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ON THE DIFFERENCE PROPERTY OF FAMILIES OF MEASURABLE FUNCTIONS

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Abstract. We show that, generally, families of measurable functions do not have the difference property under some assumption. We also show that there are natural classes of functions which do not have the difference property in ZFC. This extends the result of Erdős concerning the family of Lebesgue measurable functions.

1. Introduction. Erdős showed that the family of Lebesgue measurable functions does not have the difference property if we assume the Continuum Hypothesis (see e.g. [8]). His proof works for both the family of functions with the Baire property and the family of Borel functions. In that proof he used two key facts. The first one was the existence of a Lebesgue nonmeasurable $\mathcal{N}$-almost invariant set (under the Continuum Hypothesis). Such a set was first constructed by Sierpiński [13]. Moreover, Erdős used the fact that the family of Lebesgue measurable sets has the weak Ostrowski property.

On the other hand, M. Laczkovich [8] proved that the family of Lebesgue measurable functions has the weak difference property, and in [9] he showed, using the previous result, that the family of Lebesgue measurable functions has the difference property under some set-theoretic assumption. In [10] he proved the weak difference property for any family of real-valued functions defined on any compact metric Abelian group and measurable with respect to the Haar measure.

In this paper we extend the result of Erdős to families of functions which do not have the weak Ostrowski property.

2. Preliminaries. All functions considered in this paper are defined on some group and are real-valued. We also assume that the groups are Abelian.

Let G be a group. For a function $f\colon G \to \mathbb{R}$ and an $h \in G$ we define the *difference function* $\Delta_h f\colon G \to \mathbb{R}$ by $\Delta_h f(x) = f(x+h) - f(x)$. A func-

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tion $A\colon G \to \mathbb{R}$ is called a *homomorphism* if it satisfies Cauchy's functional equation $A(x+y) = A(x) + A(y)$ for all $x, y \in G$.

The notion of the difference property dates back to the paper [1] of de Bruijn. Recall that a class of functions $\mathcal{F} \subset \mathbb{R}^G$ has the *difference property* if every function $f\colon G \to \mathbb{R}$ such that $\Delta_h f \in \mathcal{F}$ for each $h \in G$ is of the form $f = g + A$, where $g \in \mathcal{F}$ and A is a homomorphism.

A set A has the *Baire property* if it is the symmetric difference of an open set and a meager set. We say that a function $f\colon X \to Y$ has the *Baire property* if $f^{-1}(U)$ has the Baire property for every open set U. We denote by $\mathcal{B}(X)$ the family of sets with the Baire property in X and by $\mathcal{M}(X)$ the family of meager sets. We write simply $\mathcal{B}$ and $\mathcal{M}$ if it does not lead to misunderstanding.

We say that a set A is (s)-*measurable* (Marczewski measurable) if every perfect set P has a perfect subset Q which is a subset of A or misses A (we assume that the empty set is not perfect). We denote by (s) the class of (s)-measurable sets. We write $A \in (s_0)$ if every perfect set P has a perfect subset Q which misses A. It is known that (s) is a σ-algebra and (s_0) is a σ-ideal. A function $f : \mathbb{R} \to \mathbb{R}$ is (s)-*measurable* if the preimage of any open subset is (s)-measurable. We will use the following characterization.

THEOREM 2.1 (Marczewski [11]). *A function $f : \mathbb{R} \to \mathbb{R}$ is (s)-measurable if and only if every perfect set $P \subset \mathbb{R}$ has a perfect subset Q such that $f|Q$ is continuous.*

By a *measure* on X we mean a countably additive, nonnegative, nonzero extended real-valued function defined on a σ-algebra $\mathcal{A}$ of subsets of X. By m we will denote the Lebesgue measure defined on $\mathbb{R}$. Let $\mathcal{L}(\mu)$ denote the family of μ-measurable sets and $\mathcal{N}(\mu)$ the family of μ-measure zero sets. We write simply $\mathcal{L}$ and $\mathcal{N}$ if it does not lead to misunderstandings.

A measure μ on X is called:

- *diffused* (or continuous) if $\mu(\{x\}) = 0$ for every $x \in X$;
- *finite* if $\mu(X) < \infty$;
- *σ-finite* if X is a countable union of sets of finite measure.

