THE NATURAL OPERATORS LIFTING HORIZONTAL 1-FORMS
TO SOME VECTOR BUNDLE FUNCTORS
ON FIBERED MANIFOLDS

BY

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Abstract. Let $F : \mathcal{FM} \to \mathcal{VB}$ be a vector bundle functor. First we classify all natural operators $T_{\text{proj}}^{\mathcal{FM}_{m,n}} \mapsto T^{(0,0)}(F|\mathcal{FM}_{m,n})^*$ transforming projectable vector fields on $Y$ to functions on the dual bundle $(FY)^*$ for any $\mathcal{FM}_{m,n}$-object $Y$. Next, under some assumption on $F$ we study natural operators $T_{\text{hor}}^{\mathcal{FM}_{m,n}} \mapsto T^*(F|\mathcal{FM}_{m,n})^*$ lifting horizontal 1-forms on $Y$ to 1-forms on $(FY)^*$ for any $Y$ as above. As an application we classify natural operators $T_{\text{hor}}^{\mathcal{FM}_{m,n}} \mapsto T^*(F|\mathcal{FM}_{m,n})^*$ for some vector bundle functors $F$ on fibered manifolds.

0. Introduction. In this paper we consider the following categories over manifolds: the category $\mathcal{Mf}$ of manifolds and maps, the category $\mathcal{Mf}_m$ of $m$-dimensional manifolds and embeddings, the category $\mathcal{FM}$ of fibered manifolds and fibered maps, the category $\mathcal{FM}_{m,n}$ of fibered manifolds with $m$-dimensional bases and $n$-dimensional fibers and fibered embeddings, and the category $\mathcal{VB}$ of all vector bundles and vector bundle maps.

The notions of bundle functors and natural operators can be found in the fundamental monograph [3].

In [5], given a vector bundle functor $F : \mathcal{Mf} \to \mathcal{VB}$ we classified all natural operators $A : T|\mathcal{Mf}_m \mapsto T^{(0,0)}(F|\mathcal{Mf}_m)^*$ transforming a vector field $X$ on an $m$-manifold $M$ into a function $A(X) : (FM)^* \to \mathbb{R}$ on the dual vector bundle $(FM)^*$ and proved that every natural operator $B : T^*(F|\mathcal{Mf}_m)^*$ transforming a 1-form $\omega$ on an $m$-manifold $M$ into a 1-form $B(\omega) = a\omega^V + \lambda$ for some uniquely determined canonical map $a : (FM)^* \to \mathbb{R}$ and some canonical 1-form $\lambda$ on $(FM)^*$. These results were generalizations of [1, 4].

In the present paper we study similar problems for a vector bundle functor $F : \mathcal{FM} \to \mathcal{VB}$ on a fibered manifold instead of on a manifold. Modifying methods from [5], for natural numbers $m$ and $n$ we classify all natural operators $A : T_{\text{proj}}^{\mathcal{FM}_{m,n}} \mapsto T^{(0,0)}(F|\mathcal{FM}_{m,n})^*$ transform-
ing a projectable vector field \( X \) on an \((m, n)\)-dimensional fibered manifold \( Y \) into a function \( A(X): (FY)^* \to \mathbb{R} \) on the dual vector bundle \((FY)^*\) and prove (under some assumption on \( F \)) that every natural operator \( B: T^*_{\text{hor}}(\mathcal{F}M_{m,n}) \to T^*(F|_{\mathcal{F}M_{m,n}})^* \) transforming a horizontal 1-form \( \omega \) on an \((m, n)\)-dimensional fibered manifold \( Y \) into a 1-form \( B(\omega) \) on \((FY)^*\) is of the form \( B(\omega) = a\omega + \lambda \) for some uniquely determined canonical map \( a: (FY)^* \to \mathbb{R} \) and some canonical 1-form \( \lambda \) on \((FY)^*\). As an application we describe all natural operators \( B: T^*_{\text{hor}}(\mathcal{F}M_{m,n}) \to T^*(F|_{\mathcal{F}M_{m,n}})^* \) for some vector bundle functors \( F \) on fibered manifolds.

From now on the usual coordinates on \( \mathbb{R}^{m,n} \), the trivial bundle \( \mathbb{R}^m \times \mathbb{R}^n \) over \( \mathbb{R}^m \), will be denoted by \( x^1, \ldots, x^m, y^1, \ldots, y^n \).

All manifolds are assumed to be finite-dimensional and smooth, i.e. of class \( C^\infty \). Maps between manifolds are assumed to be smooth.

