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MULTIPLIERS FOR THE TWISTED LAPLACIAN

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Abstract. We study $\mathcal{H}^1$-L^p boundedness of certain multiplier transforms associated to the special Hermite operator.

1. Introduction and main results. Consider on $\mathbb{C}^n$ the $2n$ linear differential operators

$$Z_j = \frac{\partial}{\partial z_j} + \frac{1}{2}\bar{z}_j, \quad \overline{Z}_j = \frac{\partial}{\partial \bar{z}_j} - \frac{1}{2} z_j, \quad j = 1, \ldots, n.$$

They generate a nilpotent Lie algebra isomorphic to the Heisenberg Lie algebra of dimension $2n+1$. This algebra plays for the twisted convolution (see definition below) on $\mathbb{C}^n$ a role analogous to that of the Lie algebra of left invariant vector fields on a Lie group.

Consider the operator L defined by

$$L = -\frac{1}{2}\sum_{j=1}^{n}(Z_j\overline{Z}_j + \overline{Z}_j Z_j).$$

An easy calculation shows that L can be written in the form

$$L = -\Delta_z + \frac{1}{4}|z|^2 - i\sum_{j=1}^{n}\left(x_j\frac{\partial}{\partial y_j} - y_j\frac{\partial}{\partial x_j}\right).$$

The above operator L is called the *special Hermite operator*. We remark that this operator is closely related to the sub-Laplacian on the Heisenberg group, $H^n = \mathbb{C}^n \times \mathbb{R}$. If $\mathcal{L}$ denotes the sub-Laplacian on H^n, then $\mathcal{L}(e^{it}f(z)) = e^{it}Lf(z)$. For this reason L is also called the *twisted Laplacian*.

The aim of this paper is to prove certain multiplier theorems for L on $\mathbb{C}^n$. Eigenfunction expansion associated to L is given by the so-called special Hermite expansion which has received considerable attention in the recent years. Let $L_k^\alpha(t)$ be the Laguerre polynomials of the type $\alpha > -1$, defined

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$$e^{-t}t^{\alpha}L_k^{\alpha}(t) = (-1)^k \frac{1}{k!}\frac{d^k}{dt^k}(e^{-t}t^{k+\alpha}), \qquad t > 0.$$

Then for a function f in $L^2(\mathbb{C}^n)$, its *special Hermite expansion* is given by

$$f(z) = (2\pi)^{-n}\sum_{k=0}^{\infty} f \times \varphi_k(z)$$

where $\varphi_k(z)$ are the Laguerre functions defined by $L_k^{n-1}\big(\frac{1}{2}|z|^2\big)e^{-\frac{1}{4}|z|^2}$ and the *twisted convolution* of two functions f and g on $\mathbb{C}^n$ is defined by

$$f \times g(z) = \int_{\mathbb{C}^n} f(z-w)g(w)e^{\frac{i}{2}\mathrm{Im} z.\overline{w}}\, dw.$$

The spectral projections of L are given by the operators $f \mapsto f \times \varphi_k$ and we have $L(f \times \varphi_k) = (2k+n)(f \times \varphi_k)$. The above series converges in L^2 and the following holds:

$$\|f\|_2^2 = (2\pi)^{-n}\sum_{k=0}^{\infty}\|f \times \varphi_k\|_2^2.$$

We also mention that the twisted convolution satisfies the interesting property

$$\|f \times g\|_2 \le \|f\|_2\|g\|_2.$$

For other properties of the twisted convolution and its relation to the convolution on the Heisenberg group we refer the reader to the monographs [10] and [11].

Given a bounded function m on the set $\mathbb{N}$ of natural numbers one can define an operator $m(L)$ by

$$m(L)f(z) = (2\pi)^{-n}\sum_{k=0}^{\infty} m(2k+n) f \times \varphi_k(z).$$

Such operators are always bounded on L^2. However some smoothness assumptions on m are needed to ensure the boundedness of $m(L)$ on L^p for $p \neq 2$. Using Littlewood–Paley–Stein theory the following theorem was proved by Thangavelu (see [10]). For a function m on $\mathbb{N}$ let $\Delta_+ m(k) = m(k+1) - m(k)$ and define higher powers of Δ_+ inductively.

Theorem 1.1. *Let m be a bounded function on $\mathbb{N}$ which satisfies*

$$|\Delta_+^l m(k)| \le C(1+k)^{-l}$$

for $l = 0, 1, \ldots, n+1$ or $n+2$, when n is odd or even respectively. Then $m(L)$ is bounded on $L^p(\mathbb{C}^n)$ for $1 < p < \infty$.

Note that the conditions on the multiplier m are similar to Hörmander's condition for the Fourier multipliers. See also [2] and [14]. For other properties of the special Hermite expansions we refer to the monograph [10].

We are interested in the boundedness of certain multiplier operators from the twisted Hardy space $\mathcal{H}^1(\mathbb{C}^n)$ to $L^p(\mathbb{C}^n)$. The space $\mathcal{H}^1(\mathbb{C}^n)$ was introduced and studied in [3]. Let ψ be a C^∞-function on $\mathbb{C}^n$ with compact support such that $\psi = 1$ in a neighborhood of zero. Define

$$R_j(z) = \frac{z_j}{|z|^{2n+1}}\,\psi(z), \qquad \overline{R}_j(z) = \frac{\bar{z}_j}{|z|^{2n+1}}\,\psi(z)$$

for $j = 1, \dots, n$. Then $\mathcal{H}^1$ is defined as the set of all $f \in L^1$ for which $R_j \times f$ and $\overline{R}_j \times f$ are in L^1 for all j. The norm on $\mathcal{H}^1$ is given by

$$\|f\|_{\mathcal{H}^1} = \|f\|_1 + \sum_{j=1}^{n} \|R_j \times f\|_1 + \sum_{j=1}^{n} \|\overline{R}_j \times f\|_1.$$

Basic properties such as atomic decomposition and boundedness of singular integral operators etc. were studied in [3]. The space $\mathcal{H}^1$ can also be defined as the subspace of $L^1(\mathbb{C}^n)$ containing all functions f for which the maximal function

$$f^*(z) = \sup_{t>0} |e^{-tL} f(z)|$$

is in $L^1(\mathbb{C}^n)$. Atomic decomposition can be stated as follows. Any f in $\mathcal{H}^1$ can be written as

$$f(z) = \sum_{k=1}^{\infty} \lambda_k f_k(z)$$

with $C_1\|f\|_{\mathcal{H}^1} \le \sum_{k=1}^{\infty} |\lambda_k| \le C_2\|f\|_{\mathcal{H}^1}$, where the atoms f_k satisfy the following:

(i) f_k is supported in a cube $Q(z_k, r_k)$ centered at z_k with half-side $r_k \le 2\sqrt{\pi}$,
(ii) $\|f_k\|_\infty \le (2r_k)^{-2n}$,
(iii) $\int f_k(w) e^{-\frac{i}{2}\mathrm{Im} z_k.\overline{w}}\, dw = 0$.

