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# *ON THE HOMOGENEOUS PIECES OF GRADED GENERALIZED LOCAL COHOMOLOGY MODULES*

BY

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**Abstract.** We study, in certain cases, the notions of finiteness and stability of the set of associated primes and vanishing of the homogeneous pieces of graded generalized local cohomology modules.

**1. Introduction.** Throughout the paper $R$ is a commutative Noetherian ring with identity and $M$ is an $R$-module. Also, we denote the set of positive (resp. non-negative) integers by $\mathbb{N}$ (resp. $\mathbb{N}_0$). For an ideal $\mathfrak{a}$ of $R$, the $\mathfrak{a}$-*torsion submodule* of $M$ is $\Gamma_{\mathfrak{a}}(M) := \bigcup_{n\geq 0}(0 :_M \mathfrak{a}^n)$. Let $\mathcal{C}(R)$ denote the category of $R$-modules and $R$-homomorphisms. It is well known that $\Gamma_{\mathfrak{a}}(\cdot) : \mathcal{C}(R) \to \mathcal{C}(R)$ is a covariant, $R$-linear and left exact functor. For a non-negative integer $i$, the $i$th right derived functor of $\Gamma_{\mathfrak{a}}(\cdot)$ is denoted by $H^i_{\mathfrak{a}}(\cdot)$ and is called the $i$th *local cohomology functor* with respect to $\mathfrak{a}$. Note that $H^i_{\mathfrak{a}}(\cdot) \cong \varinjlim_{n\in\mathbb{N}} \mathrm{Ext}^i_R(R/\mathfrak{a}^n, \cdot)$.

The following generalization of the local cohomology functor was given in the local cases by J. Herzog [5], and in the general case by M. H. Bijan-Zadeh [2]. The $i$th generalized local cohomology functor, $H^i_{\mathfrak{a}}(\cdot,\cdot) : \mathcal{C}(R) \times \mathcal{C}(R) \to \mathcal{C}(R)$ is defined by

$$H^i_{\mathfrak{a}}(M,N) := \varinjlim_{n\in\mathbb{N}} \mathrm{Ext}^i_R(M/\mathfrak{a}^n M, N) \quad \text{for all } M,N \in \mathcal{C}(R).$$

If $M$ is finitely generated, then, of course, there are other approaches to the construction of generalized local cohomology functors. In the following we recall two of those.

(a) Following [7, 2.1], let $M$ be finitely generated and let $\boldsymbol{J}^N$ be an injective resolution for an $R$-module $N$. Then

$$H^i_{\mathfrak{a}}(M,N) \cong H^i(\Gamma_{\mathfrak{a}}(\mathrm{Hom}_R(M,\boldsymbol{J}^N))) \cong H^i(\mathrm{Hom}_R(M,\Gamma_{\mathfrak{a}}(\boldsymbol{J}^N))).$$

In fact, for a fixed finitely generated $R$-module $M$, it is easy to see that the functors $H^i(\mathrm{Hom}_R(M,\Gamma_{\mathfrak{a}}(\boldsymbol{J}^{\cdot})))_{i\in\mathbb{N}_0}$ constitute a connected sequence of functors, for which $H^0_{\mathfrak{a}}(M,\cdot)$ and $H^0(\mathrm{Hom}_R(M,\Gamma_{\mathfrak{a}}(\boldsymbol{J}^{\cdot})))$ are isomorphic; so this

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sequence is isomorphic to the connected sequence $(H^i_{\mathfrak{a}}(M,\cdot))_{i\in\mathbb{N}_0}$ of functors. In the same way one shows that $H^i_{\mathfrak{a}}(M,N)\cong H^i(\Gamma_{\mathfrak{a}}(\operatorname{Hom}_R(M,\boldsymbol{J}^N)))$. Observe that the above isomorphisms imply that all the $H^i_{\mathfrak{a}}(M,N)$ are $\mathfrak{a}$-torsion modules.