1. **Natural operators** \( T_{\text{proj}}(\mathcal{F}M_{m,n}) \to T^{(0,0)}(F|_{\mathcal{F}M_{m,n}})^* \). Let \( F: \mathcal{F}M \to \mathcal{V}B \) be a vector bundle functor. Let \( m \) and \( n \) be natural numbers. In this section modifying methods from [5] we classify the natural operators \( A: T_{\text{proj}}(\mathcal{F}M_{m,n}) \to T^{(0,0)}(F|_{\mathcal{F}M_{m,n}})^* \) transforming a projectable vector field \( X \) on an \((m, n)\)-dimensional fibered manifold \( Y \) into a function \( A(X): (FY)^* \to \mathbb{R} \) on the dual vector bundle \((FY)^*\).

We recall that a *projectable* vector field on a fibered manifold \( Y \) over \( M \) is a vector field \( X \) on \( Y \) such that there exists an underlying vector field \( \hat{X} \) on \( M \) which is \( p \)-related with \( X \), where \( p: Y \to M \) is the bundle projection. The flow of a projectable vector field is formed by \( \mathcal{F}M \)-morphisms.

The following example is an extension of Example 1 in [5] to fibered manifolds.

**Example 1.** Let \( v \in F_0(\mathbb{R}^{1,0}) \). Consider a projectable vector field \( X \) on an \((m, n)\)-dimensional fibered manifold \( Y \) over \( M \). We define \( A^v(X): (FY)^* \to \mathbb{R} \) by \( A^v(X)_\eta = \langle \eta, F(\Phi^X_y)(v) \rangle \) for \( \eta \in (F_Y)^* \), \( y \in Y_x \), \( x \in M \).

Here \( \Phi^X_y: (\varepsilon, \varepsilon) \to Y \) with \( \Phi^X_y(t) = \text{Exp}(tX)_y \) for \( t \in (-\varepsilon, \varepsilon), \varepsilon > 0 \). We consider \( \Phi^X_x \) as a fibered map \( \mathbb{R}^{1,0} \to Y \) covering \( \Phi^X_x: (-\varepsilon, \varepsilon) \to M \), where \( \Phi^X_x(t) = \text{Exp}(tX)_x \) for \( t \in (-\varepsilon, \varepsilon) \). The correspondence \( A^v: T_{\text{proj}}(\mathcal{F}M_{m,n}) \to T^{(0,0)}(F|_{\mathcal{F}M_{m,n}})^* \) is a natural operator.

**Proposition 1.** Let \( v_1, \ldots, v_L \in F_0(\mathbb{R}^{1,0}) \) be a basis of the vector space \( F_0(\mathbb{R}^{1,0}) \). Every natural operator \( A: T_{\text{proj}}(\mathcal{F}M_{m,n}) \to T^{(0,0)}(F|_{\mathcal{F}M_{m,n}})^* \) is of the form

\[
A = H(A^{v_1}, \ldots, A^{v_L})
\]

for some uniquely determined smooth map \( H \in C^\infty(\mathbb{R}^L) \).

**Proof.** We modify the proof of Proposition 1 in [5] as follows. Let \( v^*_1, \ldots, v^*_L \in (F_0(\mathbb{R}^{1,0}))^* \) be the dual basis. Let \( q: \mathbb{R}^m \times \mathbb{R}^n \to \mathbb{R} \) be the projection
onto the first factor. It is a fibered map $\mathbb{R}^{m,n} \to \mathbb{R}^{1,0}$ over the projection $\mathbb{R}^m \to \mathbb{R}$ onto the first factor. For $A$ as above we define $H : \mathbb{R}^L \to \mathbb{R}$ by

$$H(t_1, \ldots, t_L) = A(\partial/\partial x^1)(F_0 q)^*(\sum_{s=1}^L t_s v_s^*) .$$

We prove that $A = H(A^{v_1}, \ldots, A^{v_L})$. Since any projectable vector field $X$ on an $\mathcal{FM}_{m,n}$-object $Y$ such that the underlying vector field $X$ is non-vanishing is locally $\partial/\partial x^1$ in some local fiber coordinates on $Y$, it is sufficient to show that

$$A(\partial/\partial x^1)_\eta = H(A^{v_1}(\partial/\partial x^1)_\eta, \ldots, A^{v_L}(\partial/\partial x^1)_\eta)$$

for any $\eta \in (F_0 \mathbb{R}^{m,n})^*$. By the invariance of $A$ and $A^{v_s}$ with respect to $\mathcal{FM}_{m,n}$-morphisms $(x^1, t^{-1} x^2, \ldots, t^{-1} x^m, t^{-1} y^1, \ldots, t^{-1} y^n) : \mathbb{R}^m \times \mathbb{R}^n \to \mathbb{R}^m \times \mathbb{R}^n$ for $t \neq 0$ and next by letting $t \to 0$, we can assume that $\eta = (F_0 q)^*(\sum_{s=1}^L t_s v^*_s)$. Now, it remains to observe that $A^{v_s}(\partial/\partial x^1)_\eta = t_s$ for $s = 1, \ldots, L$.