Measures μ and ν on X are called *equivalent* if

1. $(\forall A \subset X)(A$ is μ-measurable if and only if A is ν-measurable$)$,
2. $(\forall A \subset X)(\mu(A) = 0$ if and only if $\nu(A) = 0)$.

PROPOSITION 2.2. *Every σ-finite measure is equivalent to some finite measure.*

By a *universal measure* on X we mean a diffused and finite measure defined on $P(X)$.

Let κ be a cardinal. A measure μ is called *κ-additive* if $\mu(\bigcup \mathcal{F}) = \sum_{F\in\mathcal{F}} \mu(F)$ for every disjoint family $\mathcal{F}$ such that $|\mathcal{F}| < \kappa$.

We say that an uncountable cardinal κ is *real-valued measurable* if there exists a κ-additive, universal measure on κ. And κ is *measurable* if there exists a two-valued, κ-additive, universal measure on κ.

We will use the following well known theorems.

Theorem 2.3. *If there is a universal measure on a set X then there is a real-valued measurable cardinal $\le |X|$. And if there is a universal two-valued measure on a set X then there is a measurable cardinal $\le |X|$.*

Theorem 2.4. *There is no measurable cardinal less than or equal to $\mathfrak{c}$.*

We say that $\mathcal{J} \subset P(X)$ is an *ideal* on X if

1. $\emptyset \in \mathcal{J}$,
2. $(\forall A, B \in \mathcal{J})(A \cup B \in \mathcal{J})$,
3. $(\forall A \in \mathcal{J})(\forall B \subset A)(B \in \mathcal{J})$.

An ideal $\mathcal{J}$ on X is called:

- *proper* if $X \notin \mathcal{J}$;
- *prime* if for every $A \subset X$, either $A \in \mathcal{J}$ or $X \setminus A \in \mathcal{J}$;
- a *σ-ideal* (or countably complete) if $\mathcal{J}$ is closed under countable unions of sets from $\mathcal{J}$ (i.e. $\bigcup_{n=0}^{\infty} A_n \in \mathcal{J}$ whenever $A_n \in \mathcal{J}$ for every n);
- *κ-complete* if $\mathcal{J}$ is closed under unions of less than κ sets from $\mathcal{J}$ (i.e. $\bigcup \mathcal{F} \in \mathcal{J}$ for every family $\mathcal{F} \subset \mathcal{J}$ such that $|\mathcal{F}| < \kappa$);
- *κ-saturated* if every disjoint family $\mathcal{F} \subset P(X) \setminus \mathcal{J}$ has size $< \kappa$.

We say that an uncountable cardinal κ is *quasi-measurable* if there is a proper ω_1-saturated κ-complete ideal on κ containing all the singletons.

We can easily prove the following propositions.

Proposition 2.5. *If there is a prime σ-ideal on X containing all the singletons then there is a two-valued universal measure on X.*

Proposition 2.6. *If there is a proper ω_1-saturated σ-ideal on X containing all the singletons then there is a quasi-measurable cardinal $\le |X|$.*

If $\mathcal{J}$ is an ideal on X we define the following cardinal coefficient:

$$\mathrm{non}(J) = \min\{|A| : A \subset X \wedge A \notin \mathcal{J}\}.$$

We shall need one more theorem concerning measurable cardinals (see e.g. [4]).

Theorem 2.7. *If there is a real-valued measurable cardinal then $\mathrm{non}(\mathcal{N}(m)) = \omega_1$. In particular, there is no real-valued measurable cardinal $\le \mathrm{non}(\mathcal{N}(m))$.*

If $\mathcal{A}$ is a σ-algebra of subsets of X and $\mathcal{J} \subset \mathcal{A}$ is a σ-ideal on X then we say that the pair $(\mathcal{A}, \mathcal{J})$ satisfies the *c.c.c.* if every disjoint family $\mathcal{F} \subset \mathcal{A} \setminus \mathcal{J}$ is countable.

Let $\mathcal{A}$ be a family of subsets of a group G. Then we write $-A = \{-a : a \in A\}$, $A - g = \{a - g : a \in A\}$, $A - A = \{a - b : a, b \in A\}$ and $-\mathcal{A} = \{-A : A \in \mathcal{A}\}$ for $g \in G$, $A \in \mathcal{A}$. Moreover, we say that a family $\mathcal{A}$ is *invariant under translations* if $A - g \in \mathcal{A}$ for every $A \in \mathcal{A}$ and $g \in G$ and *invariant under reflections* if $-A \in \mathcal{A}$ for every $A \in \mathcal{A}$.