We shall make use of this atomic decomposition in the proofs.

We start with a Hardy–Littlewood–Sobolev result for L. In [12] it was established that $L^{-\alpha}$ is bounded from $L^p(\mathbb{C}^n)$ to $L^q(\mathbb{C}^n)$ provided $\frac{1}{p} - \frac{1}{q} = \frac{\alpha}{n}$, $0 < \alpha < n$ and $1 < p, q < \infty$. We are interested in the end point result. We have the following theorem.

THEOREM 1.2. *The operator $L^{-\alpha}$ is bounded from $\mathcal{H}^1(\mathbb{C}^n)$ to $L^q(\mathbb{C}^n)$, $1 \le q \le \infty$ provided $1 - \frac{1}{q} = \frac{\alpha}{n}$.*

To prove Theorem 1.2, we shall first prove that L^{-n} is bounded from $\mathcal{H}^1$ to L^∞, and $L^{i\beta}$, for β real, is bounded from $\mathcal{H}^1$ to itself. Then we apply Stein's analytic interpolation [8] to the family L^{-z}. The proof of the interpolation theorem for the usual Hardy space H^1, by Fefferman and Stein [1], can be modified to deal with the present situation. The sharp maximal function has to be replaced by the twisted sharp maximal function

$$f_\tau^*(z) = \sup \frac{1}{|Q|} \int\limits_Q |f(w) - \widetilde{f}_Q e^{\frac{i}{2}\mathrm{Im} z.\overline{w}}|\, dw$$

where $\widetilde{f}_Q = \frac{1}{|Q|}\int_Q f(w) e^{-\frac{i}{2}\mathrm{Im} z.\overline{w}}\, dw$. Here Q is a cube centered at z. In order to complete the proof of the analytic interpolation theorem we need the fact that

$$C_1\|f\|_p \le \|f_\tau^*\|_p \le C_2\|f\|_p.$$

This has already been proved in Phong–Stein [5]. Theorem 1.2 will be proved in the second section.

In the third section we prove an $\mathcal{H}^1$-L^p multiplier theorem for L. This result is analogous to that of Nilsson [4] on $\mathbb{R}^n$ for Fourier multipliers. In [4] Nilsson establishes the following. Let $S_j = \{2^{j-1} \le |\xi| \le 2^{j+1}\}$.

THEOREM 1.3. *Assume that* $m \in C^k(\mathbb{R}^n - \{0\})$ *and*

$$\int\limits_{S_j} \sum_{|\alpha|\le k} |2^{j|\alpha|} D_\alpha m(\xi)|^2\, d\xi \le 2^{\frac{nj(2-q)}{q}}, \qquad j \in \mathbb{Z},$$

where k is the least integer $> n\left|\frac{1}{q} - \frac{1}{2}\right|$ *and* $1 \le q \le 2$. *Then the convolution by* $K = \widehat{m}$ *maps* $H^1(\mathbb{R}^n)$ *to* $L^q(\mathbb{R}^n)$.

Before we state our result we note that the above theorem easily extends to $q > 2$ as well. This can be seen as follows. Assume that m satisfies the above for some $q > 2$. Then the function $\widetilde{m}(\xi) = m(\xi)|\xi|^{2n(\frac{1}{q'}-\frac{1}{2})}$, where q' is the conjugate exponent of q, satisfies the multiplier conditions in Theorem 1.3 with q replaced by q'. Hence the operator with multiplier $\widetilde{m}$ is bounded from H^1 to $L^{q'}$. As $m(\xi) = \widetilde{m}(\xi)|\xi|^{-2n(\frac{1}{q'}-\frac{1}{2})}$ the result follows from the above theorem and the Hardy–Littlewood–Sobolev theorem.

Our result for the operator L is the following. Let Δ_+ stand for the difference operator defined above and $\Delta_+^l = \Delta_+^{l-1}\Delta_+$.

THEOREM 1.4. *Let m be a bounded function defined on N which satisfies*

$$|\Delta_+^l m(k)| \le C(1+k)^{-\frac{n}{2}+n(\frac{1}{q}-\frac{1}{2})-l}$$

for $l = 0, 1, \ldots, M$, *where M is the least even integer* $> 2n\left|\frac{1}{q} - \frac{1}{2}\right|$. *Then* $m(L)$ *is bounded from* $\mathcal{H}^1(\mathbb{C}^n)$ *to* $L^q(\mathbb{C}^n)$.

As in Theorem 1.3, in view of the Hardy–Littlewood–Sobolev theorem for L it is enough to prove Theorem 1.4 for $q \le 2$. We also remark that for

$q = 2$ the assumptions on the derivatives of m are not needed (see Theorem 3.4 in [12]).

To motivate our next theorem consider the following Hardy–Littlewood inequalities for the Fourier transform (see [6] for a proof):

(i) $\int_{\mathbb{R}^n} |\widehat{f}(x)|^p |x|^{n(p-2)}\,dx \le \int_{\mathbb{R}^n} |f(x)|^p\,dx$ for $1 < p \le 2$,
(ii) $\int_{\mathbb{R}^n} |\widehat{f}(x)|^p \le C \int_{\mathbb{R}^n} |f(x)|^p |x|^{n(p-2)}\,dx$ for $p \ge 2$,
(iii) $\int_{\mathbb{R}^n} |\widehat{f}(x)||x|^{-n}\,dx \le C\|f\|_{H^1}$.