The uniqueness of $H$ is clear because $(A^{v_s}(\partial/\partial x^1))_{s=1}^L$ is a surjection onto $\mathbb{R}^L$.

We have a functor $i : \mathcal{M}f \to \mathcal{FM}$, $i(M) = (\text{id}_M : M \to M)$, $i(f) = f$, $M \in \text{obj}(\mathcal{M}f)$, $f : M \to N$, which is an $\mathcal{M}f$-morphism.

Thus we have a vector bundle functor $F \circ i : \mathcal{M}f \to \mathcal{VB}$. So, by [2], we can choose a basis $v_1, \ldots, v_L \in F_0 \mathbb{R}^{1,0} = (F \circ i)_0 \mathbb{R}$ such that $v_s$ is homogeneous of weight $n_s \in \mathbb{N} \cup \{0\}$, i.e. $F(\tau \text{id})(v_s) = \tau^{n_s} v_s$ for any $\tau \in \mathbb{R}$.

(*) By a permutation we assume that $v_1, \ldots, v_{k_1}$ are of weight 0, and $v_{k_1+1}, \ldots, v_{k_2}$ are of weight 1, and so on.

Then $A^{v_1}(X), \ldots, A^{v_{k_1}}(X)$ do not depend on $X$, i.e. $A^{v_1}, \ldots, A^{v_{k_1}}$ are natural functions on $(FY)^*$. Moreover $A^{v_{k_1+1}}(X), \ldots, A^{v_{k_2}}(X)$ depend linearly on $X$, i.e. $A^{v_{k_1+1}}, \ldots, A^{v_{k_2}}$ are linear operators.

The following corollaries are simple consequences of Proposition 1 and the homogeneous function theorem.

**Corollary 1.** Every natural (canonical) function $G$ on $(F|\mathcal{FM}_{m,n})^*$ is of the form

$$G = K(A^{v_1}, \ldots, A^{v_{k_1}})$$

for some uniquely determined $K \in C^\infty(\mathbb{R}^{k_1})$. If $F \circ i$ has the point property, i.e. $F \circ i(\text{pt}) = \text{pt}$, then $G = \text{const}$, where pt denotes a one-point manifold.

**Corollary 2.** Let $A : T_{\text{proj}}(\mathcal{FM}_{m,n}) \rightsquigarrow T^{(0,0)}(F|\mathcal{FM}_{m,n})^*$ be a natural linear operator. Then

$$A = \sum_{s=k_1+1}^{k_2} K_s(A^{v_1}, \ldots, A^{v_{k_1}})A^{v_s}$$

for some uniquely determined $K_s \in C^\infty(\mathbb{R}^{k_1})$.  

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**Lifting Horizontal 1-Forms**

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2. A decomposition proposition. Let \( F \) and \( v_1, \ldots, v_L \) be as in Section 1 with the assumption (*). Let \( i : Mf \to FM \) be the functor as in Section 1.

Let \( p : Y \to M \) be a fibered manifold. A 1-form \( \omega : TY \to \mathbb{R} \) on \( Y \) is called horizontal if \( \omega|_{VY} = 0 \), where \( VY \) is the vertical bundle.

**Example 2.** If \( \omega : TY \to \mathbb{R} \) is a horizontal 1-form on a fibered manifold \( Y \), we have its vertical lifting \( B^v(\omega) = \omega \circ T\pi : (FY)^* \to \mathbb{R} \) to \((FY)^* \), where \( \pi : (FY)^* \to Y \) is the bundle projection. The correspondence \( B^v : T^*_{hor|FM_{m,n}} \to T^*(FM_{m,n})^* \) is a natural operator.

**Assumption 1.** From now on we assume that there exists a basis \( w_1, \ldots, \ldots, \ldots, w_N \in F_0\mathbb{R}^{m,n} \) such that \( w_s \) is homogeneous of weight \( n_s \in \mathbb{N} \cup \{0\} \). This means that \( F(\tau \text{id}_{\mathbb{R}^m \times \mathbb{R}^n})(w_s) = \tau^{n_s}w_s \) for any \( \tau \in \mathbb{R} \).

