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ENDOMORPHISM RINGS OF REGULAR MODULES OVER WILD HEREDITARY ALGEBRAS

BY

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Dedicated to Raymundo Bautista on the occasion of his sixtieth birthday

Abstract. Let H be a connected wild hereditary path algebra. We prove that if Z is a quasi-simple regular brick, and $[r]Z$ indecomposable regular of quasi-length r and with quasi-top Z, then $\operatorname{rad}^r \operatorname{End}_H([r]Z) = 0$.

Let H be a finite-dimensional hereditary algebra over some algebraically closed field K. We will assume that H is basic and connected, hence H is isomorphic to the path algebra $K\mathcal{Q}$ of some finite connected quiver $\mathcal{Q}$ without oriented cycles, and the category A-mod of finite-dimensional left A-modules can be identified with the category $\operatorname{rep}_K \mathcal{Q}$ of finite-dimensional K-linear representations of the quiver $\mathcal{Q}$. By n we always denote the number of vertices of the quiver $\mathcal{Q}$, which coincides with the number of isomorphism classes of simple H-modules.

We additionally assume that H is wild hereditary. This means that the quiver $\mathcal{Q}$ is neither of Dynkin nor of Euclidean type, or equivalently that every finite-dimensional K-algebra B is isomorphic to the endomorphism ring of some module $X \in H$-mod. In particular each finite-dimensional local K-algebra B is the endomorphism ring of some finite-dimensional indecomposable H-module X.

We denote by $\Gamma(H)$ the Auslander–Reiten quiver of H and by $\tau = \tau_H$ the Auslander–Reiten translation in $\Gamma(H)$ as well as in H-mod. Since H is hereditary, the functor $\tau = \mathrm{D}\operatorname{Ext}^1_H(-, H)$, with $\mathrm{D} = \operatorname{Hom}_K(-, K)$, is full and left exact on H-mod. Since H is wild hereditary, the Auslander–Reiten quiver $\Gamma(H)$ contains besides the preprojective component $\mathcal{P}$ and the preinjective component $\mathcal{I}$ infinitely many regular components $\mathcal{C}$, all of them of type $\mathbb{Z}A_\infty$.

If X is an indecomposable regular module, i.e. the isomorphism class $[X]$ of X is a vertex of some regular component $\mathcal{C}$ of $\Gamma(H)$, then X is called *quasi-simple* if the Auslander–Reiten sequence starting or ending in X has

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indecomposable middle term. Clearly X is quasi-simple if and only if $\tau^i X$ is quasi-simple for each $i \in \mathbb{Z}$. If X is an arbitrary indecomposable regular module, then there exist a chain of irreducible monomorphisms

$$Y = Y(1) \xrightarrow{\varepsilon_1} Y(2) \xrightarrow{\varepsilon_2} \ldots \xrightarrow{\varepsilon_{r-1}} Y(r) = X$$

and a chain of irreducible epimorphisms

$$X = [r]Z \xrightarrow{\pi_{r-1}} [r-1]Z \xrightarrow{\pi_{r-2}} \ldots \xrightarrow{\pi_1} [1]Z = Z$$

where Y and Z are quasi-simple and are called the *quasi-socle*, respectively the *quasi-top*, of X. Moreover $Y = \tau^{r-1}Z$, and the natural number r is called the *quasi-length* of X.

Since $\dim \operatorname{Hom}_H(Z, \tau^{r-1}Z)$ grows exponentially with r, and

$$\pi_{r-1} \ldots \pi_1 \operatorname{Hom}_H(Z, \tau^{r-1}Z)\varepsilon_1 \ldots \varepsilon_{r-1}$$

is a subspace of $\operatorname{End}_H([r]Z)$, even of $\operatorname{rad}\operatorname{End}_H([r]Z)$, it follows that also $\dim \operatorname{rad}\operatorname{End}_H([r]Z)$ grows (at least) exponentially with r. We are concerned with the structure of the endomorphism rings $\operatorname{End}_H([r]Z)$.

For a finite-dimensional K-algebra B the radical $\operatorname{rad} B$ is nilpotent. Denote by

$$e(\operatorname{rad} B) = \min\{m \in \mathbb{N} \mid (\operatorname{rad} B)^m = 0\}$$

the exponent (of nilpotency) of $\operatorname{rad} B$.

For a quasi-simple regular H-module Z we will prove the following fact.

Proposition. *The set* $\{e(\operatorname{rad}\operatorname{End}_H([r]Z)) \mid r \in \mathbb{N}\}$ *is unbounded.*

Since τ is an equivalence on the category of regular H-modules, we have

$$\operatorname{End}_H([r]Z) \cong \operatorname{End}_H(\tau^i[r]Z)$$

for all integers i. Consequently, the function $e(\operatorname{rad}\operatorname{End}_H(-))$ is constant on the τ-orbits of indecomposable regular H-modules.

