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# NOTE ON ANALYTIC REGULARITY OF HEAT KERNELS ON NILPOTENT LIE GROUPS

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**Abstract.** Let $G$ be the simplest nilpotent Lie group of step 3. We prove that the densities of the semigroup generated by the sublaplacian on $G$ are not real-analytic.

**Introduction.** Let $\mathcal{G}$ be a nilpotent Lie algebra generated by $X_1, \ldots, X_k$ and let $L = X_1^2 + \ldots + X_k^2$ be the corresponding sublaplacian. Then $L$ is the infinitesimal generator of a semigroup of smooth convolution kernels $p_t$, $t > 0$, on $G = \exp \mathcal{G}$. It has been noticed in [Hu] that the formula for $p_t$ when $G$ is a Heisenberg group (cf. also [Cy], [G]) implies that the kernels $p_t$ are real-analytic functions. On the other hand, it follows from the general theory of elliptic operators that the same holds for the kernels $p_t$ in the case when $L$ is elliptic, i.e. when $X_1, \ldots, X_k$ is a linear basis of $\mathcal{G}$. The aim of this note is to show that for the sublaplacian $L$ on the simplest, step 3, nilpotent Lie group the kernels $p_t$ are not real-analytic in the neighbourhood of zero. As a consequence we get the same result for nilpotent Lie groups with algebras generated by two elements with the one exception of the Heisenberg group. Our method also works for some (higher order) Rockland operators on the Heisenberg group. The idea of our approach is based on the results of M. Christ (see [C1]–[C3]).

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**Main result.** Let $\mathcal{G}$ be the step three nilpotent Lie algebra with the commutation relations

$$[X_1, X_2] = X_3, \qquad [X_1, X_3] = X_4.$$

We will identify $\mathcal{G}$ with the corresponding connected and simply connected Lie group $G = \exp \mathcal{G}$, defining the multiplication in $\mathcal{G}$ by the Campbell–Hausdorff formula.

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Let $L$ be the sublaplacian on $\mathcal{G}$,

$$L = X_1^2 + X_2^2. \tag{1}$$

For $r > 0$, let $\delta_r$ be the linear operator in $\mathcal{G}$ defined by

$$\delta_r X_i = r^{d(i)} X_i \quad \text{with} \quad d(1) = d(2) = 1,\ d(3) = 2,\ d(4) = 3.$$

Then the $\delta_r$ form a one-parameter group of automorphisms of $G$ such that the vector field $X_i$ is homogeneous of degree $d(i)$, and the sublaplacian $L$ is homogeneous of degree 2 with respect to $\delta_r$.

We say that a smooth function $f$ satisfies the *Gevrey regularity condition* $G_{s,i}$ at zero if there is a constant $C$ such that

$$|\partial_i^n f(x)| \le C(Cn)^{sn} \tag{2}$$

for all $n \in \mathbb{N}$, and all $x$ from a fixed small neighbourhood of zero.

Our aim is to prove the following

THEOREM. *Let $s_0 < 7/6$ be fixed. Then for $i \in \{1, 2, 3, 4\}$ the kernel $p_1$ of the semigroup generated by $L$ does not satisfy the condition $G_{s_0,i}$ at zero. In fact, for a fixed $i$ there is no constant $C$ for which*

$$|\partial_i^{2n} p_1(0)| \le C(Cn)^{2s_0 n} \quad \textit{for all } n \in \mathbb{N}. \tag{3}$$

Let $\tau$ denote a riemannian distance on $\mathcal{G}$. To prove the theorem we will need the following lemma.

LEMMA. *Assume that a function $f$ satisfies*

$$\int |X_i^n f|^2\, dx \le C(Cn)^{2sn} \quad \textit{for } n \in \mathbb{N} \textit{ and fixed } s > 1, \tag{4}$$

$$\int |f(x)|^2 \exp(2\tau(x))\, dx \le C. \tag{5}$$

*Then*

$$\int |X_i^n f(x)|^2 \exp(\tau(x))\, dx \le C(2Cn)^{2sn}. \tag{6}$$

*Proof.* Let $\phi$ be a function satisfying

(i) $C_1 \exp(\tau(x)) \le \phi(x) \le C_2 \exp(\tau(x))$ for some $C_1, C_2 > 0$,

(ii) $|X_i^n \phi(x)| \le C(Cn)^{(1+\varepsilon)n} \exp(\tau(x))$
for fixed sufficiently small $\varepsilon > 0$ and all $n \ge 0$.