**Remark 1.** It seems that every vector bundle functor \( F : FM \to VB \) satisfies Assumption 1.

**Proposition 2** (Decomposition Proposition). Consider a natural operator \( B : T^*_{hor|FM_{m,n}} \to T^*(FM_{m,n})^* \). Under Assumption 1 there exists a uniquely determined natural function \( a \) on \((FM_{m,n})^* \) such that

\[
B = aB^v + \lambda
\]

for some canonical 1-form \( \lambda \) on \((FM_{m,n})^* \).

**Lemma 1.** (a) We have \( (B(\omega) - B(0))(V(F\mathbb{R}^{m,n}))_0 = 0 \) for any horizontal 1-form \( \omega \) on \( \mathbb{R}^{m,n} \), where \((V(F\mathbb{R}^{m,n}))_0 \) is the fiber over 0 in \( \mathbb{R}^m \times \mathbb{R}^n \) of the vertical subbundle in \( T(F\mathbb{R}^{m,n})^* \).

(b) If \( F \circ i \) has the point property then \( B(\omega)(V(F\mathbb{R}^{m,n}))_0 = 0 \) for any horizontal 1-form \( \omega \) on \( \mathbb{R}^{m,n} \).

**Proof.** We modify the proof of Lemma 1 in [5] as follows.

(a) We use the invariance of \( (B(\omega) - B(0))(V(F\mathbb{R}^{m,n}))_0 \) with respect to the homotheties \( t^{-1} \text{id}_{\mathbb{R}^m \times \mathbb{R}^n} \) for \( t \neq 0 \) and apply the homogeneous function theorem. We deduce that \( (B(\omega) - B(0))(V(F\mathbb{R}^{m,n}))_0 \) is independent of \( \omega \).

(b) We observe that if \( F \circ i \) has the point property then \((F_0\mathbb{R}^{m,n})^* \) has no non-zero homogeneous elements of weight 0. Next, we use the invariance of \( B(\omega)(V(F\mathbb{R}^{m,n}))_0 \) with respect to the homotheties \( t^{-1} \text{id}_{\mathbb{R}^m \times \mathbb{R}^n} \) for \( t \neq 0 \) and let \( t \to 0 \).

**Proof of Proposition 2.** We modify the proof of Proposition 2 in [5]. Replacing \( B \) by \( B - B(0) \) we can assume \( B(0) = 0 \) and \( B(\omega)(V(F\mathbb{R}^{m,n}))_0 = 0 \). Then \( B \) is determined by the values \( \langle B(\omega)\eta, F^*(\partial/\partial x^1)\eta \rangle \) for all horizontal 1-forms \( \omega = \sum_{i=1}^m \omega_i dx^i \) on \( \mathbb{R}^{m,n} \) and \( \eta \in (F_0\mathbb{R}^{m,n})^* \), with \( F^*(\partial/\partial x^1) \) the complete lifting (flow prolongation) of \( \partial/\partial x^1 \) to \((F\mathbb{R}^{m,n})^* \). Using the invariance of \( B \) with respect to the homotheties \( t^{-1} \text{id}_{\mathbb{R}^m \times \mathbb{R}^n} \) for \( t \neq 0 \) we get
the homogeneity condition
\[
t \langle B(\omega)_{\eta}, F^*(\partial/\partial x^1)_{\eta} \rangle
= \langle B((t \text{id}_{\mathbb{R}^{m} \times \mathbb{R}^{n}})^* \omega)F(t^{-1} \text{id}_{\mathbb{R}^{m} \times \mathbb{R}^{n}})^*(\eta), F^*(\partial/\partial x^1)F(t^{-1} \text{id}_{\mathbb{R}^{m} \times \mathbb{R}^{n}})^*(\eta) \rangle.
\]

Then by the non-linear Peetre theorem [3], the homogeneous function theorem and \(B(0) = 0\) we deduce that \(\langle B(\omega)_{\eta}, F^*(\partial/\partial x^1)_{\eta} \rangle\) is a linear combination of \(\omega_1(0), \ldots, \omega_m(0)\) with coefficients being smooth maps in homogeneous coordinates of \(\eta\) of weight 0. Then using the invariance of \(B\) with respect to \((x^1, t^{-1}x^2, \ldots, t^{-1}x^m, t^{-1}y^1, \ldots, t^{-1}y^n) : \mathbb{R}^m \times \mathbb{R}^n \to \mathbb{R}^m \times \mathbb{R}^n\) for \(t \neq 0\) and letting \(t \to 0\) we end the proof.