It seems to be hopeless to find a close relation between the natural numbers r and $e(\operatorname{rad}\operatorname{End}_H([r]Z))$ for an arbitrary quasi-simple H-module Z, as in the case of tame hereditary algebras A, since in that case the category A-reg of regular A-modules is abelian and all indecomposable regular modules are uniserial in A-reg. Indeed, if H is wild hereditary, for each finite-dimensional local K-algebra B there even exists a quasi-simple module $Z(B)$ with $B \cong \operatorname{End}_H(Z(B))$. Moreover, for each indecomposable regular H-module R the radical $\operatorname{rad}\operatorname{End}_H(R)$ is contained in the infinite radical $\operatorname{rad}^\infty(H\text{-mod})$ of the category H-mod, whereas the Auslander–Reiten quiver visualises the factor category $H\text{-mod}/\operatorname{rad}^\infty(H\text{-mod})$.

On the other hand, if the quasi-simple H-module Z is a brick, i.e. $\operatorname{End}_H(Z) = K$, then the irreducible epimorphisms $\pi_{r-1}\colon [r]Z \to [r-1]Z$ as well as the irreducible monomorphisms $\varepsilon_{r-1}\colon Z(r-1) \to Z(r)$ induce surjective ring homomorphisms

$$\widehat{\pi}_{r-1}\colon \operatorname{End}_H([r]Z) \to \operatorname{End}_H([r-1]Z),$$

respectively

$$\widehat{\varepsilon}_{r-1}\colon \operatorname{End}_H(Z(r)) \to \operatorname{End}_H(Z(r-1)),$$

therefore inductive arguments apply. Besides this technical advantage, bricks are of interest by themselves, for example for geometrical reasons.

For a quasi-simple brick Z, we have to distinguish two cases: either $\operatorname{Ext}^1_H(Z,Z) \neq 0$ or $\operatorname{Ext}^1_H(Z,Z) = 0$. In the latter case, there exists a natural number t with $1 \leq t \leq n-2$ and $\operatorname{Ext}^1_H([i]Z,[i]Z) = 0$ for $1 \leq i \leq t$, but $\operatorname{Ext}^1_H([t+1]Z,[t+1]Z) \neq 0$ and $\operatorname{End}_H([t+1]Z) = K$. Conversely, if for some $t > 0$ the module $[t+1]Z$ is a brick, then $[t]Z$ is a brick without self-extensions. In particular, Z is a brick without self-extensions in this case.

Theorem. *Let H be connected wild hereditary and Z be a quasi-simple regular brick.*

(a) *If Z has self-extensions, then for $r > 1$,*

$$\begin{aligned} e(\operatorname{rad}\operatorname{End}_H([r-1]Z)) &\leq e(\operatorname{rad}\operatorname{End}_H([r]Z)) \\ &\leq \min\{r, 1 + e(\operatorname{rad}\operatorname{End}_H([r-1]Z))\}. \end{aligned}$$

(b) *If $\operatorname{Ext}^1_H(Z,Z) = 0$ and $t \geq 1$ is maximal with $\operatorname{Ext}^1_H([t]Z,[t]Z) = 0$, then for $i \geq 2$ the following holds, where $[-]$ denotes the Gauss bracket:*

$$\begin{aligned} e(\operatorname{rad}\operatorname{End}_H([t+i-1]Z)) &\leq e(\operatorname{rad}\operatorname{End}_H([t+i]Z)) \\ &\leq 1 + \min\{[i/2], e(\operatorname{rad}\operatorname{End}_H([t+i-2]Z))\}. \end{aligned}$$

(c) *In both cases the dimensions of the left socle as well as the right socle of $\operatorname{End}_H([r]Z)$ grow exponentially with r.*

An indecomposable regular module E is called *elementary* if it has no nontrivial filtrations with regular subquotients, or equivalently, if $\tau^m E$ has no nontrivial regular factor modules for $m \gg 0$. Elementary modules are quasi-simple bricks, therefore the Theorem applies. It will be shown in the last section that the results of the Theorem can be improved for elementary modules.

For unexplained terminology we refer to [1] and [13], for basic results on wild hereditary algebras to [7]. Morphisms between modules will be written opposite to the scalars. If U_i, $1 \leq i \leq 3$, are modules in H-mod and $f_i\colon U_i \to U_{i+1}$ are morphisms for $i = 1, 2$, the composition will be therefore written as $f_1 f_2\colon U_1 \to U_3$. Compositions of ring homomorphisms will always be written from right to left.

1. Proof of the Proposition. Let H be connected wild hereditary and $M = [r]Z = X(r)$ an indecomposable regular H-module of quasi-length $r > 1$. Basic for the considerations in this paper is the following result, due to Ringel [12]:

Let $X = X(1) \xrightarrow{\varepsilon_1} \ldots \xrightarrow{\varepsilon_{r-1}} X(r)$ a chain of irreducible monomorphisms and $[r]Z \xrightarrow{\pi_{r-1}} \ldots \xrightarrow{\pi_1} Z = [1]Z$ be a chain of irreducible epimorphisms.

LEMMA 1.1. *For* $1 \le i < r$ *one has*:

(a) *The cokernel of the map* $\varepsilon_i \ldots \varepsilon_{r-1}$: $X(i) \to X(r)$ *is* $\tau^{-i}X(r-i) = [r-i]Z$. *The kernel of the map* $\pi_{r-1} \ldots \pi_i$: $[r]Z \to [i]Z$ *is* $\tau^i[r-i]Z = X(r-i)$.