Let $\mathcal{A}$ denote a σ-algebra of subsets of a group G (a topological group if necessary) and let $\mathcal{J} \subset \mathcal{A}$ denote a σ-ideal on G.

We say that a set $A \subset G$ is *$\mathcal{J}$-almost invariant* if $(A + g) \triangle A \in \mathcal{J}$ for every $g \in G$. Moreover, we say that a pair $(\mathcal{A}, \mathcal{J})$ has

- the *Steinhaus property* (SP) if for every set $A \in \mathcal{A} \setminus \mathcal{J}$ the set $A - A$ contains an open neighbourhood of 0,
- the *Ostrowski property* (OP) if every homomorphism bounded on a set from $\mathcal{A} \setminus \mathcal{J}$ is continuous,
- the *weak Ostrowski property* (WOP) if every homomorphism bounded on a set from $\mathcal{A} \setminus \mathcal{J}$ is $\mathcal{A}$-measurable.

We will use $S(\mathcal{A}, \mathcal{J})$ to denote the following condition:

There exists a set $A \subset G$ such that A is $\mathcal{J}$-almost invariant and $A \notin \mathcal{A}$.

Moreover, $S^*(\mathcal{A}, \mathcal{J})$ will denote the following condition:

There exists a set $A \subset G$ such that A is $\mathcal{J}$-almost invariant, $A \notin \mathcal{A}$ and $A = -A$.

3. Pairs with(out) the SP, OP, WOP. First, we have the well known theorems which explain the names of the Ostrowski property and Steinhaus property. They concern Lebesgue measure on $\mathbb{R}$.

THEOREM 3.1 (Steinhaus [14]). *The pair $(\mathcal{L}, \mathcal{N})$ has the Steinhaus property.*

THEOREM 3.2 (Ostrowski [12]). *The pair $(\mathcal{L}, \mathcal{N})$ has the Ostrowski property.*

PROPOSITION 3.3. (i) *If a pair $(\mathcal{A}, \mathcal{J})$ has the Ostrowski property then it has the weak Ostrowski property (provided all open sets are $\mathcal{A}$-measurable).*

(ii) *If a pair $(\mathcal{A}, \mathcal{J})$ has the Steinhaus property then it has the Ostrowski property.*

Proof. (i) is trivial. For (ii) suppose that f is a homomorphism bounded on a set $A \in \mathcal{A} \setminus \mathcal{J}$. Then there is an open neighbourhood $U \subset A - A$ on which f is bounded as well. The additivity of f implies that f is continuous at 0 (see e.g. [10, p. 4]). Using the additivity of f once more we conclude that f is continuous everywhere. ∎

We have extensions of the above theorems which are also well known. For the proofs of the theorems below see e.g. [2, pp. 173–174].

THEOREM 3.4. *Let G be a locally compact topological group and let μ be a left invariant Haar measure on G. The pair $(\mathcal{L},\mathcal{N})$ has the Steinhaus property.*

THEOREM 3.5. *Let G be a topological group. The pair $(\mathcal{B},\mathcal{M})$ has the Steinhaus property.*

But the SP, OP, WOP are not very common among σ-algebras.

PROPOSITION 3.6. *The pair $((s),(s_0))$ does not have the weak Ostrowski property (thus it has neither the Ostrowski nor the Steinhaus properties).*

Proof. We will prove the first statement. Then the second follows from Proposition 3.3.

Let $P\subset\mathbb{R}$ be a perfect set (i.e., nonempty, closed, with no isolated points) which is linearly independent over rationals. Let $B\subset P$ be a Bernstein subset of P (i.e., B and $P\setminus B$ do not contain a nonempty perfect (in P) set). Now we define a function A_1: $P\to\mathbb{R}$ by $A_1(x)=0$ for $x\in B$ and $A_1(x)=1$ for $x\in P\setminus B$. Since P is linearly independent over the rationals, we can extend A_1 to an additive function A: $\mathbb{R}\to\mathbb{R}$ such that $A|P=A_1$. Clearly, A is bounded on the set $P\in(s)\setminus(s_0)$. Suppose that A is (s)-measurable. Then, by Theorem 2.1, there is a perfect set $D\subset P$ such that $A|D$ is continuous. Take an $x\in D\cap B$ (this can be done because B is a Bernstein set in P). Then we can find a sequence $(x_n)_{n\in\omega}$ such that $\lim_{n\to\infty}x_n=x$ and $x_n\in D\setminus B$ since $P\setminus B$ is also a Bernstein set. But $A(x_n)=1$ for all n and $A(x)=0$, so $A|D$ is not continuous, a contradiction. Thus A is not (s)-measurable. ∎