3. On canonical 1-forms on \((F|_{\mathcal{F}M_{m,n}})^*\). The injectivity in the following proposition is a consequence of Lemma 1(b).

**Proposition 3.** Every natural (canonical) 1-form \(\lambda\) on \((F|_{\mathcal{F}M_{m,n}})^*\) induces a natural linear operator \(A^{(\lambda)} : T_{\text{proj}|\mathcal{F}M_{m,n}} \to T^{(0,0)}((F|_{\mathcal{F}M_{m,n}})^*)\) such that \(A^{(\lambda)}(X)_{\eta} = \langle \lambda_{\eta}, F^*(X)_{\eta} \rangle\) for \(\eta \in (FY)^*, X \in \mathcal{X}_{\text{proj}}(Y)\), where \(F^*(X)\) is the complete lifting (flow operator) of \(X\) to \((FY)^*\). If \(F \circ i\) has the point property, then (under Assumption 1) the correspondence \(\lambda \mapsto A^{(\lambda)}\) is a linear injection.

4. A corollary. Let \(i : M \to \mathcal{F}M\) be the functor as in Section 1.

**Corollary 3.** Assume that \(F \circ i\) has the point property and there are no non-zero elements from \(F_0 \mathbb{R}^{1,0}\) of weight 1. (For example, let \(F = F_1 \otimes F_2 : \mathcal{F}M \to VB\) be the tensor product of two vector bundle functors \(F_1, F_2 : \mathcal{F}M \to VB\) such that \(F_1 \circ i, F_2 \circ i\) have the point property.) Then (under Assumption 1) every natural operator \(B : T_{\text{hor}|\mathcal{F}M_{m,n}}^* \to T^*((F|_{\mathcal{F}M_{m,n}})^*)\) is a constant multiple of the vertical lifting.

**Proof.** Since there are no non-zero elements from \(F_0 \mathbb{R}^{1,0}\) of weight 1, we see that every canonical 1-form on \((F|_{\mathcal{F}M_{m,n}})^*\) is zero because of Corollary 2 and Proposition 3. Then Proposition 2 together with Corollary 1 ends the proof.

5. Applications. From now on let \(r, s, q\) be natural numbers with \(s \geq r \leq q\).

**Application 1.** The concept of \(r\)-jets can be generalized as follows (see [3]). Let \(Y \to M\) and \(Z \to N\) be fibered manifolds. We recall that two fibered maps \(f, g : Y \to Z\) with base maps \(\tilde{f}, \tilde{g} : M \to N\) determine the same \((r, s, q)\)-jet \(j_y^{(r, s, q)} f = j_y^{(r, s, q)} g\) at \(y \in Y_x, x \in M\), if \(j_y^r f = j_y^r g, j_y^s f = j_y^s g, j_y^q f = j_y^q g\). The space of all \((r, s, q)\)-jets of \(Y\) into \(Z\)
is denoted by $J^{(r,s,q)}(Y,Z)$. The composition of fibered maps induces the composition of $(r,s,q)$-jets [3, p. 126].

The vector $r$-tangent bundle functor $T^{(r)} = (J^r(\cdot, \mathbb{R})_0)^* : \mathcal{M}f \to \mathcal{VB}$ can be generalized as follows. Let $\mathbb{R}^{1,1} = \mathbb{R} \times \mathbb{R}$ be the trivial bundle over $\mathbb{R}$. The space $J^{(r,s,q)}(Y, \mathbb{R}^{1,1})_0$, $0 \in \mathbb{R}^2$, has an induced structure of a vector bundle over $Y$. Every fibered map $f : Y \to Z$, $f(y) = z$, induces a linear map $\lambda(j_y^{(r,s,q)} f) : J_y^{(r,s,q)}(Z, \mathbb{R}^{1,1})_0 \to J_y^{(r,s,q)}(Y, \mathbb{R}^{1,1})_0$ by means of the jet composition. If we denote by $T^{(r,s,q)}Y$ the dual vector bundle of $J^{(r,s,q)}(Y, \mathbb{R}^{1,1})_0$ and define $T^{(r,s,q)}f : T^{(r,s,q)}Y \to T^{(r,s,q)}Z$ by using the dual maps to $\lambda(j_y^{(r,s,q)} f)$, we obtain a vector bundle functor $T^{(r,s,q)} : \mathcal{F}M \to \mathcal{VB}$.