(b) *If* Y_1 *is indecomposable with* $Y_1 \not\cong \tau^{-i}X(r-1-i)$ *for* $0 \le i < r-1$, *then each morphism* f: $X(r-1) \to Y_1$ *factorises through* ε_{r-1}.

If Y_2 *is indecomposable with* $Y_2 \not\cong \tau^i[r-1-i]Z$ *for* $0 \le i < r-1$, *then each morphism* g: $Y_2 \to [r-1]Z$ *factorises through* π_{r-1}.

If Z is a quasi-simple regular H-module contained in the regular component $\mathcal{C}$ and r is a natural number, the mesh-complete full subquiver $\mathcal{W}([r]Z)$ of $\mathcal{C}$ defined by the vertices $[\tau^i[j]Z]$ with $i \ge 0$, $j \ge 1$ and $i + j \le r$ is called the *wing* of length r and top $[r]Z$. The quasi-simple modules $\tau^i Z$ with $0 \le i \le r-1$ are called the *basis* of the wing $\mathcal{W}([r]Z)$. If one chooses irreducible epimorphisms $\pi_i^0 = \pi_i$: $[i+1]Z \to [i]Z$ for $1 \le i \le r-1$, one can always find a consistent choice for the irreducible maps inside the wing $\mathcal{W}([r]Z)$. This means that there are irreducible monomorphisms

$$\varepsilon_j^i\colon \tau^i[j]Z \to \tau^{i-1}[j+1]Z, \qquad i \ge 1,\ i+j \le r,$$

and irreducible epimorphisms

$$\pi_{j-1}^i\colon \tau^i[j]Z \to \tau^i[j-1]Z, \qquad j > 1,\ i+j \le r,$$

such that

$$\varepsilon_1^i \pi_1^{i-1} = 0, \qquad \varepsilon_j^i \pi_j^{i-1} = \pi_{j-1}^i \varepsilon_{j-1}^i \qquad \text{for } j > 1.$$

We will always assume that the irreducible maps in a sufficiently large wing are chosen consistently.

Let Z be a quasi-simple regular H-module, $i, j, r \in \mathbb{N}$, and choose the irreducible maps in the wing $\mathcal{W}([j+ri]Z)$ consistently.

For $1 \le t \le r$ denote by $\pi_{(t)}$: $[j + (r-t+1)i]Z \to [j + (r-t)i]Z$ the chain of irreducible epimorphisms, and by $\varepsilon_{(t)}$: $\tau^{i(r-t+1)}[j + (t-1)i]Z \to \tau^{i(r-t)}[j + it]Z$ the corresponding chain of irreducible monomorphisms, both at the borders of the wing $\mathcal{W}([j+ri]Z)$. With these notations the following holds.

LEMMA 1.2. *Let α_t: $\tau^{i(t-1)}[j]Z \to \tau^{it}[j]Z$ be morphisms for $1 \le t \le r$. Then there exist morphisms β_t: $[j+(r-1)i]Z \to \tau^i[j+(r-1)i]Z$ such that*

$$\pi_{(1)} \ldots \pi_{(r)}\alpha_1 \ldots \alpha_r \varepsilon_{(1)} \ldots \varepsilon_{(r)} = \prod_{t=1}^{r}(\pi_{(1)}\beta_t\varepsilon_{(r)}).$$

Proof. The proof is by induction on r; the case $r = 1$ is trivial. Assume that the statement holds for $r-1 \ge 1$.

Let $\varepsilon'_{(t)}$: $\tau^{i(r-t)}[j+(t-1)i]Z \to \tau^{i(r-t-1)}[j+it]Z$ be a chain of irreducible monomorphisms for $1 \le t \le r-1$ and choose

$$\gamma_t\colon [j+(r-2)i]Z \to \tau^i[j+(r-2)i]Z$$

such that $\pi_{(2)} \ldots \pi_{(r)}\alpha_1 \ldots \alpha_{r-1}\varepsilon'_{(1)} \ldots \varepsilon'_{(r-1)} = \prod_{t=1}^{r-1}(\pi_{(2)}\gamma_t\varepsilon'_{(r-1)})$. From Lemma 1.1 we infer that there are morphisms

$$\beta_t\colon [j+(r-1)i]Z \to \tau^i[j+(r-1)i]Z$$

for $1 \le t \le r$ such that $\alpha_r\varepsilon_{(1)} \ldots \varepsilon_{(r-1)} = \varepsilon'_{(1)} \ldots \varepsilon'_{(r-1)}\beta_r$ and $\pi_{(2)}\gamma_t = \beta_t\pi'_{(2)}$ for $1 \le t < r$, where $\pi'_{(2)}$: $\tau^i[j+(r-1)i]Z \to \tau^i[j+(r-2)i]Z$ is a chain of irreducible epimorphisms.