THEOREM 3.7. *There exists an extension of the Lebesgue measure which is invariant under translations and reflections and which does not have the Steinhaus property.*

Proof. See [6, p. 148, Proposition 2].

THEOREM 3.8. *There exists an extension of the Lebesgue measure which is invariant under reflections and which does not have the weak Ostrowski property.*

Proof. This extension was constructed in [7, Example 2]. That extension is not complete and the completion of this measure may have the weak Ostrowski property. But one can change a little the definition of that measure to get a complete measure which does not have the weak Ostrowski property. ∎

But we do not know if there exists an extension of the Lebesgue measure which is invariant under translations and which does not have the (weak) Ostrowski property.

4. Generalization of Erdős's result. Let $\mathcal{A}$ denote a σ-algebra of subsets of a group G and $\mathcal{J} \subset \mathcal{A}$ a σ-ideal on G.

4.1. *A trivial generalization.* In order to repeat Erdős's proof (showing that the family of Lebesgue measurable functions does not have the difference property) for the family of $\mathcal{A}$-measurable functions we only need to know that the pair $(\mathcal{A}, \mathcal{J})$ has the weak Ostrowski property and the condition $S(\mathcal{A}, \mathcal{J})$ holds. Thus we get a theorem, essentially due to Erdős.

THEOREM 4.1. *Suppose that a pair $(\mathcal{A}, \mathcal{J})$ has the weak Ostrowski property. If $S(\mathcal{A}, \mathcal{J})$ holds then the family of $\mathcal{A}$-measurable functions does not have the difference property.*

Proof. The proof is the same as Erdős's but we sketch it for the sake of completeness. By $S(\mathcal{A}, \mathcal{J})$ we have a set $S \subset G$ such that S is $\mathcal{J}$-almost invariant and $S \notin \mathcal{A}$. Put $f = \chi_S$. Now we show that f is a witness for the lack of the difference property for the family of $\mathcal{A}$-measurable functions.

Since $\{x \in G : \Delta_h f \neq 0\} = \{x \in G : \chi_{S-h}(x) \neq \chi_S(x)\} = (S-h) \triangle S \in \mathcal{J}$, the function $\Delta_h f$ is $\mathcal{A}$-measurable for every $h \in G$. Now suppose that $f = g + A$, where g is $\mathcal{A}$-measurable and A is a homomorphism. Then there is n such that the set $B = g^{-1}([-n, n])$ is in $\mathcal{A} \setminus \mathcal{J}$ and thus the homomorphism A is bounded on B since f and g are bounded on B. By the weak Ostrowski property, the function A is $\mathcal{A}$-measurable. Thus $f = \chi_S$ is $\mathcal{A}$-measurable as a sum of two $\mathcal{A}$-measurable functions. But the set S is not $\mathcal{A}$-measurable, a contradiction. ■

COROLLARY 4.2. *Let G be a locally compact topological group and let μ be a left invariant Haar measure on G. If $S(\mathcal{L}, \mathcal{N})$ holds then the family of $\mathcal{L}$-measurable functions does not have the difference property.*

Proof. This follows from Theorems 3.4 and 4.1. ■

COROLLARY 4.3. *Let G be a topological group. If $S(\mathcal{B}, \mathcal{M})$ holds then the family of $\mathcal{B}$-measurable functions does not have the difference property.*

Proof. This follows from Theorems 3.5 and 4.1. ■

4.2. *A less trivial generalization.* Now we shall show that (in some cases) we do not need the weak Ostrowski property for $(\mathcal{A}, \mathcal{J})$ in order to prove that the family of $\mathcal{A}$-measurable functions does not have the difference property.