To see that such a function $\phi$ exists, we take a nonzero, nonnegative function $\psi \in C_c^\infty(\mathcal{G})$ which belongs to the Gevrey class of order $1 + \varepsilon$ and we put $\phi(x) = \exp(\tau) * \psi(x)$. Then $\phi$ satisfies (ii) because the vector fields $X_i$ are real-analytic and so

$$|X_i^n \psi(x)| \le C(Cn)^{(1+\varepsilon)n}, \qquad X_i^n \phi = \exp(\tau) * (X_i^n \psi)$$

(we omit a simple inductive proof that $|\partial^k X_i^n \psi(x)| \le C(C(n+k))^{(1+\varepsilon)(n+k)}$). Now for $1+\varepsilon \le s$ we have

$$
\begin{aligned}
C_1 \int |X_i^n f(x)|^2 \exp(\tau(x))\, dx &\le \int |X_i^n f(x)|^2 \phi(x)\, dx \\
&= (-1)^n \int f(x) X_i^n (X_i^n f(x)\phi(x))\, dx \\
&= (-1)^n \sum_{k=0}^{n} \binom{n}{k} \int f(x) X_i^{n+k} f(x) X_i^{n-k}\phi(x)\, dx \\
&\le 2^n \sum_{k=0}^{n} \Big(\int |f(x)|^2 (X_i^{n-k}\phi)^2(x)\, dx\Big)^{1/2} \Big(\int |X_i^{n+k} f(x)|^2\, dx\Big)^{1/2} \\
&\le 2^n \sum_{k=0}^{n} (n-k)^{(1+\varepsilon)(n-k)} \Big(\int |f(x)|^2 \phi(x)^2\, dx\Big)^{1/2} \Big(\int |X_i^{n+k} f(x)|^2\, dx\Big)^{1/2} \\
&\le C 2^n \sum_{k=0}^{n} (n-k)^{(1+\varepsilon)(n-k)} (C(n+k))^{s(n+k)} \le (2Cn)^{2sn},
\end{aligned}
$$

which completes the proof of the lemma.

*Proof of the Theorem.* Assume that (3) holds. By the identification of $G$ and $\mathcal{G}$, and the symmetry of $p_{1/2}(x)$, for fixed $i$ we have

$$(-1)^n \partial_i^{2n} p_1(0) = (-1)^n X_i^{2n} p_1(0) = \int |X_i^n p_{1/2}(x)|^2\, dx \le C(Cn)^{2sn}. \tag{7}$$

By (7) and the following (well-known, cf. [V]) easy estimate for the kernel $p_{1/2}$:

$$p_{1/2}(x) \le C \exp(-4\tau(x)) \tag{8}$$

and the lemma, we obtain

$$\int |X_i^n p_{1/2}(x)|^2 \exp(\tau(x))\, dx \le C(Cn)^{2sn}. \tag{9}$$

Hence

$$|X_i^n p_1(x)| \exp(\tau(x)) \le p_{1/2} * X_i^n p_{1/2}(x) \exp(\tau(x)) \tag{10}$$

$$\le \Big(\int |X_i^n p_{1/2}(x)|^2 \exp(\tau(x))\, dx\Big)^{1/2} \cdot \Big(\int p_{1/2}(x)^2 \exp(\tau(x)\Big)\, dx)^{1/2} \le C(Cn)^{sn}$$

with a different constant $C$. Let $R$ be a convolution kernel of $(\mathrm{Id} - L)^{-k-1}$,

$$R(x) = (k!)^{-1} \int_0^\infty t^k p_t(x) e^{-t}\, dt. \tag{11}$$

By the homogeneity of $L$ and $X_i$ we have

$$X_i^{2n} (\mathrm{Id} - L)^{-nd(i)-1} f(x) = f * K(x) \tag{12}$$

for $f \in \mathcal{S}(\mathcal{G})$, where

$$K(x) = ((nd(i))!)^{-1} \int_0^\infty (X_i^{2n} p_1)(\delta_{t^{-1/2}} x) t^{-Q/2} e^{-t}\, dt.$$

This together with (10) implies

$$\int |K(x)|\, dx \le C(Cn)^{(2s-d(i))n}, \tag{13}$$

and consequently,

$$\int |X_i^{2n} f(x)|^2\, dx \le C(Cn)^{2(2s-d(i))n} \int |(\mathrm{Id} - L)^{d(i)n+1} f(x)|^2\, dx \quad \text{for } f \in \mathcal{S}(\mathcal{G}). \tag{14}$$

We are going to show that for our group $G$, estimate (13) fails for large $n$. Observe that (13) implies the same operator norm estimate for the unitary representation image $\pi_K$ of $K$. Let $\pi$ be the unitary representation of $G$ such that the corresponding representation $d\pi$ of the Lie algebra $\mathcal{G}$ is defined for $f \in C_c^\infty(\mathbb{R})$ by the formulas

$$d\pi(X_1) f(t) = \frac{d}{dt} f(t), \tag{15}$$

$$d\pi(X_2) f(t) = it^2 f(t), \tag{16}$$

$$d\pi(X_3) f(t) = 2it f(t), \tag{17}$$

$$d\pi(X_4) f(t) = 2i f(t).$$

Let $\phi$ be a first eigenfunction of $d\pi(L)$,

$$d\pi(L)\phi = \Big(-\frac{d^2}{dt^2} + t^4\Big)\phi = \lambda\phi, \tag{18}$$

and $\lambda$ the corresponding eigenvalue. Then $\phi \in \mathcal{S}(\mathbb{R})$. The theory of ODE (see [CL]) yields

$$\left|\frac{d}{dt}\phi(t)\right| + |\phi(t)| \ge C_1 \exp(-C_2 |t|^3) \quad \text{for sufficiently large } |t|.$$

