*_{\mathrm{hor}})^* f : (J^{(r,s)}T^*_{\mathrm{hor}})^* Y \to (J^{(r,s)}T^*_{\mathrm{hor}})^* Z$ covering f such that

$$\langle (J^{(r,s)}T^*_{\mathrm{hor}})^* f(\eta), j^{(r,s)}_{f(y)}\omega\rangle = \langle \eta, j^{(r,s)}_y(f^*\omega)\rangle$$

for $\eta \in (J^{(r,s)}T^*_{\mathrm{hor}})^*_y Y$, $j^{(r,s)}_{f(y)}\omega \in (J^{(r,s)}T^*_{\mathrm{hor}})_{f(y)} Z$, $y \in Y$. Then $(J^{(r,s)}T^*_{\mathrm{hor}})^* : \mathcal{FM} \to \mathcal{VB}$ is a vector bundle functor.

Given an $\mathcal{FM}_{m,n}$-object Y we have a canonical 1-form $\Theta^{(r,s)}$ on $J^{(r,s)}T^*_{\mathrm{hor}}Y$ such that

$$\langle \Theta^{(r,s)}_w, v\rangle = \langle \omega_y, T\pi(v)\rangle$$

for $v \in T_w(J^{(r,s)}T^*_{\mathrm{hor}}Y)$, $w = j^{(r,s)}_y\omega$, $y \in Y$, $\omega \in \Omega^1_{\mathrm{hor}}(Y)$, where $\pi : J^{(r,s)}T^*_{\mathrm{hor}}Y \to Y$ is the bundle projection.

Corollary 9. *Every natural operator $B : T^*_{\mathrm{hor}|\mathcal{FM}_{m,n}} \rightsquigarrow T^*(J^{(r,s)}T^*_{\mathrm{hor}})$ is a linear combination of the vertical lifting B^V and $\Theta^{(r,s)}$ with real coefficients.*

Proof. We observe that the subspace of elements from $(J^{(r,s)}T^*_{\mathrm{hor}})^*_0\mathbb{R}^{1,0}$ of weight 1 is 1-dimensional. ■

Of course, other applications are also possible. For example we can study liftings to $J^{(r,s)}(\wedge^k T^*_{\mathrm{hor}})$, $J^{(r,s)}(\otimes^k T^*_{\mathrm{hor}})$, $J^{(r,s)}(\odot^k T^*_{\mathrm{hor}})$, $J^r(T^*)$, $J^{(r,s)}(T^*)$, etc.

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