THEOREM 4.4. *Let $\mathcal{A}$ be a σ-algebra invariant under reflections on a group G and $\mathcal{J} \subset \mathcal{A}$ be a σ-ideal on G. If $S^*(\mathcal{A}, \mathcal{J})$ holds then the family of $\mathcal{A}$-measurable functions does not have the difference property.*

Proof. Let $A \subset G$ be a set such that $A \notin \mathcal{A}$, $A = -A$ and $(A+g) \triangle A \in \mathcal{J}$ for all $g \in G$. Such a set A exists by $S^*(\mathcal{A}, \mathcal{J})$. Let $f = \chi_A$. We will show that f witnesses the lack of the difference property of the family of $\mathcal{A}$-measurable functions.

First, it is easy to see that $\Delta_g f$ is $\mathcal{A}$-measurable for every $g \in G$. Now suppose that $f = k + h$, where k is $\mathcal{A}$-measurable and h is a homomorphism. Define $F(x) = f(x) + f(-x)$. Then $F(x) = (k(x) + h(x)) + (k(-x) + h(-x)) = k(x) + k(-x)$ so F is $\mathcal{A}$-measurable (since $\mathcal{A} = -\mathcal{A}$).

But on the other hand $F(x) = f(x) + f(-x) = \chi_A(x) + \chi_A(-x) = \chi_A(x) + \chi_{-A}(x) = 2\chi_A(x)$, and since $A \notin \mathcal{A}$, we deduce that F is not $\mathcal{A}$-measurable. This contradiction completes the proof. ∎

5. On the conditions $S^*(\mathcal{A}, \mathcal{J})$ and $S(\mathcal{A}, \mathcal{J})$. We have seen that in both generalizations (Sections 4.1 and 4.2), a weak point is the assumption that $S(\mathcal{A}, \mathcal{J})$ or $S^*(\mathcal{A}, \mathcal{J})$ holds. Hence we now examine these conditions.

Let $\mathcal{A}$ be a σ-algebra on a group G and $\mathcal{J}$ be a σ-ideal on G. Let $|G| = \kappa$ and $G = \{g_\alpha : \alpha < \kappa\}$ be an enumeration of elements of G. We denote by G_α the group generated by $\{g_\beta : \beta < \alpha\}$. For any $\alpha < \kappa$ let $Q_\alpha = G_{\alpha+1} \setminus G_\alpha$, and for every $T \subset \kappa$ let $A_T = \bigcup_{\alpha \in T} Q_\alpha$.

LEMMA 5.1. *If* $\mathrm{non}(\mathcal{J}) = |G|$ *then the set* A_T *is* $\mathcal{J}$*-almost invariant and* $A_T = -A_T$ *for every* $T \subset \kappa$.

Proof. It is easy to see that $(A_T + g) \setminus A_T \in \mathcal{J}$ for all $T \subset \kappa$ and $g \in G$. Indeed,

$$(A_T + g) \setminus A_T = \bigcup_{\alpha \in T} (Q_\alpha + g) \setminus \bigcup_{\alpha \in T} Q_\alpha = \bigcup_{\alpha \in T} [(G_{\alpha+1} + g) \setminus (G_\alpha + g)] \setminus \bigcup_{\alpha \in T} Q_\alpha.$$

But there is $\beta < \kappa$ such that $g \in G_\alpha$ for every $\alpha > \beta$, so

$$(A_T + g) \setminus A_T = \Big[\bigcup_{T \ni \alpha \le \beta} (Q_\alpha + g) \cup \bigcup_{T \ni \alpha > \beta} (G_{\alpha+1} \setminus G_\alpha) \Big] \setminus \bigcup_{\alpha \in T} Q_\alpha \subset \bigcup_{\alpha \le \beta} (Q_\alpha + g).$$

The last set is in $\mathcal{J}$, by the assumption that $\mathrm{non}(\mathcal{J}) = |G|$, so $(A_T + g) \setminus A_T \in \mathcal{J}$. Similarly we show that $A_T \setminus (A_T + g)$ is in $\mathcal{J}$, hence $(A_T + g) \triangle A_T \in \mathcal{J}$.

Now one can easily check that $A_T = -A_T$ for every $T \subset \kappa$: this follows from the fact that $Q_\alpha = -Q_\alpha$ for every $\alpha < \kappa$. ∎

The construction of almost invariant sets appeared in papers devoted to extensions of invariant measures, and the above construction appeared e.g. in [5] and [15].