We can rewrite the above inequalities in the following way. Consider the fractional powers of the Laplacian defined as follows:

$$((-\Delta)^{-\alpha} f)^\wedge(x) = |x|^{-2\alpha}\widehat{f}(x).$$

If we let $T(\alpha)f = (-\Delta)^{-\alpha} f$, then the above inequalities take the following form with $\alpha = n\left|\frac{1}{p} - \frac{1}{2}\right|$:

$$\begin{aligned} \|\{T(\alpha)f\}^\wedge\|_p &\le C\|f\|_p \quad \text{for } 1 < p \le 2,\\ \|f\|_p &\le C\|\{T(\alpha)f\}^\wedge\|_p \quad \text{for } p \ge 2,\\ \|\{T(\alpha)f\}^\wedge\|_1 &\le C\|f\|_{H^1}. \end{aligned}$$

Such inequalities have been proved in [13] when $-\Delta$ is replaced by the Hermite operator. We can ask the same questions for the special Hermite operator. Let us define the operators $T_t(\alpha)$ given by

$$T_t(\alpha)f(z) = (2\pi)^{-n}\sum_{k=0}^{\infty}(2k+n)^{-\alpha}e^{it(2k+n)} f\times\varphi_k(z).$$

Note that when $t = \pi/2$, we have $T_t^\alpha f = i^n \mathcal{F}_{\mathrm{s}} L^{-\alpha} f$ where $\mathcal{F}_{\mathrm{s}}$ is the symplectic Fourier transform defined by

$$\mathcal{F}_{\mathrm{s}} f(z) = 2^{-n}\int_{\mathbb{C}^n} f(w)e^{-\frac{i}{2}\mathrm{Im}\, z.\overline{w}}\,dw.$$

To see this, observe that $\mathcal{F}_{\mathrm{s}} f(z) = 2^{-n} f\times 1(z)$ where 1 is the constant function 1. Since $\mathcal{F}_{\mathrm{s}}$ commutes with L and $\mathcal{F}_{\mathrm{s}}\varphi_k = (-1)^k\varphi_k$ (see [11]) we have

$$\mathcal{F}_{\mathrm{s}} L^{-\alpha} f(z) = e^{i\frac{n\pi}{2}}\sum_{k=0}^{\infty}(2k+n)^{-\alpha}e^{i(2k+n)\frac{\pi}{2}} f\times\varphi_k(z).$$

For the operators $T_t(\alpha)$ we prove the following.

THEOREM 1.5. *When $\alpha = 2n\left|\frac{1}{p} - \frac{1}{2}\right|$, $1 < p < \infty$, the operators $T_t(\alpha)$ are bounded on $L^p(\mathbb{C}^n)$. When $p = 1$ and $\alpha = n$ the operator $T_t(\alpha)$ is bounded from $\mathcal{H}^1$ into L^1.*

We note that the above theorem gives the Hardy–Littlewood inequalities for the operator L. Theorem 1.5 has another application to the solutions of the Schrödinger equation. Let $u(z,t)$ denote the solution to the initial value problem

$$i\partial_t u(z,t) = Lu(z,t), \qquad u(z,0) = f(z).$$

The solution to this problem has the expansion

$$u(z,t) = \sum_{k=0}^{\infty} e^{i(2k+n)t} f \times \varphi_k(z).$$

As in the Euclidean case it is not possible to have an inequality of the form

$$\|u(\cdot,t)\|_p \le C(t)\|f\|_p \qquad \text{for } p \ne 2.$$

Indeed, $u(z,t)$ is nothing but a fractional power of the symplectic Fourier transform defined earlier and we know that no fractional power of the Fourier transform is bounded on L^p, for $p \ne 2$. Therefore, following Sjöstrand [7] we define the Riesz means for the solution

$$G_\tau(\alpha)f(z) = \alpha\tau \int_0^\tau (\tau - t)^{\alpha-1} u(z,t)\, dt.$$

Using the Hardy–Littlewood inequalities for L we can prove the following

COROLLARY 1.6. *If $\alpha \ge 2n\left|\frac{1}{p} - \frac{1}{2}\right|$, then the operators $G_\tau(\alpha)$ are bounded on $L^p(\mathbb{C}^n)$. When $p = 1$ and $\alpha = n$, $G_\tau(\alpha)$ is bounded from $\mathcal{H}^1$ into L^1.*

2. Hardy–Littlewood–Sobolev theorem for L. In this section we prove Theorem 1.2. We start with a simple lemma.

LEMMA 2.1. *Let K be a kernel such that both $K(z)$ and $|z|K(z)$ are in $L^1(\mathbb{C}^n)$. Then the map $f \mapsto K \times f$ is bounded from $\mathcal{H}^1$ to itself.*

Proof. Let $\mathbf{K}f = K \times f$. Let $\mathbf{R}_j$ and $\overline{\mathbf{R}}_j$ stand for the operators $f \mapsto R_j \times f$ and $f \mapsto \overline{R}_j \times f$ respectively, which were defined in the introduction. Since $\mathbf{K}$ maps $L^1(\mathbb{C}^n)$ to $L^1(\mathbb{C}^n)$ it is enough to show that the commutators $[\mathbf{K}, \mathbf{R}_j]$ and $[\mathbf{K}, \overline{\mathbf{R}}_j]$ map L^1 to L^1 (see [3]). Now the estimate

$$\begin{aligned}
|R_j \times K(z) - K \times R_j(z)| &\le \int |R_j(z-w)|\,|K(w)|\,|e^{\frac{i}{2}\mathrm{Im} z.\overline{w}} - e^{-\frac{i}{2}\mathrm{Im} z.\overline{w}}|\, dw \\
&\le \int |R_j(z-w)|\,|K(w)|\,|1 - e^{i\mathrm{Im} z.\overline{w}}|\, dw \\
&\le \int |R_j(z-w)|\,|z-w|\,|K(w)|\,|w|\, dw
\end{aligned}$$

shows that $[\mathbf{K}, \mathbf{R}_j]$ is twisted convolution with an L^1 kernel and hence bounded on $L^1(\mathbb{C}^n)$. A similar estimate holds for $[\mathbf{K}, \overline{\mathbf{R}}_j]$. This finishes the proof.

Next we proceed to prove that $L^{i\beta}$ is bounded on $\mathcal{H}^1$. We shall make use of the following result proved in [3].

THEOREM 2.2. *Let K be a function with compact support such that*

$$\int\limits_{|z|>2|w|} |K(z-w)-K(z)|\,dz \le A,$$

and assume either $\|K\times f\|_2 \le B\|f\|_2$ or $|\widehat{K}(\xi)| \le B$. Then $\mathbf{K}f = K\times f$ is a bounded operator from $\mathcal{H}^1$ into itself.