(b) There is another approach to the construction of the generalized local cohomology functors which uses double complexes. Following [2, 4.2], let $a_1,\ldots,a_n$ $(n>0)$ be a generating set of $\mathfrak{a}$ and let $K^t_{\cdot}$ denote the Koszul complex of $R$ with respect to $a_1^t,\ldots,a_n^t$. Let $P_{\cdot}$ be a projective resolution for $M$. If $\boldsymbol{C}^t_{\cdot}$ denotes the single complex associated to the double complex $K^t_{\cdot}\otimes P_{\cdot}$, then

$$H^i_{\mathfrak{a}}(M,N)\cong \varinjlim_{t\in\mathbb{N}} H^i(\operatorname{Hom}_R(\boldsymbol{C}^t_{\cdot},N))$$

for all $i\in\mathbb{N}_0$.

Since each term in the complex $\boldsymbol{C}^t_{\cdot}$ is a projective $R$-module, one can see, by the standard arguments in homological algebra, that the sequence $(\varinjlim_{t\in\mathbb{N}} H^i(\operatorname{Hom}_R(\boldsymbol{C}^t_{\cdot},\cdot)))_{i\in\mathbb{N}_0}$ is a connected sequence of functors and that if $M$ is finitely generated then the sequences

$$(H^i_{\mathfrak{a}}(M,\cdot))_{i\in\mathbb{N}_0} \quad\text{and}\quad (\varinjlim_{t\in\mathbb{N}} H^i(\operatorname{Hom}_R(\boldsymbol{C}^t_{\cdot},\cdot)))_{i\in\mathbb{N}_0}$$

of functors are isomorphic.

In this paper, we obtain some results on graded generalized local cohomology modules. Let $R=\bigoplus_{i\in\mathbb{Z}}R_i$ be a graded (Noetherian) ring. For graded $R$-modules $X=\bigoplus_{n\in\mathbb{Z}}X_n$ and $Y=\bigoplus_{n\in\mathbb{Z}}Y_n$, a homomorphism $f:X\to Y$ is said to be *homogeneous* if $f(X_n)\subseteq Y_n$ for all $n\in\mathbb{Z}$. The category of all graded $R$-modules and homogeneous $R$-homomorphisms is denoted by ${}^*\mathcal{C}(R)$. For $t\in\mathbb{Z}$, we shall denote the $t$-shift functor by $(\cdot)(t):{}^*\mathcal{C}(R)\to{}^*\mathcal{C}(R)$: thus, for a graded $R$-module $X=\bigoplus_{n\in\mathbb{Z}}X_n$, we have $X(t)_n=X_{n+t}$ for all $n\in\mathbb{Z}$; also $f(t)\lceil_{X(t)_n}=f\lceil_{X_{n+t}}$ for each morphism $f$ in ${}^*\mathcal{C}(R)$ and all $n\in\mathbb{Z}$. An $R$-module homomorphism $f:X\to Y$ is called *homogeneous of degree* $i$ if $f(X_n)\subseteq Y_{n+i}$ for all $n\in\mathbb{Z}$. Note that $f$ may be considered as a morphism $f:X(-i)\to Y$ in ${}^*\mathcal{C}(R)$. We denote by $\operatorname{Hom}_i(X,Y)$ the set of homogeneous homomorphisms of degree $i$. Then the $\mathbb{Z}$-submodules $\operatorname{Hom}_i(X,Y)$ of $\operatorname{Hom}_R(X,Y)$ form a direct sum, and it is obvious that ${}^*\operatorname{Hom}_R(X,Y):=\bigoplus_{n\in\mathbb{Z}}\operatorname{Hom}_i(X,Y)$ is a graded $R$-module which is a submodule of $\operatorname{Hom}_R(X,Y)$. One can prove that ${}^*\operatorname{Hom}_R(X,Y)=\operatorname{Hom}_R(X,Y)$ whenever $X$ is finitely generated (see [4, 1,5.19]).