Since $\pi'_{(2)}\varepsilon'_{(r-1)} = \varepsilon_{(r)}\pi_{(1)}$ by the consistent choice of the irreducible maps, we get

$$\begin{aligned}
&\pi_{(1)} \ldots \pi_{(r)}\alpha_1 \ldots \alpha_r\varepsilon_{(1)} \ldots \varepsilon_{(r)} \\
&\qquad = \pi_{(1)} \ldots \pi_{(r)}\alpha_1 \ldots \alpha_{r-1}\varepsilon'_{(1)} \ldots \varepsilon'_{(r-1)}\beta_r\varepsilon_{(r)} \\
&\qquad = \pi_{(1)}\Big(\prod_{t=1}^{r-1}\pi_{(2)}\gamma_t\varepsilon'_{(r-1)}\Big)\beta_r\varepsilon_{(r)} \\
&\qquad = \pi_{(1)}\Big(\prod_{t=1}^{r-1}\beta_t\varepsilon_{(r)}\pi_{(1)}\Big)\beta_r\varepsilon_{(r)} = \prod_{t=1}^{r}(\pi_{(1)}\beta_t\varepsilon_{(r)}).
\end{aligned}$$

We now prove the Proposition: Let $\mathcal{C}$ be a regular component in $\Gamma(H)$ and Z a quasi-simple module in $\mathcal{C}$. By Lukas [11, 2.3] there exists a monomorphism α_1: $Z \to \tau^i Z$ for some $i > 0$. Consequently, all the morphisms $\alpha_t = \tau^{i(t-1)}\alpha_1$: $\tau^{i(t-1)}Z \to \tau^{it}Z$ are injective for $t \ge 0$, since τ is a left exact functor.

We apply Lemma 1.2 for $j = 1$ and i as above. For any $r \in \mathbb{N}$ the map $0 \ne \pi_{(1)} \ldots \pi_{(r)}\alpha_1 \ldots \alpha_r\varepsilon_{(1)} \ldots \varepsilon_{(r)} = \prod_{t=1}^{r}(\pi_{(1)}\beta_t\varepsilon_{(r)})$ is contained in $\operatorname{rad}^r \operatorname{End}_H([1+ri]Z) \setminus \{0\}$. Hence we have

$$e(\operatorname{rad}\operatorname{End}_H([1+ri]Z)) > r.$$

2. Preliminaries. Let $H = KQ$ be a connected wild hereditary algebra and Z be a quasi-simple regular H-module. Then $\operatorname{Hom}_H(Z, \tau^i Z) \ne 0$

for $i \gg 0$ [2, 3.1]. Moreover, $\dim \mathrm{Hom}_H(Z, \tau^i Z)$ grows exponentially (see for example [7, 10.6]), whereas $\mathrm{Hom}_H(Z, \tau^{-i} Z) = 0$ for $i \gg 0$ ([5]), but frequently $\mathrm{Hom}_H(Z, \tau^- Z) \neq 0$ [9, 3.1]. For regular components containing bricks, the following lemma [6, 1.2] holds.

LEMMA 2.1. *If Z is a quasi-simple brick, then* $\mathrm{Hom}_H(Z, \tau^{-i} Z) = 0$ *for all* $i > 0$.

Notice that there always exist infinitely many regular components in $\Gamma(H)$, containing quasi-simple bricks. If the number n of pairwise nonisomorphic representatives of simple H-modules is greater than 2, there are even infinitely many components containing quasi-simple regular modules without self-extensions [14]. They are bricks by [3, 4.1], since H is hereditary. As an immediate consequence of Lemma 2.1 we get the following:

Let Z be a quasi-simple regular brick. Then:

(a) $\mathrm{Hom}_H(Z, \tau^{-i} Z(j)) = 0$ for all $i, j \geq 1$.
(b) $\mathrm{Hom}_H(\tau^i [j] Z, Z) = 0$ for all $i, j \geq 1$.
(c) $\mathrm{Hom}_H(\tau^i [j] Z, \tau^{-i'} Z(j')) = 0$ for all $i, j, i', j' \geq 1$.

For a quasi-simple brick Z, we consider the wing $\mathcal{W}([r]Z)$ and denote the consistently chosen irreducible maps inside the wing as in Section 1. Then one has:

LEMMA 2.2. *Let Z be a quasi-simple brick and $r > 1$. Then*:

(a) $\mathrm{Hom}_H([r]Z, Z) = K\pi_{r-1} \dots \pi_1$ *and* $\pi_{r-1} \dots \pi_1 \notin \mathrm{rad}^r([r]Z, Z)$.
(b) $\mathrm{Hom}_H(\tau^{r-1} Z, [r]Z) = K\varepsilon_1^{r-1} \dots \varepsilon_{r-1}^1$ *and* $\varepsilon_1^{r-1} \dots \varepsilon_{r-1}^1$ *is not contained in* $\mathrm{rad}^r(\tau^{r-1} Z, [r]Z)$.