We mention that the above two results hold for the operators $f\mapsto f\times K$ as well.

THEOREM 2.3. *The operator $L^{i\beta}$ for $\beta\in\mathbb{R}$ is bounded from $\mathcal{H}^1$ to itself.*

Note that $L^{i\beta} = m(L)$ where $m(t) = t^{i\beta}$. Let ϕ be a C^∞ function on $\mathbb{R}$ such that $\operatorname{supp}\phi\subset\left(\frac{1}{2},2\right)$ and $\sum_{j=0}^{\infty}\phi(2^j t) = 1$ for every $t\ge 1$. Let $m_j(t) = \phi(2^{-j}t)m(t)$. Then we have $m(L) = \sum_{j=0}^{\infty} m_j(L)$. Let $k_j(z)$ be the kernel of $m_j(L)$. Then

$$k_j(z) = (2\pi)^{-n}\sum_{k=0}^{\infty} m_j(2k+n)\varphi_k(z).$$

We first obtain estimates for the kernels k_j away from the origin. We need the following proposition. Let Δ_- denote the backward finite difference operator defined by

$$\Delta_-\psi(k) = \psi(k)-\psi(k-1)$$

and let D stand for the operator $D\psi(k) = -(k\Delta_-\Delta_+\psi(k) + n\Delta_-\psi(k))$.

PROPOSITION 2.4. *If $M_\psi(z) = \sum_{k=0}^{\infty}\psi(k)\varphi_k(z)$ then*

$$\frac{1}{2}\,|z|^2 M_\psi(z) = \sum_{k=0}^{\infty} D\psi(k)\varphi_k(z) = M_{D\psi}(z).$$

Proof. See Lemma 2.4.2 in [10].

PROPOSITION 2.5. *Let $\alpha(z)$ be a C_{c}^∞ function such that $\alpha = 1$ in a neighborhood of the origin. Then there exists a $\delta>0$ such that*

$$\int\limits_{\mathbb{C}^n} |(1-\alpha(z))k_j(z)|\,dz \le C2^{-\delta j}, \qquad \int\limits_{\mathbb{C}^n} \big||z|(1-\alpha(z))k_j(z)\big|\,dz \le C2^{-\delta j}$$

with C independent of j.

Proof. A repeated application of Proposition 2.4 gives

$$\left(\tfrac{1}{2}|z|^2\right)^N k_j(z) = \sum_{k=0}^{\infty} D^N m_j(2k+n)\varphi_k(z).$$

Hence

$$|k_j(z)| \le C|z|^{-2N}\Big|\sum_{k=0}^{\infty} D^N m_j(2k+n)\varphi_k(z)\Big|.$$

Note that the function $m(t) = t^{i\beta}$ satisfies the estimates $|m^{(j)}(t)| \le C|t|^{-j}$ for every j. So $|D^N m_j(2k+n)| \le C_N(2k+n)^{-N}$ where C_N depends only on N. Hence using the Cauchy–Schwarz inequality and the orthogonality of φ_k (see [10]) we have

$$\int_{\mathbb{C}^n} |(1-\alpha(z))k_j(z)|\,dz \le C_N\Big(\sum_{2^{j-1}\le 2k+n\le 2^{j+1}} (2k+n)^{-2N}\|\varphi_k\|_2^2\Big)^{\frac{1}{2}}$$

where C_N depends only on N. Since $\|\varphi_k\|_2^2 \le Ck^{n-1}$ (see [10]), choosing N large enough we prove the first estimate in the proposition. The other estimate can also be proved similarly.

This takes care of the part at infinity. To deal with the local part we look at the operators $L^{-\varepsilon+i\beta}$ for $0<\varepsilon<1$. Let p_t and K_ε stand for the kernels of the operators e^{-tL} and $L^{-\varepsilon+i\beta}$ respectively. Then

$$p_t(z) = (2\pi)^{-n}\sum_{k=0}^{\infty} e^{-t(2k+n)}\varphi_k(z) = (4\pi)^{-n}(\sinh t)^{-n}e^{-\frac{1}{4}|z|^2\coth t}. \tag{2.1}$$

Using the the identity

$$L^{-\varepsilon+i\beta} = \frac{1}{\Gamma(\varepsilon-i\beta)}\int_0^\infty t^{\varepsilon-1-i\beta}e^{-tL}\,dt$$

we have

$$K_\varepsilon(z) = C\,\frac{1}{\Gamma(\varepsilon-i\beta)}\int_0^\infty t^{\varepsilon-1-i\beta}e^{-\frac{1}{4}|z|^2\coth t}(\sinh t)^{-n}\,dt.$$

An easy computation shows that

$$|\alpha(z)K_\varepsilon(z)| \le C|z|^{-2n}, \qquad |\nabla(\alpha K_\varepsilon)(z)| \le C|z|^{-2n-1}$$

with C independent of $0<\varepsilon<1$. These estimates imply that

$$\int_{|z|>2|w|} |(\alpha K_\varepsilon)(z-w) - (\alpha K_\varepsilon)(z)|\,dz \le A,$$

with A independent of ε.

Now we can complete the proof of Theorem 2.3. Let $K(z)$ stand for the kernel of the operator $L^{i\beta}$. From Proposition 2.5 it follows that both $(1-\alpha(z))K(z)$ and $|z|(1-\alpha(z))K(z)$ are in L^1 and so the operator $f \mapsto f \times (1-\alpha)K$ is bounded from $\mathcal{H}^1$ to $\mathcal{H}^1$ by Lemma 2.1. Note that the operators $f \mapsto f \times K_\varepsilon$ are all bounded on $L^2(\mathbb{C}^n)$, uniformly in $\varepsilon \ge 0$. Proceeding as in Proposition 2.5 we can easily show that the kernels $(1-\alpha(z))K_\varepsilon(z)$ are in $L^1(\mathbb{C}^n)$ with norms uniformly bounded in $0<\varepsilon\le 1$. Hence it follows that the operators $f \mapsto f \times \alpha K_\varepsilon$ are all bounded on $L^2(\mathbb{C}^n)$ uniformly in $0<\varepsilon\le 1$. Now using the above observations and Theorem 2.2 we see that the operators $f \mapsto f \times \alpha K_\varepsilon$ are all bounded from $\mathcal{H}^1$ to itself uniformly in $0 < \varepsilon < 1$. Letting $\varepsilon \to 0$ we have the boundedness of the operator $f \mapsto f \times \alpha K$. Putting together we get Theorem 2.3.