We see that to prove $S^*(\mathcal{A}, \mathcal{J})$ we have to show that there is a set T such that A_T is not in $\mathcal{A}$. This will be done for each case separately. We had to assume that $\mathrm{non}(\mathcal{J}) = |G|$. For some cases we will need to assume something more. It is no wonder that we need some assumptions since the Erdős theorem is proved under the Continuum Hypothesis (and our assumptions will be fulfilled under CH). Moreover, we prove that there are some families for which those assumptions are fulfilled in ZFC.

5.1. *Measure case*

THEOREM 5.2. *Let μ be a σ-finite measure on a group G. If $|G|$ is less than the first real-valued measurable cardinal and* $\mathrm{non}(\mathcal{N}) = |G|$ *then $S^*(\mathcal{L}, \mathcal{N})$ holds.*

Proof. By Lemma 5.1 we must find $T \subset \kappa$ such that A_T is not $\mathcal{L}$-measurable. Suppose that A_T is $\mathcal{L}$-measurable for every $T \subset \kappa$. By Proposition 2.2 we have a finite measure ν equivalent to μ. We define a measure $\tau\colon P(\kappa) \to [0, +\infty]$ by $\tau(T) = \nu(A_T)$. It is a diffused finite measure which measures all subsets of κ, so by Theorem 2.3 there is a real-valued measurable cardinal $\leq \kappa = |G|$, a contradiction. ∎

COROLLARY 5.3. *Under the assumptions of Theorem* 5.2, *if the measure μ is invariant under reflections then the family of $\mathcal{L}$-measurable functions does not have the difference property.*

Proof. Apply Theorems 4.4 and 5.2. ∎

5.2. *Category case*

THEOREM 5.4. *Let G be a topological group. If $|G|$ is less than the first measurable cardinal and* $\mathrm{non}(\mathcal{M}) = |G|$ *then $S^*(\mathcal{B}, \mathcal{M})$ holds.*

Proof. Again we must find $T \subset \kappa$ such that A_T is not $\mathcal{B}$-measurable. First we show that there is $T \subset \kappa$ such that $A_T \notin \mathcal{M}$ and $G \setminus A_T \notin \mathcal{M}$. Suppose that for every $T \subset \kappa$ we have $A_T \in \mathcal{M}$ or $G \setminus A_T \in \mathcal{M}$. Then $\{T \subset \kappa : A_T \in \mathcal{M}\}$ is a prime σ-ideal on κ containing all the singletons. Thus by Proposition 2.5 and Theorem 2.3 there is a measurable cardinal $\leq \kappa = |G|$, a contradiction. So there is $T_0 \subset \kappa$ such that $A_{T_0} \notin \mathcal{M}$ and $G \setminus A_{T_0} \notin \mathcal{M}$. Then A_{T_0} does not have the Baire property. Indeed, otherwise there is $g \in G$ such that $(A_{T_0} + g) \cap (G \setminus A_{T_0}) \notin \mathcal{M}$, hence $(A_{T_0} + g) \setminus A_{T_0} \notin \mathcal{M}$, a contradiction since A_{T_0} is $\mathcal{M}$-almost invariant by Lemma 5.1.

COROLLARY 5.5. *Under the assumptions of Theorem* 5.4, *the family of $\mathcal{B}$-measurable functions does not have the difference property.*

Proof. Apply Theorems 4.4 and 5.4. ∎

5.3. *c.c.c. case.* Now we generalize the measure and category cases and consider the case where on G there is given a σ-algebra $\mathcal{A}$ and a σ-ideal $\mathcal{J}$ such that the pair $(\mathcal{A}, \mathcal{J})$ satisfies the c.c.c.

THEOREM 5.6. *Let $\mathcal{A}$ be a σ-algebra on a group G and $\mathcal{J}$ be a proper σ-ideal on G such that the pair $(\mathcal{A}, \mathcal{J})$ satisfies the c.c.c. If $|G|$ is less than the first quasi-measurable cardinal and* $\mathrm{non}(\mathcal{J}) = |G|$ *then $S^*(\mathcal{A}, \mathcal{J})$ holds.*