Let $\mathfrak{a}$ be a homogeneous ideal of $R$ and suppose that the $R$-module $M$ is graded and finitely generated. Let $N$ be any graded $R$-module. In the following, we indicate three methods of grading $H^i_{\mathfrak{a}}(M,N)$.

(i) If we consider a projective resolution of $M/\mathfrak{a}^nM$ in the category ${}^*\mathcal{C}(R)$ and apply the functor ${}^*\operatorname{Hom}_R(-,N)$ to that complex, the $i$th homology module of the resulting complex is ${}^*\mathrm{Ext}^i_R(M/\mathfrak{a}^n M, N)$, which is a graded $R$-module. Since $M$ is finitely generated we have ${}^*\mathrm{Ext}^i_R(M/\mathfrak{a}^n M, N) = \mathrm{Ext}^i_R(M/\mathfrak{a}^n M, N)$. (See [4, p. 33].) Hence the $R$-module

$$H^i_{\mathfrak{a}}(M,N) = \varinjlim_{n\in\mathbb{N}} \mathrm{Ext}^i_R(M/\mathfrak{a}^n M, N)$$

has the structure of a graded $R$-module.

(ii) As mentioned in (a), the connected sequences

$$(H^i_{\mathfrak{a}}(M,\cdot))_{i\in\mathbb{N}_0} \quad \text{and} \quad H^i(\mathrm{Hom}_R(M, \Gamma_{\mathfrak{a}}(\boldsymbol{J}^{\boldsymbol{\cdot}})))_{i\in\mathbb{N}_0}$$

of functors are isomorphic. But $\Gamma_{\mathfrak{a}}(N)$ is a graded submodule of $N$ for any graded $R$-module $N$. So, if we consider an ${}^*$injective resolution ${}^*\boldsymbol{J}^N$ of $N$ and apply $\Gamma_{\mathfrak{a}}(\cdot)$ and the functor $\mathrm{Hom}_R(M,-)$ to that complex, the $i$th homology module $H^i(\mathrm{Hom}_R(M, \Gamma_{\mathfrak{a}}({}^*\boldsymbol{J}^{\boldsymbol{\cdot}})))$ of the resulting complex is a graded $R$-module. Thus, in view of (a), $H^i_{\mathfrak{a}}(M,N)$ can be equipped with a grading.

(iii) Finally, in order to establish a graded $R$-module structure on $H^i_{\mathfrak{a}}(M,N)$, let $a_1,\dots,a_n$ $(n>0)$ be homogeneous elements which generate $\mathfrak{a}$. Then, by [3, 12.4.4], $K^t_{\cdot}$ is in ${}^*\mathcal{C}(R)$ and so, for a fixed projective resolution $P_{\cdot}$ of $M$ in ${}^*\mathcal{C}(R)$, the complex $K^t_{\cdot} \otimes_R P_{\cdot}$ is in ${}^*\mathcal{C}(R)$. Hence $\boldsymbol{C}^t_{\cdot}$, the single complex associated to the double complex $K^t_{\cdot} \otimes_R P_{\cdot}$, is in ${}^*\mathcal{C}(R)$. Therefore $\varinjlim_{t\in\mathbb{N}} H^i(\mathrm{Hom}_R(\boldsymbol{C}^t_{\cdot}, N))$ has a graded $R$-module structure; and so $H^i_{\mathfrak{a}}(M,N)$ inherits a grading in view of (b).

Although we have mentioned three methods of grading $H^i_{\mathfrak{a}}(M,N)$, the same arguments as in [3, Chapter 12] show that these are the same, and are precisely those with respect to which $(H^i_{\mathfrak{a}}(M,\cdot))_{i\in\mathbb{N}_0}$ has the ${}^*$restriction property [3, 12.2.5]. Thus in the rest of this note we endow the module $H^i_{\mathfrak{a}}(M,N)$ with one of the above gradings and we use the notation $H^i_{\mathfrak{a}}(M,N)_n$ for the $n$th homogeneous piece of that module.