Proof. (a) Consider the short exact sequence

$$0 \to \tau[r-1]Z \xrightarrow{\varepsilon_{r-1}^1} [r]Z \xrightarrow{\pi_{r-1}\dots\pi_1} Z \to 0.$$

Since $\mathrm{Hom}_H(\tau[r-1]Z, Z) = 0$, by Lemma 2.1, we get an isomorphism

$$(\pi_{r-1} \dots \pi_1, Z)\colon \mathrm{Hom}_H([r]Z, Z) \to \mathrm{Hom}_H(Z, Z) = K,$$

hence $\mathrm{Hom}_H([r]Z, Z) = K\pi_{r-1} \dots \pi_1$. Since the maps $\pi_{r-1}, \dots, \pi_1$ are irreducible maps on a sectional path, the composition $\pi_{r-1} \dots \pi_1 \notin \mathrm{rad}^r([r]Z, Z)$ (see [4]).

Dually one shows part (b).

LEMMA 2.3. *Let Z be a quasi-simple regular brick*, $r > 1$ *an integer and* $1 \leq i < r$. *Then*:

(a) *The map* $\pi = \pi_{r-1} \dots \pi_i\colon [r]Z \to [i]Z$ *induces an epimorphism of rings*

$$\widehat{\pi}\colon \mathrm{End}_H([r]Z) \to \mathrm{End}_H([i]Z)$$

such that $\pi\widehat{\pi}(f) = f\pi$ *for* $f \in \mathrm{End}_H([r]Z)$.

(b) *The map* $\varepsilon = \varepsilon_i^{r-i} \dots \varepsilon_{r-1}^1\colon \tau^{r-i}[i]Z \to [r]Z$ *induces an epimorphism of rings*

$$\widehat{\varepsilon}\colon \operatorname{End}_H([r]Z) \to \operatorname{End}_H(\tau^{r-i}[i]Z)$$

with $\widehat{\varepsilon}(f)\varepsilon = \varepsilon f$ *for* $f \in \operatorname{End}_H([r]Z)$.

(c) $\operatorname{rad}\operatorname{End}_H([r-1]Z) = \{f \mid f = \pi_{r-1}\alpha\varepsilon_{r-1}^1$ for $\alpha\colon [r-1]Z \to \tau[r-1]Z\}$.

(d) *For* $f = \pi_{r-1}\alpha\varepsilon_{r-1}^1 \in \operatorname{rad}\operatorname{End}_H([r]Z)$ *one has*:

$$\widehat{\varepsilon}_{r-1}^1(f) = \varepsilon_{r-1}^1\pi_{r-1}\alpha \in \operatorname{rad}\operatorname{End}_H(\tau^{r-1}Z),$$
$$\widehat{\pi}_{r-1}(f) = \alpha\varepsilon_{r-1}^1\pi_{r-1} \in \operatorname{rad}\operatorname{End}_H([r-1]Z).$$

Proof. (a,b) Let $\varepsilon' = \varepsilon_{r-i}^i \dots \varepsilon_{r-1}^1\colon \tau^i[r-i]Z \to [r]Z$ and consider the short exact sequence

$$0 \to \tau^i[r-i]Z \xrightarrow{\varepsilon'} [r]Z \xrightarrow{\pi} [i]Z \to 0.$$

Since $\operatorname{Hom}_H(\tau^i[r-i]Z, [i]Z) = 0$ by Lemma 2.1, the maps ε' and π induce homomorphisms $\widehat{\varepsilon}'\colon \operatorname{End}_H([r]Z) \to \operatorname{End}_H(\tau^i[r-i]Z)$ and $\widehat{\pi}\colon \operatorname{End}_H([r]Z) \to \operatorname{End}_H([i]Z)$ such that $\varepsilon' f = \widehat{\varepsilon}'(f)\varepsilon'$, respectively $f\pi = \pi\widehat{\pi}(f)$. It is easy to check that $\widehat{\varepsilon}'$ and $\widehat{\pi}$ are ring homomorphisms. They are surjective by Lemma 1.1(b).

(c) If $f = \pi_{r-1}\alpha\varepsilon_{r-1}^1$, then $\operatorname{Ker} f$ contains $\operatorname{Ker}\pi_{r-1} \neq 0$, hence $f \in \operatorname{rad}\operatorname{End}_H([r]Z)$.

Let $\varepsilon^*\colon \tau^{r-1}Z \to [r]Z$, respectively $\pi^*\colon [r]Z \to Z$, be the canonical maps. Since the ideal $\operatorname{rad}\operatorname{End}_H([r]Z)$ is nilpotent and Z, respectively $\tau^{r-1}Z$, are bricks, we get $\widehat{\varepsilon}^*(f) = 0$ and $\widehat{\pi}^*(f) = 0$ for $f \in \operatorname{rad}\operatorname{End}_H([r]Z)$. But $\widehat{\varepsilon}^*(f) = 0$ means that f factorises through π_{r-1}. Since $\widehat{\pi}^*(f) = 0$, the map f has a factorisation through ε_{r-1}^1, hence $f = \pi_{r-1}\alpha\varepsilon_{r-1}^1$ for some $\alpha\colon [r-1]Z \to \tau[r-1]Z$.

The proof of (d) is straightforward.

Remark. The surjectivity of $\widehat{\pi}_{r-1}$ implies

$$e(\operatorname{rad}\operatorname{End}_H([s+1]Z)) \geq e(\operatorname{rad}\operatorname{End}_H([s]Z))$$

for all $s \geq 1$.