LEMMA 2.6. *The operator L^{-n} is bounded from $\mathcal{H}^1(\mathbb{C}^n)$ into $L^\infty(\mathbb{C}^n)$.*

Proof. The operator L^{-n} is given by twisted convolution with the kernel $K(z) = (2\pi)^{-n} \sum_{k=0}^\infty (2k+n)^{-n}\varphi_k(z)$ so that

$$L^{-n}f(z) = f \times K(z).$$

We shall show that all the atoms are mapped into a fixed ball in L^∞. Since this operator commutes with twisted translations we can assume that atoms are supported in a cube centered at the origin. So let f be such an atom. Then we have:

(i) $\operatorname{supp} f \subset Q(0,r)$ with $r \le 2\sqrt{\pi}$,
(ii) $\|f\|_\infty \le (2r)^{-2n}$,
(iii) $\int f = 0$.

We need to show that $|\sum (2k+n)^{-n} f \times \varphi_k(z)| \le C$. We consider two cases. First assume that $|z| \le 2r$. Now

$$\Big| \sum_{2k+n \ge r^{-2}} (2k+n)^{-n} f \times \varphi_k(z)\Big| \le \|f\|_2 \Big\| \sum_{2k+n\ge r^{-2}} (2k+n)^{-n}\varphi_k\Big\|_2. \tag{2.2}$$

Using the fact that φ_k's are orthogonal, $\|\varphi_k\|_2^2 \le C(2k+n)^{n-1}$ and $\|f\|_2 \le Cr^{-n}$, we see that the sum in (2.2) is bounded by a constant. Now, as the mean value of f is 0, we can write

$$\begin{aligned} f \times \varphi_k(z) = & \int_{\mathbb{C}^n} f(w)[\varphi_k(z-w) - \varphi_k(z)] e^{-\frac{i}{2}\operatorname{Im} z.\bar{w}}\, dw \\ & + \varphi_k(z) \int_{\mathbb{C}^n} f(w) [e^{-\frac{i}{2}\operatorname{Im} z.\bar{w}} - 1]\, dw. \end{aligned} \tag{2.3}$$

Writing $g(t) = \varphi_k(z - tw)$ we see that

$$\begin{aligned} |\varphi_k(z-w) - \varphi_k(z)| &= |g(1) - g(0)| = |g'(t)| \quad \text{for some } 0 < t < 1 \\ &\le |w| \sum_{j=0}^n \Big| \frac{\partial \varphi_k}{\partial z_j}(z - tw)\Big|. \end{aligned}$$

Using the formula (see [9])

$$\frac{d}{dt} L_k^{n-1}(t) = -L_{k-1}^n(t)$$

and the estimate

$$\sup_{t>0} |L_k^n(t) e^{-\frac{1}{2}t}| \le C(2k+n)^n$$

we see that

$$\begin{aligned} |\varphi_k(z-w) - \varphi_k(z)| &\le C(2k+n)^n |z - tw|\, |w| \quad \text{for some } 0 < t < 1 \\ &\le Cr^2 (2k+n)^n \quad \text{as } |z| \le 2r. \end{aligned}$$

So the first term in (2.3) is bounded by $Cr^2(2k+n)^n$. Since $|e^{-\frac{i}{2}\mathrm{Im} z.\overline{w}}-1| \le |z|\,|w| \le Cr^2$ and $|\varphi_k(z)| \le C(2k+n)^{n-1}$, we get the same estimate for the second term. Hence for $|z| \le 2r$ we have $|f \times \varphi_k(z)| \le Cr^2(2k+n)^n$, which shows that $\sum_{2k+n\le r^{-2}}(2k+n)^{-n} f \times \varphi_k(z)$ is bounded by a constant. This finishes the case when $|z| \le 2r$.

When $|z| \ge 2r$ we use the formula

$$L^{-n} = \frac{1}{\Gamma(n)} \int_0^\infty t^{n-1} e^{-tL}\, dt.$$

Hence the kernel K of L^{-n} can be written as

$$\begin{aligned} K(z) &= \frac{1}{\Gamma(n)} \int_0^\infty t^{n-1} \sum_{k=0}^\infty e^{-(2k+n)t} \varphi_k(z)\, dt \\ &= C_n \int_0^\infty t^{n-1} (\sinh t)^{-n} e^{-\frac{1}{4}|z|^2 \coth t}\, dt. \end{aligned}$$

Since $\sinh t \sim e^t$ and $\coth t \sim 1$ for t large, it is easy to see that the integral from 1 to ∞ defines a bounded function. So it is enough to consider the twisted convolution of f with $K_1(z) = \int_0^1 t^{n-1}(\sinh t)^{-n} e^{-\frac{1}{4}|z|^2 \coth t}\, dt$. Since the mean value of f is 0 we have

$$\begin{aligned} |f \times K_1(z)| \le {} & \int |f(w)[K_1(z-w) - K_1(z)]|\, dw \\ & + |K_1(z)| \int |f(w)[e^{-\frac{i}{2}\mathrm{Im} z.\overline{w}} - 1]|\, dw. \end{aligned}$$

Hence it is enough to show that $K_1(z-w) - K_1(z)$ and $K_1(z)[e^{-\frac{i}{2}\mathrm{Im} z.\overline{w}} - 1]$ are both bounded for $|z| \ge 2r$. Now

$$|K_1(z-w) - K_1(z)| = \Big| \int_0^1 t^{n-1} (\sinh t)^{-n} (e^{-\frac{1}{4}|z-w|^2 \coth t} - e^{-\frac{1}{4}|z|^2 \coth t})\, dt \Big|,$$

which is bounded by

$$C \int_0^1 |z - sw|\,|w| \Big(\int_0^1 t^{n-1} (\sinh t)^{-n} e^{-\frac{1}{4}|z-sw|^2 \coth t} \coth t\, dt \Big)\, ds.$$