Proof. Once more we must find $T \subset \kappa$ such that A_T is not $\mathcal{A}$-measurable. Suppose that $A_T \in \mathcal{A}$ for every $T \subset \kappa$. Let $\mathcal{I} = \{T \subset \kappa : A_T \in \mathcal{J}\}$. Then it is not difficult to check that $\mathcal{I}$ is a proper σ-ideal containing all the singletons. Since $(\mathcal{A}, \mathcal{J})$ satisfies the c.c.c., so does $(P(\kappa), \mathcal{I})$. But this means that the ideal $\mathcal{I}$ is ω_1-saturated. So by Proposition 2.6 there is a quasi-measurable cardinal $\leq |G|$, a contradiction. ∎

COROLLARY 5.7. *Under the assumptions of Theorem* 5.6, *if $\mathcal{A}$ is invariant under reflections then the family of $\mathcal{A}$-measurable functions does not have the difference property.*

Proof. Apply Theorems 4.4 and 5.6. ∎

REMARK. Although the family of Lebesgue measurable sets and the family of sets with the Baire property satisfy c.c.c., Theorem 5.6 does not imply in general Theorems 5.2 and 5.4. Indeed, if there exists a model with a measurable cardinal then there exists a model for Martin's Axiom and with a quasi-measurable cardinal $< \mathfrak{c}$ (see e.g. [4]). In that model there is no real-valued measurable cardinal $\leq \mathfrak{c}$.

5.4. *Results in ZFC.* Now we show that there are σ-algebras and σ-ideals for which the condition $S^*(\cdot,\cdot)$ holds in ZFC and, by Theorem 4.4, suitable families of functions do not have the difference property (in ZFC).

5.4.1. (s)*-measurable sets*

THEOREM 5.8. $S^*((s),(s_0))$ *holds in ZFC.*

Proof. We have to construct an (s)-nonmeasurable set A such that $A = -A$ and $(A + x) \setminus A \in (s_0)$ for every $x \in \mathbb{R}$. We slightly modify Sierpiński's construction [13]. Let $\mathbb{R} = \{r_\alpha : \alpha < \mathfrak{c}\}$ and $\{P_\alpha : \alpha < \mathfrak{c}\}$ be an enumeration of the reals and perfect sets respectively. Let L_α be the linear space over the rationals spanned by $\{r_\beta : \beta < \alpha\}$. We construct two sequences $\{x_\alpha : \alpha < \mathfrak{c}\}$ and $\{y_\alpha : \alpha < \mathfrak{c}\}$ with

$$x_\alpha \in P_\alpha \setminus (L_\alpha + (\{\pm x_\beta : \beta < \alpha\} \cup \{\pm y_\beta : \beta < \alpha\})),$$
$$y_\alpha \in P_\alpha \setminus (L_\alpha + (\{\pm x_\beta : \beta \leq \alpha\} \cup \{\pm y_\beta : \beta < \alpha\})).$$

Now we put $S = \bigcup_{\alpha<\mathfrak{c}} (L_\alpha \pm x_\alpha)$.

We see that $S = -S$. To show that S is not (s)-measurable we check that S is a Bernstein set. One can see that $S \cap P \neq \emptyset$ for every perfect set P since $x_\alpha \in S$ for every $\alpha < \mathfrak{c}$. On the other hand, suppose that there is $\beta < \mathfrak{c}$ such that $y_\beta \notin \mathbb{R} \setminus S$. Thus there is α such that $y_\beta \in L_\alpha \pm x_\alpha$. If $\beta \geq \alpha$ then we get a contradiction with the definition of the points y_α. So $\alpha > \beta$. But in that case $x_\alpha \in L_\alpha \pm y_\beta$, a contradiction.

Now we show that S is (s_0)-almost invariant. It suffices to show that $|(S + r) \setminus S| < \mathfrak{c}$ for every $r \in \mathbb{R}$ since all sets of cardinality less than $\mathfrak{c}$ are in (s_0). Take $r \in \mathbb{R}$. Then there is $\beta < \mathfrak{c}$ such that $r = r_\beta$. Since $r_\beta \in L_\alpha$ for every $\alpha > \beta$ we can write

$$\begin{aligned}(S + r_\beta) \setminus S &= \Big(\bigcup_{\alpha<\mathfrak{c}} (L_\alpha \pm x_\alpha + r_\beta)\Big) \setminus S \\ &= \Big(\bigcup_{\alpha\le\beta} (L_\alpha \pm x_\alpha + r_\beta) \cup \bigcup_{\alpha>\beta} (L_\alpha \pm x_\alpha)\Big) \setminus S \\ &= \Big(\bigcup_{\alpha\le\beta} (L_\alpha \pm x_\alpha + r_\beta) \setminus S\Big) \cup \Big(\bigcup_{\alpha>\beta} (L_\alpha \pm x_\alpha) \setminus S\Big) \\ &= \Big(\bigcup_{\alpha\le\beta} (L_\alpha \pm x_\alpha + r_\beta) \setminus S\Big) \subset \bigcup_{\alpha\le\beta} (L_\alpha \pm x_\alpha + r_\beta)\end{aligned}$$

and the last set is of cardinality less than $\mathfrak{c}$.