Lemma 2.4. *Let Z be a quasi-simple regular brick and $i > 1$. For $\alpha\colon [i]Z \to \tau[i]Z$ there exist unique morphisms $\beta\colon \tau[i-1]Z \to \tau^2[i-1]Z$ and $\gamma\colon [i-1]Z \to \tau[i-1]Z$ such that $\varepsilon_{i-1}^1\alpha = \beta\varepsilon_{i-1}^2$ and $\alpha\pi_{i-1}^1 = \pi_{i-1}\gamma$.*

Proof. Since $\alpha \in \operatorname{rad}^\infty([i]Z, \tau[i]Z)$, also the composition

$$\varepsilon_{i-1}^1\alpha\pi_{i-1}^1 \dots \pi_1^1 \in \operatorname{Hom}_H(\tau[i-1]Z, \tau Z)$$

belongs to the infinite radical. But $\operatorname{rad}^\infty(\tau[i-1]Z, \tau Z) = 0$ by Lemma 2.2. Hence $\varepsilon_{i-1}^1\alpha$ factorises through $\varepsilon_{i-1}^2 = \operatorname{Ker}\pi_{i-1}^1 \dots \pi_1^1$, that is, $\varepsilon_{i-1}^1\alpha = \beta\varepsilon_{i-1}^2$. The induced map $\beta\colon \tau[i-1]Z \to \tau^2[i-1]Z$ is unique, since ε_{i-1}^2 is injective. The existence of γ is shown dually.

Lemma 2.5. *Let Z be a quasi-simple brick and $i > 1$. If*

$$0 \to \tau[i-1]Z \xrightarrow{(\varepsilon^1_{i-1},\pi^1_{i-2})} [i]Z \oplus \tau[i-2]Z \xrightarrow{(\pi_{i-1},\varepsilon^1_{i-2})^t} [i-1]Z \to 0$$

is the Auslander–Reiten sequence ending in $[i-1]Z$, then

$$\widehat{\pi}^1_{i-2}\widehat{\varepsilon}^1_{i-1} = \widehat{\varepsilon}^1_{i-2}\widehat{\pi}_{i-1}\colon \mathrm{End}_H([i]Z) \to \mathrm{End}_H([i-2]Z).$$

Proof. (i) Take first $f \in \operatorname{rad}\mathrm{End}_H([i]Z)$. By Lemma 2.3 there exists a unique morphism $\alpha \in \mathrm{Hom}_H([i-1]Z, \tau[i-1]Z)$ with $f = \pi_{i-1}\alpha\varepsilon^1_{i-1}$ and

$$\widehat{\varepsilon}^1_{i-1}(f) = \varepsilon^1_{i-1}\pi_{i-1}\alpha = \pi^1_{i-2}\varepsilon^1_{i-2}\alpha = \pi_{i-2}\beta\varepsilon^2_{i-2}$$

for some $\beta\colon \tau[i-2]Z \to \tau^2[i-2]Z$. Hence $\widehat{\pi}^1_{i-2}(\widehat{\varepsilon}^1_{i-1}(f)) = \beta\varepsilon^2_{i-2}\pi^1_{i-2}$. Similarly there exists a morphism $\gamma\colon [i-2]Z \to \tau[i-2]Z$ with $\widehat{\pi}_{i-1}(f) = \alpha\pi^1_{i-2}\varepsilon^1_{i-2} = \pi_{i-2}\gamma\varepsilon^1_{i-2}$. Therefore $\widehat{\varepsilon}^1_{i-2}(\widehat{\pi}_{i-1}(f)) = \varepsilon^1_{i-2}\pi_{i-2}\gamma$.

Since $\varepsilon^1_{i-2}(\pi_{i-2}\gamma) = \varepsilon^1_{i-2}(\alpha\pi^1_{i-2}) = \beta\varepsilon^2_{i-2}\pi^1_{i-2}$, the maps $\widehat{\pi}^1_{i-2}\widehat{\varepsilon}^1_{i-1}$ and $\widehat{\varepsilon}^1_{i-2}\widehat{\pi}_{i-1}$ coincide on $\operatorname{rad}\mathrm{End}_H([i]Z)$.

(ii) For any $g \in \mathrm{End}_H([i]Z)$ there exist $x \in K$ and $f \in \operatorname{rad}\mathrm{End}_H([i]Z)$ with $g = x1_{[i]Z} + f$. Since $\widehat{\pi}^1_{i-2}\widehat{\varepsilon}^1_{i-1}(1_{[i]Z}) = \widehat{\varepsilon}^1_{i-2}\widehat{\pi}_{i-1}(1_{[i]Z}) = 1_{\tau[i-2]Z}$, we get $\widehat{\pi}^1_{i-2}\widehat{\varepsilon}^1_{i-1}(g) = \widehat{\varepsilon}^1_{i-2}\widehat{\pi}_{i-1}(g)$ for all $g \in \mathrm{End}_H([i]Z)$, by (i).