Example 3. We have canonical 1-forms $\lambda^{(r,s,q)}_\alpha : TJ^{(r,s,q)}(Y, \mathbb{R}^{1,1})_0 \to \mathbb{R}$ on $J^{(r,s,q)}(Y, \mathbb{R}^{1,1})_0$ for $\alpha = 1, 2$ defined by $\lambda^{(r,s,q)}_\alpha (v) = d\gamma(T\pi(v))$ for $v \in T_yJ^{(r,s,q)}(Y, \mathbb{R}^{1,1})_0$, $w = j_y^{(r,s,q)}(\gamma_1, \gamma_2)$, $y \in Y$, where $\pi : J^{(r,s,q)}(Y, \mathbb{R}^{1,1})_0 \to Y$ is the bundle projection.

Corollary 4. Every natural operator

$$B : T^*_{hor}|\mathcal{F}M_{m,n} \to T^*(J^{(r,s,q)}(\cdot, \mathbb{R}^{1,1})_0)$$

is a linear combination of the vertical lifting $B^V$ and the canonical 1-forms $\lambda^{(r,s,q)}_1$ and $\lambda^{(r,s,q)}_2$ with real coefficients.

Proof. The vector bundle functor $T^{(r,s,q)}$ satisfies Assumption 1. Moreover, $T^{(r,s,q)} \circ i$ has the point property and the subspace of elements from $T_0^{(r,s,q)}\mathbb{R}^{1,0}$ of weight 1 is 2-dimensional. Then by Proposition 3 together with Corollaries 1 and 2, the space of canonical 1-forms on $J^{(r,s,q)}(\cdot, \mathbb{R}^{1,1})_0$ is at most 2-dimensional. Now, Proposition 2 ends the proof. □

Application 2. Let $r, s$ be integers such that $s \geq r \geq 0$. The concept of $r$-jets can also be generalized as follows (see [3]). Let $Y \to M$ be a fibered manifold and $Q$ be a manifold. We recall that two maps $f, g : Y \to Q$ determine the same $(r, s)$-jet $j_y^{(r,s)} f = j_y^{(r,s)} g$ at $y \in Y_x$, $x \in M$, if $j_y^r f = j_y^r g$ and $j_y^s (f|_{Y_x}) = j_y^s (g|_{Y_x})$. The space of all $(r,s)$-jets of $Y$ into $Q$ is denoted by $J^{(r,s)}(Y,Q)$.

The vector $r$-tangent bundle functor $T^{(r)} = (J^r(\cdot, \mathbb{R})_0)^* : \mathcal{M}f \to \mathcal{VB}$ can be generalized as follows. The space $J^{(r,s)}(Y, \mathbb{R})_0$, $0 \in \mathbb{R}$, has an induced structure of a vector bundle over $Y$. Every fibered map $f : Y \to Z$, $f(y) = z$, induces a linear map $\lambda(j_y^{(r,s)} f) : J_y^{(r,s)}(Z, \mathbb{R})_0 \to J_y^{(r,s)}(Y, \mathbb{R})_0$ by means of the jet composition. If we denote by $T^{(r,s)}Y$ the dual vector bundle of $J^{(r,s)}(Y, \mathbb{R})_0$ and define $T^{(r,s)}f : T^{(r,s)}Y \to T^{(r,s)}Z$ by using the dual maps to $\lambda(j_y^{(r,s)} f)$, we obtain a vector bundle functor $T^{(r,s)} : \mathcal{F}M \to \mathcal{VB}$.

Example 4. Assume additionally $r \geq 1$. We have a canonical 1-form $\lambda^{(r,s)} : TJ^{(r,s)}(Y, \mathbb{R})_0 \to \mathbb{R}$ on $J^{(r,s)}(Y, \mathbb{R})_0$ defined by $\lambda^{(r,s)}(v) = d\gamma(T\pi(v))$
for \( v \in T_wJ^{(r,s)}(Y, \mathbb{R})_0, \ w = j^{(r,s)}_y(\gamma), \ y \in Y, \) where \( \pi : J^{(r,s)}(Y, \mathbb{R})_0 \to Y \) is the bundle projection.

**Corollary 5.** Let \( r, s \) be as above. Every natural operator
\[
B : T^*_{\text{hor}}[\mathcal{F}M_{m,n}] \to T^*(J^{(r,s)}(\cdot, \mathbb{R})_0)
\]
is a linear combination of the vertical lifting \( B^V \) and the canonical 1-form \( \lambda^{(r,s)} \) with real coefficients. If \( r = 0 \), then the \( \lambda^{(0,s)} \) do not occur.