The aim of this note is to establish, in certain cases, some finiteness results for the homogeneous pieces of the module $H^i_{\mathfrak{a}}(M,N)$, and to prove a vanishing theorem under certain hypotheses.

**2. The results.** Throughout this section, we assume that the $R$-module $M$ is finitely generated. The following lemma is needed in the proof of the main results of this note.

2.1. LEMMA. *Let $\mathfrak{a}$ be an ideal of $R$. Then*:

(i) *Let $N$ be any $\mathfrak{a}$-torsion $R$-module. Then $H^i_{\mathfrak{a}}(M,N) \cong \mathrm{Ext}^i_R(M,N)$ for all $i \geq 0$. Moreover, if $N$ is finitely generated, then $H^i_{\mathfrak{a}}(M,N)$ is finitely generated for all $i \geq 0$.*

(ii) *If $\mathrm{pd}(M)$, the projective dimension of $M$, is finite, then $H^i_{\mathfrak{a}}(M,N) = 0$ for all $i > \mathrm{pd}(M) + \mathrm{ara}(\mathfrak{a})$. Here $\mathrm{ara}(\mathfrak{a})$ is the least number of elements of $R$ required to generate an ideal which has the same radical as $\mathfrak{a}$* [3, 3.3.2].

*Proof.* Since $N$ is an $\mathfrak{a}$-torsion $R$-module, by [3, 2.1.6] there is an injective resolution $\boldsymbol{J}^N$ of $N$ in which each term is an $\mathfrak{a}$-torsion $R$-module. By applying $\Gamma_{\mathfrak{a}}(\cdot)$ to $\boldsymbol{J}^N$, we still have an injective resolution for $N$. Hence $H^i(\operatorname{Hom}_R(M, \Gamma_{\mathfrak{a}}(\boldsymbol{J}^N))) = \operatorname{Ext}^i_R(M, N)$ for all $i \geq 0$. Therefore the result follows by (a).

(ii) This follows from (b) and elementary facts on double complexes. ∎

In the remaining part of this note we assume that $R = \bigoplus_{i\geq 0} R_i$ is a positively graded (Noetherian) ring and that $R_+ = \bigoplus_{i\geq 1} R_i$. We say that $R$ is *homogeneous* if it is generated as an $R_0$-algebra by homogeneous elements of degree one; that is, $R = R_0[R_1]$. Also in the rest of the paper we assume that $N$ is another finitely generated graded $R$-module.

The following well known lemma will be used several times:

2.2. Lemma. *Let $X = \bigoplus_{n\in\mathbb{Z}} X_n$ be a finitely generated graded $R$-module. Then, for all $n \in \mathbb{Z}$, $X_n$ is a finitely generated $R_0$-module. Moreover if $R_+^t X = 0$ for some $t \in \mathbb{N}$, then $X_{-n} = X_n = 0$ for all sufficiently large values of $n$.*

2.3. Proposition. *Suppose that the projective dimension of $M$ is finite. Then*:

(i) *For all $i \in \mathbb{N}_0$ and all $n \in \mathbb{Z}$, the $R_0$-module $H^i_{R_+}(M, N)_n$ is finitely generated.*

(ii) *There exists $r \in \mathbb{Z}$ such that $H^i_{R_+}(M, N)_n = 0$ for all $i \in \mathbb{N}_0$ and all $n \geq r$.*