Let Z be a quasi-simple brick, $r > 1$, and $\mathcal{W}([r]Z)$ the wing with top $[r]Z$ and a consistent choice of the irreducible maps in the wing. If $[X(i)]$ is one of the vertices of the wing $\mathcal{W}([r]Z)$, all the possible surjective ring homomorphisms

$$\mathrm{End}_H([r]Z) \to \mathrm{End}_H(X(i))$$

which are compositions of $\widehat{\varepsilon}$'s and $\widehat{\pi}$'s, where the ε's and π's are irreducible maps, coincide by Lemma 2.5. We denote this epimorphism by

$$\varrho^{[r]Z}_{X(i)}\colon \mathrm{End}_H([r]Z) \to \mathrm{End}_H(X(i)).$$

If $[U(j)]$ is a vertex in $\mathcal{W}(X(i))$, then clearly $\varrho^{X(i)}_{U(j)}\varrho^{[r]Z}_{X(i)} = \varrho^{[r]Z}_{U(j)}$.

If $\mathcal{C}$ is a regular component in $\Gamma(H)$, containing (quasi-simple) bricks, something like the converse of Lemma 1.2 holds, which is an explicit description of $\operatorname{rad}^t\mathrm{End}_H([r]Z)$.

Proposition 2.6. *Let Z be a quasi-simple regular brick*, $r > 1$ *and* $1 \le t < r$. *Then* $\operatorname{rad}^t\mathrm{End}_H([r]Z)$ *is generated as a vector space by the set*

$$\{f \in \mathrm{End}_H([r]Z) \mid f = \pi_{r-1}\ldots\pi_{r-t}\beta_1\ldots\beta_t\varepsilon^t_{r-t}\ldots\varepsilon^1_{r-1}\},$$

where $\beta_i\colon \tau^{i-1}[r-t]Z \to \tau^i[r-t]Z$. Moreover, if $f = f_1\ldots f_t$ with $f_i \in$ $\operatorname{rad}\mathrm{End}_H([r]Z)$, *then* $\varrho^{[r]Z}_{\tau^{i-1}[r-t+1]Z}(f_i) = \pi^{i-1}_{r-t}\beta_i\varepsilon^i_{r-t}$.

Proof. By Lemma 1.2 there exist morphisms α_i: $[r-1]Z \to \tau[r-1]Z$ such that

$$\pi_{r-1}\dots\pi_{r-t}\beta_1\dots\beta_t\varepsilon^t_{r-t}\dots\varepsilon^1_{r-1} = \prod_{i=1}^{t}(\pi_{r-1}\alpha_i\varepsilon^1_{r-1}) \in \operatorname{rad}^t \operatorname{End}_H([r]Z).$$

Conversely, let $f_i = \pi_{r-1}\alpha_i\varepsilon^1_{r-1} \in \operatorname{rad}\operatorname{End}_H([r]Z)$ for $1 \le i \le t$ and define β_i by $\varrho^{[r]Z}_{\tau^{i-1}[r-t+1]Z}(f_i) = \pi^{i-1}_{r-t}\beta_i\varepsilon^i_{r-t}$. We will show by induction on t that

$$f_1\dots f_t = \pi_{r-1}\dots\pi_{r-t}\beta_1\dots\beta_t\varepsilon^t_{r-t}\dots\varepsilon^1_{r-1}.$$

The statement is trivial for $t = 1$. By induction, we can assume that $\prod_{i=1}^{t-1}\widehat{\pi}_{r-1}(f_i) = \pi_{r-2}\dots\pi_{r-t}\beta_1\dots\beta_{t-1}\varepsilon^{t-1}_{r-t}\dots\varepsilon^1_{r-2}$.

Since $(\prod_{i=1}^{t-1} f_i)\pi_{r-1} = \pi_{r-1}\prod_{i=1}^{t-1}\widehat{\pi}_{r-1}(f_i)$, we get

$$\begin{aligned}\prod_{i=1}^{t} f_i &= \Big(\prod_{i=1}^{t-1} f_i\Big)\pi_{r-1}\alpha_t\varepsilon^1_{r-1}\\ &= \pi_{r-1}(\pi_{r-2}\dots\pi_{r-t}\beta_1\dots\beta_{t-1}\varepsilon^{t-1}_{r-t}\dots\varepsilon^1_{r-2})\alpha_t\varepsilon^1_{r-1}.\end{aligned}$$

By Lemma 2.4 there exists β_t: $\tau^{t-1}[r-t]Z \to \tau^t[r-t]Z$ with

$$\varepsilon^{t-1}_{r-t}\dots\varepsilon^1_{r-2}\alpha_t = \beta_t\varepsilon^t_{r-t}\dots\varepsilon^2_{r-2}.$$

Moreover, it can be checked easily that $\pi^{t-1}_{r-1}\beta_t\varepsilon^t_{r-t} = \varrho^{[r]Z}_{\tau^{t-1}[r-t+1]Z}(f_t)$, which shows the inductive step.