*Proof.* Note that the subspace of elements from \( T_0^{(r,s)}\mathbb{R}^{1,0} \) of weight 1 is 1-dimensional, and use the same arguments as in the proof of Corollary 4. ■

**Application 3.** For any fibered manifold \( Y \) we have the vertical bundle \( VY \) of \( Y \) and for every \( \mathcal{F}M \)-map \( f : Y \to Z \) we have the induced map \( Vf : VY \to VZ \). The functor \( V : \mathcal{F}M \to \mathcal{V}B \) is a vector bundle functor. Let \( V^* = (V[\mathcal{F}M_{m,n}])^* \) be the dual bundle functor.

**Corollary 6.** Every natural operator \( B : T^*_\text{hor}[\mathcal{F}M_{m,n}] \to T^*V^* \) is a constant multiple of the vertical lifting \( B^V \).

*Proof.* We observe that \( V \cong T^{(0,1)} \) and apply Corollary 5. Given a \( \mathcal{F}M \)-object \( p : Y \to M \), an isomorphism \( i : VY \to T^{(0,1)}Y \) is given by \( i(v)(j_y^{(0,1)}\gamma) = d_y(\gamma|_{Y_p(y)})(v) \).

**Application 4.** For any fibered manifold \( Y \) we have a vector bundle
\[
J^rT^*_\text{hor}Y = \{j^r_y\omega \mid \omega \text{ is a horizontal 1-form on } Y, \ y \in Y\}
\]
over \( Y \). Let \( (J^rT^*_\text{hor})^*Y = (J^rT^*_\text{hor}Y)^* \) be the dual bundle. Every \( \mathcal{F}M \)-map \( f : Y \to Z \) induces a vector bundle map \( (J^rT^*_\text{hor})^*f : (J^rT^*_\text{hor})^*Y \to (J^rT^*_\text{hor})^*Z \) covering \( f \) such that \( \langle (J^rT^*_\text{hor})^*f(\eta), j^r_{f(y)}\omega \rangle = \langle \eta, j^r_{y}(f^*\omega) \rangle \) for \( \eta \in (J^rT^*_\text{hor})^*Y, \ j^r_{f(y)}\omega \in (J^rT^*_\text{hor})f(y)Z, \ y \in Y \). The functor \( (J^rT^*_\text{hor})^* : \mathcal{F}M \to \mathcal{V}B \) is a vector bundle functor.

Given an \( \mathcal{F}M_{m,n} \)-object \( Y \) we have a canonical 1-form \( \theta^r \) on \( J^rT^*_\text{hor}Y \) such that
\[
\langle \theta^r_w, v \rangle = \langle \omega_y, T(\pi(v)) \rangle
\]
for \( v \in T_w(J^rT^*_\text{hor}Y), \ w = j^r_y\omega, \ y \in Y, \ \omega \in \Omega^1_{\text{hor}}(Y), \) where \( \pi : J^rT^*_\text{hor}Y \to Y \) is the bundle projection.

**Corollary 7.** Every natural operator \( B : T^*_\text{hor}[\mathcal{F}M_{m,n}] \to T^*(J^rT^*_\text{hor}) \) is a linear combination of the vertical lifting \( B^V \) and \( \theta^r \) with real coefficients.

*Proof.* We observe that the subspace of elements from \( (J^rT^*_\text{hor})_0^*\mathbb{R}^{1,0} \) of weight 1 is 1-dimensional. ■

**Application 5.** We can generalize Application 4 as follows. For any fibered manifold \( Y \) we have a vector bundle
\[
J^r(\Lambda^kT^*_\text{hor})Y = \{j^r_y\omega \mid \omega \text{ is a horizontal } k \text{-form on } Y, \ y \in Y\}
\]
over $\mathcal{Y}$. Let $(J^r(\wedge^k T^*_\text{hor}))^*Y = (J^r(\wedge^k T^*_\text{hor}) Y)^*$ be the dual bundle. Every $\mathcal{FM}$-map $f : \mathcal{Y} \to \mathcal{Z}$ induces a vector bundle map $(J^r(\wedge^k T^*_\text{hor}))^* f : (J^r(\wedge^k T^*_\text{hor}))^* Y \to (J^r(\wedge^k T^*_\text{hor}))^* Z$ covering $f$ such that

$$\langle (J^r(\wedge^k T^*_\text{hor}))^* f(\eta), \tilde{j}_f(y) \omega \rangle = \langle \eta, j^r(\omega) \rangle$$

for $\eta \in (J^r(\wedge^k T^*_\text{hor}))^* Y$, $\tilde{j}_f(y) \omega \in (J^r(\wedge^k T^*_\text{hor}))^* f(y) Z$, $y \in \mathcal{Y}$. Then $(J^r(\wedge^k T^*_\text{hor}))^* : \mathcal{FM} \to \mathcal{VB}$ is a vector bundle functor.