Since $\sinh t \sim t$ and $\coth t \sim t^{-1}$ for t near 0 we have

$$\begin{aligned} |K_1(z-w) - K_1(z)| &\le \int_0^1 |z - sw|\,|w| \Big(\int_0^1 t^{-2} e^{-\frac{1}{4} \frac{|z-sw|^2}{t}}\, dt \Big)\, ds \\ &\le C \int_0^1 \frac{|z-sw|\,|w|}{|z-sw|^2}\, ds \\ &\le C \quad (\text{as } |z| \ge 2r \text{ and } |w| \le r). \end{aligned}$$

As for the other term,

$$|K_1(z)[e^{-\frac{i}{2}\mathrm{Im} z.\overline{w}} - 1]| \le \int_0^1 t^{-1} e^{-\frac{1}{4}\frac{|z|^2}{t}} |z|\,|w|\,dt$$
$$= \int_0^1 t^{-\frac{1}{2}} \left(\frac{|z|}{\sqrt{t}} e^{-\frac{1}{4}\frac{|z|^2}{t}} \right) |w|\,dt \le Cr \le 2C\sqrt{\pi}.$$

Note that the functions of the form $\sum_{j=0}^{N} c_j a_j$, where a_j are atoms, form a dense subset of $\mathcal{H}^1$. Since these functions are in L^2 the operator L^{-n} is well defined on them and they are mapped boundedly into L^∞ by the above estimates. Hence it follows that this operator has a unique bounded extension to the whole of $\mathcal{H}^1$. This finishes the proof of Lemma 2.6.

Now the proof of Theorem 1.2 is easy to complete. We consider the analytic family of operators defined by $T_\alpha f(z) = L^{-\alpha} f(z)$ for $-n \le \mathrm{Re}\,\alpha \le 0$. It can be checked that they form an admissible family of analytic operators in the sense of Stein [8]. When $\mathrm{Re}\,\alpha = 0$ we see that $T_\alpha : \mathcal{H}^1 \to \mathcal{H}^1$ is bounded, by Theorem 2.3. When $\mathrm{Re}\,\alpha = -n$ we see that $L^{-\alpha}$ is bounded from $\mathcal{H}^1$ to L^∞, by Theorem 2.3 and Lemma 2.6. Applying Stein's analytic interpolation theorem we finish the proof.

3. $\mathcal{H}^1$-L^p multipliers for L. In this section we prove Theorem 1.4. The proof of the case $q = 2$ is contained in [12].

Assume that m satisfies the estimate $|m(k)| \le C(1+k)^{-\frac{n}{2}}$. We need to show that $m(L)$ is bounded from $\mathcal{H}^1$ to L^2. Let f be an atom supported in a cube centered at the origin; we need to prove that

$$\sum_{k=0}^{\infty} (2k+n)^{-n} \|f \times \varphi_k\|_2^2 \le C.$$

This has been established in [12] (see Theorem 3.4). This inequality also follows from Theorem 1.2, for $q = 2$ and $\alpha = \frac{n}{2}$. The case $q = 1$ has been considered in [14].

So we assume that $1 < q < 2$. We closely follow the method in [4]. As in the previous section it is enough to prove that atoms are mapped into a ball. Let a be an atom in $\mathcal{H}^1$ supported in $Q(0,r)$. Let K be the kernel of $m(L)$. We decompose m using the partition of unity used in Theorem 2.3 so that $m = \sum_{j=0}^{\infty} m_j$ with m_j supported in $[2^{j-1}, 2^{j+1}]$. Let K_j be the kernel of the operator $m_j(L)$ so that

$$K_j(z) = (2\pi)^{-n} \sum m_j(2k+n)\varphi_k(z).$$

Note that

$$\|K_j\|_2^2 = \sum |m_j(2k+n)|^2 \|\varphi_k\|_2^2$$
$$\leq C \sum_{2^{j-1} \leq 2k+n \leq 2^{j+1}} (1+k)^{-n+2n(\frac{1}{q}-\frac{1}{2})} k^{n-1} \leq C 2^{2nj(\frac{1}{q}-\frac{1}{2})}.$$

Using Proposition 2.4, proceeding as in the proof of Theorem 2.3 and using the assumption on m we have

$$\int_{\mathbb{C}^n} |(1+2^j|z|^2)^l K_j(z)|^2 dz \leq C 2^{2jl} \sum_{2k+n \sim 2^j} (1+k)^{-n-2l+2n(\frac{1}{q}-\frac{1}{2})} k^{n-1}$$
$$\leq C 2^{2nj(\frac{1}{q}-\frac{1}{2})}$$

if $2l \leq M$ where M is the least even integer $> 2n\left(\frac{1}{q}-\frac{1}{2}\right)$. The above estimate implies

$$\Big(\int_{\mathbb{C}^n} |K_j(z)|^q \, dz\Big)^{\frac{1}{q}}$$
$$\leq \Big(\int_{\mathbb{C}^n} |K_j(z)|^2 (1+2^j|z|^2)^M \, dz\Big)^{\frac{1}{2}} \Big(\int_{\mathbb{C}^n} (1+2^j|z|^2)^{-\frac{Mq}{2-q}} \, dz\Big)^{\frac{2-q}{2q}} \leq C.$$

In the same way we have

$$\Big(\int_{|z|\geq 2r} |K_j(z)|^q \, dz\Big)^{\frac{1}{q}} \leq 2^{nj(\frac{1}{q}-\frac{1}{2})} \Big(\int_{|z|\geq 2r} (1+2^j|z|^2)^{-\frac{Mq}{2-q}} \, dz\Big)^{\frac{2-q}{2q}}$$
$$\leq C(2^{\frac{j}{2}} r)^{\frac{2n(2-q)}{2q}-M},$$

which shows that if $|w| \leq r$ then

$$\Big(\int_{|z|\geq 2r} |K_j(z-w)e^{-\frac{i}{2}\mathrm{Im} z.\overline{w}} - K_j(z)|^q \, dz\Big)^{\frac{1}{q}} \leq C(2^{\frac{j}{2}} r)^{\frac{2n(2-q)}{2q}-M}. \tag{3.1}$$

We need another estimate on the kernel. Write

$$K_j(z-w)e^{-\frac{i}{2}\mathrm{Im} z.\overline{w}} - K_j(z) \tag{3.2}$$
$$= (K_j(z-w) - K_j(z))e^{-\frac{i}{2}\mathrm{Im} z.\overline{w}} + K_j(z)(e^{-\frac{i}{2}\mathrm{Im} z.\overline{w}} - 1).$$