*Proof.* In view of Lemma 2.1(ii), it is enough to show that, for fixed $i \in \mathbb{N}_0$, $H^i_{R_+}(M, N)_n$ is a finitely generated $R_0$-module for all $n \in \mathbb{Z}$ and that it is zero for all sufficiently large $n$. We prove this by induction on $i$. For $i = 0$, the result is true by (a) and Lemma 2.2. (Note that, by (a), $H^0_{R_+}(M, N)$ is finitely generated.) Now suppose that $i > 0$ and that the result has been proved for $i-1$. Consider the exact sequence $0 \to \Gamma_{R_+}(N) \to N \to N/\Gamma_{R_+}(N) \to 0$ in ${}^*\mathcal{C}(R)$, leading to the exact sequence

$$H^i_{R_+}(M, \Gamma_{R_+}(N)) \to H^i_{R_+}(M, N) \to H^i_{R_+}(M, N/\Gamma_{R_+}(N)). \tag{1}$$

Now, by (a), each element of $H^i_{R_+}(M, \Gamma_{R_+}(N))$ is annihilated by some power of $R_+$. So, in view of Lemma 2.1(i), there exists $h \in \mathbb{N}_0$ such that $(R_+)^h H^i_{R_+}(M, \Gamma_{R_+}(N)) = 0$. Hence, it follows from Lemma 2.2 that $H^i_{R_+}(M, \Gamma_{R_+}(N))_n$ is a finitely generated $R_0$-module for all $n \in \mathbb{Z}$, and it is zero for all sufficiently large $n$. Therefore, by using the exact sequence (1), it is enough to prove the induction step for $H^i_{R_+}(M, N/\Gamma_{R_+}(N))$. Thus we may assume that $0 \neq N$ is $R_+$-torsion free. Then by [4, 1.5.10], there exists a homogeneous element $a \in R_+$ which is a non-zerodivisor on $N$. Let $a$ be of degree $t$. Consider the exact sequence $0 \to N \xrightarrow{a} N(t) \to (N/aN)(t) \to 0$ in ${}^*\mathcal{C}(R)$, yielding the exact sequence

$$H^{i-1}_{R_+}(M, N/aN)_{n+t} \to H^i_{R_+}(M,N)_n \xrightarrow{a} H^i_{R_+}(M,N)_{n+t} \tag{2}$$

of $R_0$-modules for all $n \in \mathbb{Z}$. By the inductive hypothesis, there exists $s \in \mathbb{Z}$ such that $H^{i-1}_{R_+}(M, N/aN)_j = 0$ for all $j \geq s$. Hence it follows from the exact sequence (2) that $H^i_{R_+}(M,N)_n = 0$ for all $n \geq s - t$, because the $R$-modules $H^i_{R_+}(M,N)$ are $R_+$-torsion. Next, fix $n \in \mathbb{Z}$ and let $k \in \mathbb{N}_0$ be such that $n + kt \geq s - t$. Let $j = 0, 1, \ldots, k-1$. Successively using the exact sequences

$$H^{i-1}_{R_+}(M, N/aN)_{n+(j+1)t} \to H^i_{R_+}(M,N)_{n+jt} \to H^i_{R_+}(M,N)_{n+(j+1)t}$$

for $j = k-1, k-2, \ldots, 0$, and the inductive hypothesis, we finally deduce that $H^i_{R_+}(M,N)_n$ is a finitely generated $R_0$-module. This completes the inductive step. ■

It is well known (see for example [8, Theorem 2]) that whenever $R$ is homogeneous and, for some $l \in \mathbb{N}$ and $r \in \mathbb{Z}$, $H^i_{R_+}(N)_{r+1-i} = 0$ for all $i \geq l$, then $H^i_{R_+}(N)_{s-i} = 0$ for all $i \geq l$ and $s > r$. The next theorem establishes a similar result for generalized local cohomology modules.