**Corollary 8.** Let $k \geq 2$. Every natural operator $B : T^*_{\text{hor}}|_{\mathcal{FM}_{m,n}} \rightsquigarrow T^*(J^r(\wedge^k T^*_\text{hor}))$ is a constant multiple of the vertical lifting $B^V$.

**Proof.** We observe that the subspace of elements from $(J^r(\wedge^k T^*_\text{hor}))^* \mathbb{R}^{1,0}$ of weight 1 is 0-dimensional. □

Similar facts hold for

$$J^r(\otimes^k T^*_\text{hor}) Y = \{ j^r y \tau \mid \tau \text{ is a horizontal tensor field of type } (0,k) \text{ on } \mathcal{Y}, y \in \mathcal{Y} \},$$

$$J^r(\odot^k T^*_\text{hor}) Y = \{ j^r y \tau \mid \tau \text{ is a horizontal symmetric tensor field of type } (0,k) \text{ on } \mathcal{Y}, y \in \mathcal{Y} \}$$

in place of $J^r(\wedge^k T^*_\text{hor}) Y$.

**Application 6.** We can also generalize Application 4 as follows. Let $r$ and $s$ be two integers with $s \geq r \geq 0$. For any fibered manifold $\mathcal{Y}$ we have a vector bundle

$$J^{(r,s)}T^*_\text{hor} Y = \{ j^r y \omega \mid \omega \text{ is a horizontal } 1\text{-form on } \mathcal{Y}, y \in \mathcal{Y} \}$$

over $\mathcal{Y}$. Let $(J^{(r,s)}T^*_\text{hor})^* Y = (J^{(r,s)}T^*_\text{hor} Y)^*$ be the dual bundle. Every $\mathcal{FM}$-map $f : \mathcal{Y} \to \mathcal{Z}$ induces a vector bundle map $(J^{(r,s)}T^*_\text{hor})^* f : (J^{(r,s)}T^*_\text{hor})^* Y \to (J^{(r,s)}T^*_\text{hor})^* Z$ covering $f$ such that

$$\langle (J^{(r,s)}T^*_\text{hor})^* f(\eta), \tilde{j}_f(y) \omega \rangle = \langle \eta, j^{(r,s)} y (\omega) \rangle$$

for $\eta \in (J^{(r,s)}T^*_\text{hor})^* Y$, $\tilde{j}_f(y) \omega \in (J^{(r,s)}T^*_\text{hor})^* f(y) Z$, $y \in \mathcal{Y}$. Then $(J^{(r,s)}T^*_\text{hor})^* : \mathcal{FM} \to \mathcal{VB}$ is a vector bundle functor.

Given an $\mathcal{FM}_{m,n}$-object $\mathcal{Y}$ we have a canonical 1-form $\Theta^{(r,s)}$ on $J^{(r,s)}T^*_\text{hor} Y$ such that

$$\langle \Theta^{(r,s)}_v, v \rangle = \langle \omega_y, T \pi(v) \rangle$$

for $v \in T_w(J^{(r,s)}T^*_\text{hor} Y)$, $w = j^r_y \omega$, $y \in \mathcal{Y}$, $\omega \in \Omega^1_{\text{hor}}(\mathcal{Y})$, where $\pi : J^{(r,s)}T^*_\text{hor} Y \to \mathcal{Y}$ is the bundle projection.

**Corollary 9.** Every natural operator $B : T^*_{\text{hor}}|_{\mathcal{FM}_{m,n}} \rightsquigarrow T^*(J^{(r,s)}T^*_\text{hor})$ is a linear combination of the vertical lifting $B^V$ and $\Theta^{(r,s)}$ with real coefficients.
Proof. We observe that the subspace of elements from $(J^{(r,s)} T^*_\text{hor})^* \mathbb{R}^{1,0}$ of weight 1 is 1-dimensional.

Of course, other applications are also possible. For example we can study liftings to $J^{(r,s)} (\wedge^k T^*_\text{hor})$, $J^{(r,s)} (\otimes^k T^*_\text{hor})$, $J^{(r,s)} (\circ^k T^*_\text{hor})$, $J^r (T^*)$, $J^{(r,s)} (T^*)$, etc.

REFERENCES