Hence, using (18) and an integral Taylor formula, one gets

$$\int_{t \le x \le t+1} |\phi(x)|\, dx \ge C_3 \exp(-C_4 |t|^3) \quad \text{for sufficiently large } |t|. \tag{19}$$

Consequently,

$$\int |d\pi(X_2)^{2n} \phi(t)|^2\, dt = \int |t^{4n} \phi(t)|^2\, dt \ge (Cn)^{8n/3}, \tag{20}$$

$$\int |d\pi(X_3)^{2n} \phi(t)|^2\, dt = \int |t^{2n} \phi(t)|^2\, dt \ge (Cn)^{4n/3}.$$

On the other hand we have $\|d\pi((\mathrm{Id} - L)^{nd(i)+1})\phi\|_2 = (1+\lambda)^{nd(i)+1}$. This contradicts (13) for $s < 7/6$ for $i = 2$ and $s < 4/3$ for $i = 3$.

In order to show that the hypothesis (13) fails for $i = 1$ we observe that the Fourier transform $\Phi$ of $\phi$ satisfies the equation

$$\left(\frac{d^4}{dt^4} + t^2\right)\Phi(t) = \lambda\Phi(t). \tag{21}$$

By standard theory of ODE (see [CL]) we have

$$|\Phi(t)| + \left|\frac{d}{dt}\Phi(t)\right| + \left|\frac{d^2}{dt^2}\Phi(t)\right| + \left|\frac{d^3}{dt^3}\Phi(t)\right| \geq C_3 \exp(-C_4|t|^{3/2}) \quad \text{for large } |t|,$$

and consequently (using (21))

$$\int\limits_{t<x<t+1} |\Phi(x)|\, dx \geq C_3 \exp(-C_4|t|^{3/2}) \quad \text{for large } |t|, \tag{22}$$

Hence

$$\int |d\pi(X_1)^{2n}\phi(t)|^2\, dt = \int \left|\frac{d^{2n}}{dt^{2n}}\phi(t)\right|^2 dt = \int |t^{2n}\Phi(t)|^2\, dt \geq (Cn)^{8n/3},$$

which contradicts (13) for $i = 1$ and $s < 4/3$.

In order to disprove (13) for $i = 4$ and $s < 3/2$ we will use (18). It suffices to observe that the estimate

$$\begin{aligned}\int |\phi(t)|^2\, dt &\leq (Cn)^{2(2s-d(4))n} \left\|\left(-\frac{d^2}{dt^2} + t^4\right)^{nd(4)+1} \phi(t)\right\|_{L^2}^2 \\ &= (Cn)^{-\delta n}(1+\lambda)^{6n+2}\end{aligned}$$

is false for $\delta = 2(3 - 2s) > 0$ and $n \to \infty$. This finishes the proof of the theorem.

Remark. Assume that for some $s > 0$ the subelliptic estimate (14) holds for all $n \in \mathbb{N}$. Then as an immediate consequence of analyticity of $p_t$ on $L^2$ we get $p_t \in G_{s,i}$.

Corollary 1. *Let $\mathcal{H}$ be a nilpotent stratified Lie group whose Lie algebra is generated by two vectors $X$ and $Y$ of degree 1. Then the densities of the semigroup generated by $X^2 + Y^2$ are not real-analytic at zero.*

*Proof.* The only fact about the group $\mathcal{G}$ used in the proof of the main theorem was the existence of the representation satisfying (15), (16), (17). Since $\mathcal{G}$ is a homomorphic image of $\mathcal{H}$, such a representation of $\mathcal{H}$ exists and the corollary follows.

Corollary 2. *The densities $k_t$ of the semigroup generated by $X^2 - Y^4$, where $X$ and $Y$ generate the Heisenberg Lie algebra, are not analytic at zero. (On the contrary the densities of the semigroup generated by $\partial_1^2 - \partial_2^4$ are real-analytic on $\mathbb{R}^2$.)*

*Sketch of the proof.* Assume that $|X^{2n}k_t(0)| \le (Cn)^{2sn}$. Then by (13) with $d = 1$ we get

$$(*) \qquad \|X^{2n}(\mathrm{Id} - L)^{-n-1}\|_{L^1} \le (Cn)^{(2s-1)n}.$$

Now we choose the unitary representation of the Heisenberg Lie algebra defined by $d\pi(X)f(t) = itf(t)$, $d\pi(Y)f(t) = \frac{d}{dt}f(t)$ and we consider the image of $(*)$ in $\pi$. In order to disprove analyticity of $k_1$ it suffices to observe that by (21) and (22) the estimate

$$\|d\pi(X^{2n})d\pi((\mathrm{Id} - L)^{-n-1})\|_{L^2 \to L^2} \le (Cn)^{(2s-1)n}$$

is false for $s < 7/6$ and $n \to \infty$.

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