2.4. Theorem. *Suppose that $R$ is homogeneous and* $\mathrm{pd}(M)$ *is finite. Let $r, l \in \mathbb{Z}$ with $l \geq pd(M) + 1$ and assume that $H^i_{R_+}(M,N)_{r+1-i} = 0$ for all $i \geq l$. Then $H^i_{R_+}(M,N)_{s-i} = 0$ for all $s > r$ and $i \geq l$.*

*Proof.* By employing a similar argument to the one used in [3, pp. 282–284], we may assume that $R_0$ is local with infinite residue field. We prove the theorem by induction on $\dim N$ (the Krull dimension of $N$), which is finite (note that since we may assume that $R_0$ is local, $R$ is a ${}^*$local ring). If $\dim N = 0$, the result follows from [2, 5.1]. Now suppose that $\dim N > 0$ and the result has been proved for all finitely generated graded $R$-modules of smaller dimension. Considering the exact sequence $0 \to \Gamma_{R_+}(N) \to N \to N/\Gamma_{R_+}(N) \to 0$ in ${}^*\mathcal{C}(R)$ and using 2.1(i) we obtain the homogeneous isomorphism $H^i_{R_+}(M, N/\Gamma_{R_+}(N)) \cong H^i_{R_+}(M,N)$ for all $i \geq l$. Therefore we may replace $N$ by $N/\Gamma_{R_+}(N)$. Then, in view of [4, 1.5.12], there exists $a \in R_1$ which is a non-zerodivisor on $N$. Now consider the exact sequence $0 \to N \xrightarrow{a} N(1) \to (N/aN)(1) \to 0$, yielding the exact sequence

$$H^i_{R_+}(M,N)_{n-1} \xrightarrow{a} H^i_{R_+}(M,N)_n \to H^i_{R_+}(M, N/aN)_n \to H^{i+1}_{R_+}(M,N)_{n-1}$$

of $R_0$-modules for all $i \in \mathbb{N}$ and $n \in \mathbb{Z}$. Using the above exact sequences, we see that if $n = r + 1 - i$, then $H^i_{R_+}(M, N/aN)_{r+1-i} = 0$ for all $i \geq l$. Since $\dim(N/aN) < \dim N$, we deduce, by the induction hypothesis, that

$H^i_{R_+}(M, N/aN)_{s-i} = 0$ for all $s > r$ and $i \geq l$. Now, successive use of the exact sequence

$$H^i_{R_+}(M, N)_{n-1-i} \xrightarrow{a} H^i_{R_+}(M, N)_{n-i} \to H^i_{R_+}(M, N/aN)_{n-i}$$

with $H^i_{R_+}(M, N)_{r+1-i} = 0 = H^i_{R_+}(M, N/aN)_{r+2-i}$, and of the inductive hypothesis, yields $H^i_{R_+}(M, N)_{s-i} = 0$ for all $s > r$ and $i \geq l$ and the inductive step is complete. ∎

For a finitely generated graded $R$-module $N$ over a positively graded homogeneous ring $R$, the *Castelnuovo–Mumford regularity* of $N$ at and above level $l$ ($l \geq 0$) is defined by $\operatorname{reg}^l(N) := \sup\{\operatorname{end}(H^i_{R_+}(N)) + i \mid i \geq l\}$, where for any graded $R$-module $X = \bigoplus_{n\in\mathbb{Z}} X_n$, $\operatorname{end}(X) = \sup\{n \in \mathbb{Z} \mid X_n \neq 0\}$ (see [3, 15.2.9]). It is well known [3, 15.2.12 and 15.2.13] that $\operatorname{reg}^1(N) = -\infty$ if and only if $N_n = 0$ for all sufficiently large values of $n$ and that $\operatorname{reg}^0(N) = -\infty$ if and only if $N = 0$. The following theorem generalizes the above results in a certain case for general local cohomology modules.