Therefore

$$\Big(\int_{|z|\geq 2r} |K_j(z-w)e^{-\frac{i}{2}\mathrm{Im} z.\overline{w}} - K_j(z)|^q \, dz\Big)^{\frac{1}{q}} \tag{3.3}$$
$$\leq |w| \Big(\Big\| \sum_{i=1}^n \Big| \frac{\partial K_j}{\partial z_j} \Big| \Big\|_q + \| |z| K_j \|_q \Big).$$

As $\frac{d}{dt} L_k^{n-1}(t) = -L_{k-1}^n(t)$ we see that it is enough to estimate the L^q norms of $B_1(z)$ and $B_2(z)$ (we suppress the dependence on j) where

$$B_1(z) = |w|\,|z|\,|K_j(z)|, \qquad B_2(z) = |w|\,|z|\,\Big|\sum m_j(2k+n)\varphi_{k-1}^n(z)\Big|.$$

Here $\varphi_k^n(z) = L_k^n\big(\frac{1}{2}|z|^2\big)e^{-\frac{1}{4}|z|^2}$.

First we take care of B_2. Note that

$$\begin{aligned}\int_{\mathbb{C}^n} |B_2(z)|^2\,dz &= |w|^2 \int_{\mathbb{C}^n} |z|^2\Big|\sum m_j(2k+n)\varphi_{k-1}^n(z)\Big|^2 dz \\ &= C|w|^2\int_0^\infty \Big|\sum m_j(2k+n)\varphi_{k-1}^n(s)\Big|^2 s^{2n+1}\,ds \\ &\le C|w|^2\sum |m_j(2k+n)|^2 k^n\end{aligned}$$

(as φ_k^n are orthogonal w.r.t. $s^{2n+1}ds$ and $\|\varphi_k^n\|_2^2 \le Ck^n$; see [10])

$$\begin{aligned}&\le C|w|^2 \sum_{k\sim 2^j} k^{-n+2n(\frac{1}{q}-\frac{1}{2})}k^n \\ &\le C(2^j r^2)2^{2nj(\frac{1}{q}-\frac{1}{2})}.\end{aligned}$$

Hence $\|B_2\|_2 \le C(2^{\frac{j}{2}}r)2^{nj(\frac{1}{q}-\frac{1}{2})}$. Now we can repeat the method used to estimate the kernels K_j above, as Proposition 2.4 is true with φ_k replaced by φ_k^n. We only have to replace n by $n+1$ in the definition of the operator D there (see [10]). Thus we have

$$\Big(\int_{\mathbb{C}^n} |B_2(z)|^q\,dz\Big)^{\frac{1}{q}} \le C2^{\frac{j}{2}}r.$$

To estimate B_1 we will use Weyl transform. Let us recall some basic facts about the Weyl transform.

Let $\pi(z)$ be the unitary operator defined on $L^2(\mathbb{R}^n)$ as follows:

$$\pi(z)\phi(\xi) = e^{i(x.\xi+\frac{1}{2}x.y)}\varphi(\xi+y), \qquad z = x+iy.$$

Then the Weyl transform is defined as the integrated representation of $\pi(z)$. That is, if f is in $L^1(\mathbb{C}^n)$ the *Weyl transform* $W(f)$ of f is defined to be the operator

$$W(f) = \int_{\mathbb{C}^n} f(z)\pi(z)\,dz.$$

Note that $W(f)$ is a bounded operator on $L^2(\mathbb{R}^n)$ and $\|W(f)\| \le \|f\|_1$. We also have the Plancherel theorem which states that if f is in $L^2(\mathbb{C}^n)$, then $W(f)$ is a Hilbert–Schmidt operator and

$$\|W(f)\|_{\mathrm{HS}} = c_n\|f\|_2.$$

We refer the reader to [10] for details.

We proceed with the estimation of B_1. Note that it is enough to estimate the L^q norm of $|w||z_l||K_j(z)|$ for $l = 1, \ldots, n$. As in the other case we will start with the L^2 norm. If $A_j = \frac{\partial}{\partial x_j} + x_j$ and $A_j^* = -\frac{\partial}{\partial x_j} + x_j$ then for a function f on $\mathbb{C}^n$ we have (see [10])

$$W(z_l f) = W(f)A_l^* - A_l^* W(f).$$

Hence by the Plancherel theorem for the Weyl transform

$$|w|^2 \int |z_l K_j(z)|^2\, dz = |w|^2 \|W(K_j)A_l^* - A_l^* W(K_j)\|_{\mathrm{HS}}^2. \tag{3.4}$$

To compute the Hilbert–Schmidt norm we use the Hermite basis. Let $\{\Phi_\alpha(x) : \alpha \in \mathbb{N}^n\}$ be the orthonormal basis of normalized Hermite functions for $L^2(\mathbb{R}^n)$. Let P_k be the projection defined on $L^2(\mathbb{R}^n)$ as follows:

$$P_k g = \sum_{|\alpha|=k} (g, \Phi_\alpha)\Phi_\alpha \quad \text{for } g \in L^2(\mathbb{R}^n).$$

Here $|\alpha| = \alpha_1 + \ldots + \alpha_n$. Then it is well known that

$$(2\pi)^{-n} W(\varphi_k) = P_k.$$

We also have the relations

$$A_l \Phi_\alpha = (2\alpha_l)^{\frac{1}{2}} \Phi_{\alpha - e_l}, \qquad A_l^* \Phi_\alpha = \{2(\alpha_l + 1)\}^{\frac{1}{2}} \Phi_{\alpha + e_l}.$$

We refer the reader to the monograph [10] for details. Using the above it is easy to see that (3.4) is bounded by $|w|^2 \sum k |m_j(2k+n)|^2 k^{n-1}$, which in turn, by the assumption on m, is bounded by $(2^j r^2) 2^{2nj(\frac{1}{q} - \frac{1}{2})}$. Proceeding as in the previous case we conclude that $\|B_1\|_q \le C 2^{\frac{j}{2}} r$. Putting together all the estimates we have

$$\begin{aligned}\Big(\int_{|z|\ge 2r} |K(z-w)e^{-\frac{i}{2}\mathrm{Im} z.\overline{w}} - K(z)|^q\Big)^{\frac{1}{q}} \\ \le C \sum \min(2^{\frac{j}{2}} r, (2^{\frac{j}{2}} r)^{2n(\frac{1}{q}-\frac{1}{2}) - M}) \le C.\end{aligned} \tag{3.5}$$

We just need one more inequality to complete the proof. If a is an atom in $\mathcal{H}^1$ supported in $Q(0,r)$ we need to prove that

$$\sum (2k+n)^{-n+2n(\frac{1}{q}-\frac{1}{2})} \|a \times \varphi_k\|_2^2 \le C r^{-2n(\frac{1}{q}-\frac{1}{2})} \tag{3.6}$$

for $1 \le q \le 2$ with C independent of a. We start with the case $q = 1$. We get

$$\sum \|a \times \varphi_k\|_2^2 = \|a\|_2^2 \le C r^{-2n},$$

which is the required estimate. When $q = 2$ we need to show that

$$\sum (2k+n)^{-n} \|a \times \varphi_k\|_2^2 \le C,$$

which has already been proved. Hence we obtain (3.6) from interpolation. Now we can finish the proof of Theorem 1.4 as in [4]. We omit the details.