COROLLARY 5.9. *The family of* (s)*-measurable functions does not have the difference property.*

Proof. Apply Theorems 4.4 and 5.8. ∎

REMARK. Corollary 5.9 was proved directly in [3]. Here we use a general theorem (Theorem 4.4) which is useful in many other cases as well.

REMARK. Since $S^*((s),(s_0))$ holds, so does $S((s),(s_0))$. But we cannot use Theorem 4.1 instead of Theorem 4.4 to prove Corollary 5.9 since by Proposition 3.6 the pair $((s),(s_0))$ does not have the weak Ostrowski property.

5.4.2. *A subgroup of the real line with measure.* Let X be a subset of the real line which is of positive outer Lebesgue measure. Define $\mathcal{L}_X = \{A \cap X : A \text{ is Lebesgue measurable}\}$ and $m_X\colon \mathcal{L}_X \to [0, +\infty]$ by $m_X(B) = m^o(B)$ where m^o denotes the Lebesgue outer measure and $B \in \mathcal{L}_X$. Then it is not difficult to check that $\mathcal{L}_X$ is a σ-algebra and m_X is a measure on $\mathcal{L}_X$. Moreover, $\mathrm{non}(\mathcal{N}(m_X)) = \mathrm{non}(\mathcal{N}(m))$.

THEOREM 5.10. *There is a subgroup G of $\mathbb{R}$ such that $S^*(\mathcal{L}_G, \mathcal{N}(m_G))$ holds* (*in ZFC*).

Proof. Let $X \subset \mathbb{R}$ be a Lebesgue nonmeasurable set of cardinality $\kappa = \mathrm{non}(\mathcal{N}(m))$. Let G be the group generated by X. Then $|G| = \kappa$.

Since $|G| = \mathrm{non}(\mathcal{N}(m))$, Theorem 2.7 shows that $|G|$ is less than the first real-valued measurable cardinal. Moreover, $\mathrm{non}(\mathcal{N}(m_G)) = \mathrm{non}(\mathcal{N}(m)) = |G|$. Thus $S^*(\mathcal{L}_G, \mathcal{N}(m_G))$ holds by Theorem 5.2. ∎

COROLLARY 5.11. *The family of m_G-measurable functions does not have the difference property* (*in ZFC*).

Proof. Apply Theorems 4.4 and 5.10. ∎

Remark. One can see that every Lebesgue nonmeasurable group of cardinality non($\mathcal{N}(m)$) witnesses the assertion of Theorem 5.10.

5.4.3. *Subgroup of the real line with topology*

Theorem 5.12. *There is a subgroup G of the real line such that $S^*(\mathcal{B}, \mathcal{M})$ holds (in ZFC).*

Proof. Let $X \subset \mathbb{R}$ be a set without the Baire property of cardinality $\kappa = \mathrm{non}(\mathcal{M}(\mathbb{R}))$. Let G be the group generated by $X \cup \mathbb{Q}$. Then $|G| = \kappa$.

Since $|G| \le \mathfrak{c}$, Theorem 2.4 implies that $|G|$ is less than the first measurable cardinal. Moreover, $\mathrm{non}(\mathcal{M}(G)) = \mathrm{non}(\mathcal{M}(\mathbb{R})) = |G|$ since G is dense in $\mathbb{R}$. Thus $S^*(\mathcal{B}(G), \mathcal{M}(G))$ holds by Theorem 5.4. ■

Corollary 5.13. *The family of functions with the Baire property on G does not have the difference property (in ZFC).*

Proof. Apply Theorems 4.4 and 5.12. ■

Remark. One can see that every dense group without the Baire property of cardinality non($\mathcal{M}(\mathbb{R})$) witnesses the assertion of Theorem 5.12.

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