2.5. Theorem. *Assume that $(R, \mathfrak{m})$ is homogeneous $^*$local with unique $^*$maximal ideal $\mathfrak{m}$ and that the finitely generated graded $R$-module $M$ is non-zero and projective. Then*:

(i) $H^i_{R_+}(M, N) = 0$ *for all $i \geq 1$ if and only if $N_n = 0$ for sufficiently large $n$.*

(ii) $H^i_{R_+}(M, N) = 0$ *for all $i \geq 0$ if and only if $N = 0$.*

*Proof.* (i) First suppose that $N_n = 0$ for sufficiently large $n$. Then there exists $n_0 \in \mathbb{N}$ such that $(R_+)^{n_0} N = 0$. Hence, by 2.1(i), $H^i_{R_+}(M, N) = 0$ for all $i \geq 1$.

Conversely, assume that $H^i_{R_+}(M, N) = 0$ for all $i \geq 1$. Then one can easily see that $H^i_{R_+}(M, N/\Gamma_{R_+}(N)) = 0$ for all $i \geq 0$. Hence, by [2, 5.3 and 5.5], we have $(M/R_+M) \otimes_R (N/\Gamma_{R_+}(N)) = 0$, which in turn implies $M \otimes_R N/\Gamma_{R_+}(N) = 0$. Now $R/\mathfrak{m}$ is either a field, or else $R/\mathfrak{m} \cong k[x, x^{-1}]$, where $k$ is a field and $x$ is a homogeneous element of positive degree which is transcendental over $k$; see [4, 1.5.7]. Hence by the same argument as in [1, p. 31, Exercise 3], it follows that $N = \Gamma_{R_+}(N)$. Therefore, $N_n = 0$ for sufficiently large $n$.

(ii) ($\Leftarrow$) is clear.

($\Rightarrow$) In view of [2, 5.3 and 5.5], we have $(M/R_+M) \otimes_R N = 0$. Hence $M \otimes_R N = 0$, which implies that $N = 0$. ∎

2.6. Corollary. *Assume that $R$ is homogeneous $^*$local and that the projective dimension of $M$ is finite. Let $i \in \mathbb{N}_0$ be such that the $R$-module $H^j_{R_+}(M, N)$ is finitely generated for $j < i$. Then $\operatorname{Ass}_{R_0}(H^i_{R_+}(M, N)_n)$ is asymptotically stable as $n \to -\infty$.*

*Proof.* It is clear that $R_0$ is a local ring. Let $\mathfrak{m}_0$ be the maximal ideal of $R_0$. Let $R'_0 := R_0[x]_{\mathfrak{m}_0 R[x]}$, $R' := R \otimes_{R_0} R'_0$, $M' := M \otimes_{R_0} R'_0$ and $N' := N \otimes_{R_0} R'_0$, where $x$ is an indeterminate. Then a similar argument to that in [3, 15.2.2] gives rise to isomorphisms of $R'_0$-modules

$$H^t_{R'_+}(M', N')_n \cong H^t_{R_+}(M, N)_n \otimes_{R_0} R'_0 \quad \text{for all } t \in \mathbb{N}_0 \text{ and all } n \in \mathbb{Z}.$$

These show that the $R'_0$-module $H^j_{R'_+}(M', N')_n$ is finitely generated for all $j < i$ and $\mathrm{Ass}_{R_0}(H^i_{R_+}(M, N)_n) = \{P'_0 \cap R_0 \mid P'_0 \in \mathrm{Ass}_{R'_0}(H^i_{R'_+}(M', N')_n)\}$ for all $n \in \mathbb{Z}$ (see [6, (23.2)(ii)]). This shows that we can replace $R$, $M$ and $N$ respectively by $R', M'$ and $N'$ and hence we may assume that $R_0/\mathfrak{m}_0$ is infinite.