Note that an application of Theorem 1.4 to the multiplier $m(k) = k^{-a}$ gives a different proof of Theorem 1.2 for $q < \infty$, though the $q = \infty$ case and Theorem 2.3 do not follow from this.

4. Hardy–Littlewood inequalities. In this section we prove Theorem 1.5. Consider the operator $T_t(\alpha)$ which is defined by

$$T_t(\alpha)f(z) = \sum_{k=0}^{\infty}(2k+n)^{-\alpha}e^{i(2k+n)it} f \times \varphi_k(z).$$

The operator $T_t(\alpha)$ has the following kernel:

$$K^t(z) = \sum(2k+n)^{-\alpha}e^{(2k+n)it}\varphi_k(z),$$

which can be written as

$$K^t(z) = \frac{1}{\Gamma(\alpha)}\int_0^\infty \lambda^{\alpha-1}K_t^*(z,\lambda)\,d\lambda$$

where we have set

$$K_t^*(z,\lambda) = \sum e^{-(2k+n)(\lambda-it)}\varphi_k(z).$$

In view of (2.1), a simple calculation shows that

$$K_t^*(z,\lambda) = c(\sinh(\lambda-it))^{-n}e^{-A_t(z,\lambda)}e^{iB_t(z,\lambda)}$$

where

$$A_t(z,\lambda) = \frac{1}{4}\,\frac{\sinh 2\lambda}{\cosh 2\lambda - \cos 2t}\,|z|^2, \qquad B_t(z,\lambda) = \frac{1}{4}\,\frac{\sin 2t}{\cosh 2\lambda - \cos 2t}\,|z|^2.$$

We also note that

$$|\sinh(\lambda-it)|^2 = c(\sinh^2\lambda + \sin^2 t) = \cosh 2\lambda - \cos 2t.$$

Observe that, in view of Stein's analytic interpolation theorem, it is enough to prove Theorem 1.5 for $\alpha = n$. So we choose $\alpha = n$. It is easy to check that the integral

$$\int_1^\infty \lambda^{n-1}K_t^*(z,\lambda)\,d\lambda$$

defines an L^1 kernel. Assuming $t \le 1$ consider the integral

$$\int_t^1 \lambda^{n-1}K_t^*(z,\lambda)\,d\lambda.$$

Note that $\sinh^2\lambda + \sin^2 t \sim \lambda^2$, because $t \le \lambda$. So the modulus of the above integral is bounded by a constant times

$$\int_0^1 \lambda^{n-1}\lambda^{-n}e^{-\delta\frac{|z|^2}{\lambda}}\,d\lambda$$

for some $\delta > 0$, which is easily seen to be integrable on $\mathbb{C}^n$. So we can very well assume that $T_t(\alpha)$ is given by the kernel

$$\int_0^t \lambda^{n-1} K_t^*(z,\lambda)\, d\lambda.$$

A simple calculation shows that the above is not integrable unless $\sin t = 0$. We shall denote $T_t(n)$ by T_t. To prove that T_t is bounded from $\mathcal{H}^1$ to L^1 we only need to show the following.

PROPOSITION 4.1. *There exists a constant C such that $\int |T_t f(z)|\, dz \le C$ whenever f is an atom.*

In proving this we closely follow [5] and [13]. We will make use of the following estimates on the kernel. Let $K_t(z,\lambda) = \{\sinh(\lambda - it)\}^{-n} e^{-A_t(z,\lambda)}$.

LEMMA 4.2. *There exists a constant C such that*

$$\Big|\int_0^t \lambda^{n-1} K_t(z,\lambda) e^{iB_t(z,\lambda)}\, d\lambda\Big| \le C|z|^{-2n},$$

$$\Big|\int_0^t \lambda^{n-1} \frac{\partial}{\partial z_j} K_t(z,\lambda) e^{iB_t(z,\lambda)}\, d\lambda\Big| \le C|z|^{-2n-1},$$

$$\Big|\int_0^t \lambda^{n-1} K_t(z,\lambda) \frac{\partial}{\partial z_j} e^{iB_t(z,\lambda)}\, d\lambda\Big| \le C|z|^{-2n+1},$$

$$\Big|\int_0^t \lambda^{n-1} K_t(z,\lambda) \lambda \partial_\lambda e^{iB_t(z,\lambda)}\, d\lambda\Big| \le C|z|^{-2n-2}.$$

Proof. Recall that we are assuming $0 < t < 1$ and $\lambda \le t$. Then $\sinh^2\lambda + \sin^2 t \sim t^2$. So

$$\Big|\int_0^t \lambda^{n-1} K_t(z,\lambda) e^{iB_t(z,\lambda)}\, d\lambda\Big| \le C\int_0^1 \lambda^{n-1} t^{-n} e^{-\frac{\lambda}{4t^2}|z|^2}\, d\lambda \le C|z|^{-2n},$$

which proves the first estimate. Now $\frac{\partial}{\partial z_j} K_t(z,\lambda)$ brings in a factor of $\sinh 2\lambda$, which accounts for an extra $|z|^{-1}$. Here we are using the fact that $\sinh 2\lambda \sim \lambda$ for λ near the origin. The other estimates can be proved similarly.

Once the above estimates are established we may follow the method in [13] to complete the proof of Proposition 4.1 and hence of Theorem 1.5. As the proof is very similar we omit the details. Corollary 1.6 can be obtained with the help of Theorem 1.5 in a routine fashion as in [13].

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