Now we prove the corollary by induction on $i$. The case $i = 0$ follows from (a) and Lemma 2.2. So let $i > 0$. Since $\Gamma_{R_+}(N)$ has only finitely many non-zero homogeneous components, in view of Lemma 2.1(i) and using induction on the projective dimension on $M$, one can see that $H^t_{R_+}(M, \Gamma_{R_+}(N))_n = 0$ and so $H^t_{R_+}(M, N)_n \cong H^t_{R_+}(M, N/\Gamma_{R_+}(N))_n$ for all $t \geq 1$, and for all but finitely many $n \in \mathbb{Z}$. Thus, we can replace $N$ by $N/\Gamma_{R_+}(N)$, and so we may assume that there exists $a \in R_1$ which is a non-zerodivisor on $N$. Now, from the short exact sequence $0 \to N \xrightarrow{a} N(1) \to (N/aN)(1) \to 0$ (in ${}^*\mathcal{C}(R)$), we obtain the exact sequence

$$H^{t-1}_{R_+}(M, N)(1) \to H^{t-1}_{R_+}(M, N/aN)(1) \to H^t_{R_+}(M, N) \xrightarrow{a} H^t_{R_+}(M, N)(1)$$

for all $t \in \mathbb{N}$ and all $n \in \mathbb{Z}$.

Using the above exact sequences, we deduce that $H^{j-1}_{R_+}(M, N/aN)$ is finitely generated for all $j < i$. So, by induction, there exists $n_1 \in \mathbb{Z}$ such that

$$\mathrm{Ass}_{R_0}(H^{i-1}_{R_+}(M, N/aN)_n) = \mathrm{Ass}_{R_0}(H^{i-1}_{R_+}(M, N/aN)_{n_1}) =: B$$

for all $n \leq n_1$. On the other hand, by Lemma 2.2, there is $n_2 < n_1$ such that $H^{i-1}_{R_+}(M, N)_{n+1} = 0$ for all $n \leq n_2$. Therefore, for each $n \leq n_2$, the above exact sequence induces the exact sequence

$$0 \to H^{i-1}_{R_+}(M, N/aN)_{n+1} \to H^i_{R_+}(M, N)_n \to H^i_{R_+}(M, N)_{n+1}$$

of $R_0$-modules. From this we deduce that

$$B \subseteq \mathrm{Ass}_{R_0}(H^i_{R_+}(M, N)_n) \subseteq B \cup \mathrm{Ass}_{R_0}(H^i_{R_+}(M, N)_{n+1})$$

for all $n \leq n_2$. Hence

$$B \subseteq \mathrm{Ass}_{R_0}(H^i_{R_+}(M, N)_n) \subseteq \mathrm{Ass}_{R_0}(H^i_{R_+}(M, N)_{n+1})$$

for $n < n_2$, because then

$$B \cup \mathrm{Ass}_{R_0}(H^i_{R_+}(M, N)_{n+1}) = \mathrm{Ass}_{R_0}(H^i_{R_+}(M, N)_{n+1}).$$

Since $B$ is a finite set the result follows from this, and the inductive step is complete. ∎

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*REFERENCES*

[1] M. F. Atiyah and I. G. Macdonald, *Introduction to Commutative Algebra*, Addison-Wesley, 1969.

[2] M. H. Bijan-Zadeh, *A common generalization of local cohomology theories*, Glasgow Math. J. 21 (1980), 173–181.

[3] M. P. Brodmann and R. Y. Sharp, *Local Cohomology: an Algebraic Introduction with Geometric Applications*, Cambridge Stud. Adv. Math. 60, Cambridge Univ. Press, Cambridge, 1998.

[4] W. Bruns and J. Herzog, *Cohen–Macaulay Rings*, Cambridge Stud. Adv. Math. 39, Cambridge Univ. Press, Cambridge 1993.

[5] J. Herzog, *Komplexe, Auflösungen und Dualität in der lokalen Algebra*, preprint, Univ. Essen.

[6] M. Matsumura, *Commutative Ring Theory*, Cambridge Univ. Press, Cambridge, 1986.

[7] N. Suzuki, *On the generalized local cohomology and its duality*, J. Math. Kyoto Univ. 18 (1978), 71–85.

[8] A. Ooishi, *Castelnuovo's regularity of graded rings and modules*, Hiroshima Math. J. 12 (1982), 627–644.

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