LEMMA 2.7. *Let Z be a regular quasi-simple brick, $s \in \mathbb{N}$, and assume that, for some $m \in \mathbb{N}$, we have*

$$\prod_{i=0}^{m-1}\operatorname{Hom}_H(\tau^i[s]Z, \tau^{i+1}[s]Z) = 0.$$

Then each morphism $f = \prod_{i=1}^{m}\beta_i$ such that $\beta_i \in \operatorname{Hom}_H(\tau^{i-1}[s+1]Z, \tau^i[s+1]Z)$ has a factorisation $f = \pi_s\dots\pi_1\gamma\varepsilon^{m+s}_1\dots\varepsilon^{m+1}_s$ for some γ: $Z \to \tau^{m+s}Z$.

Proof. By Lemma 2.4 there exist morphisms γ_i: $\tau^{i-1}[s]Z \to \tau^i[s]Z$ with $\beta_i\pi^i_s = \pi^{i-1}_s\gamma_i$. Hence we get $\beta_1\dots\beta_m\pi^m_s = \beta_1\dots\beta_{m-1}\pi^{m-1}_s\gamma_m = \dots = \pi_s\gamma_1\dots\gamma_m = 0$. Therefore $\beta_1\dots\beta_m$ factorises through $\varepsilon^{m+s}_1\dots\varepsilon^{m+1}_s = \operatorname{Ker}\pi^m_s$. Similarly one shows $\varepsilon^1_s\beta_1\dots\beta_m = 0$. Consequently, $\beta_1\dots\beta_m$ also factorises through $\operatorname{Coker}\varepsilon^1_s = \pi_s\dots\pi_1$.

PROPOSITION 2.8. *If Z is a quasi-simple brick, then*

$$e(\operatorname{rad}\operatorname{End}_H([r]Z)) \le e(\operatorname{rad}\operatorname{End}_H([r+1]Z)) \le 1 + e(\operatorname{rad}\operatorname{End}_H([r]Z)).$$

Proof. Let $e = e(\operatorname{rad}\operatorname{End}_H([r]Z))$ and take

$$f_1,\dots,f_{e+1} \in \operatorname{rad}\operatorname{End}_H([r+1]Z).$$

Since $g = f_2 \dots f_{e+1} \in \operatorname{rad}^e \operatorname{End}_H([r+1]Z)$, we get $\widehat{\varepsilon}_r^1(g) = 0$. Therefore we have the following commutative diagram, since Z is a brick:

$$\begin{tikzcd} 0 \arrow[r] & \tau[r]Z \arrow[r, "\varepsilon_r^1"] \arrow[d, "0"] & {[r+1]Z} \arrow[r, "\pi"] \arrow[d, "g"] & Z \arrow[r] \arrow[d, "0"] & 0 \\ 0 \arrow[r] & \tau[r]Z \arrow[r, "\varepsilon_r^1"] & {[r+1]Z} \arrow[r, "\pi"] & Z \arrow[r] & 0 \end{tikzcd}$$

where $\pi = \pi_r \dots \pi_1$. Consequently, there exists β: $Z \to \tau[r]Z$ with $g = \pi\beta\varepsilon_r^1$. But $f_1 = \pi_r\alpha_1\varepsilon_r^1$ for some α: $[r]Z \to \tau[r]Z$. Hence $f_1 g = \pi_r\alpha_1\varepsilon_r^1\pi\beta\varepsilon_r^1 = 0$, since $\varepsilon_r^1\pi = 0$.

3. Proof of the Theorem. Let $\mathcal{C}$ be a regular component in the Auslander–Reiten quiver $\Gamma(H)$ and assume Z is a quasi-simple brick in $\mathcal{C}$.

(a) Suppose that $\operatorname{Ext}_H^1(Z,Z) \cong \operatorname{D}\operatorname{Hom}_H(Z,\tau Z) \neq 0$. In this case $\operatorname{Hom}_H(Z,\tau Z) \to \operatorname{rad}\operatorname{End}_H([2]Z)$, given by $\alpha \mapsto \pi_1\alpha\varepsilon_1^1$, defines an isomorphism. Since $\varepsilon_1^1\pi_1 = 0$, we get $e(\operatorname{rad}\operatorname{End}_H([2]Z)) = 2$. The proof of part (a) of the Theorem now follows from Proposition 2.8.

(b) Suppose that $\operatorname{Ext}_H^1([i]Z,[i]Z) = 0$ for $1 \le i \le t$, but

$$\operatorname{Ext}_H^1([t+1]Z,[t+1]Z) \cong \operatorname{D}\operatorname{Hom}_H([t+1]Z,\tau[t+1]Z) \neq 0.$$

We know that $[t+1]Z$ is a brick [6]. By Lemma 2.7, $\operatorname{Hom}_H(Z,\tau^{t+1}Z) \to \operatorname{Hom}_H([t+1]Z,\tau[t+1]Z)$ given by

$$\gamma \mapsto \pi_r \dots \pi_1\gamma\varepsilon_1^{r+1} \dots \varepsilon_r^2$$

is an isomorphism. Moreover $\operatorname{rad}\operatorname{End}_H([t+2]Z) \cong \operatorname{Hom}_H([t+1]Z,\tau[t+1]Z)$, by Lemma 2.3(c).

Let $c([j]Z) = \min\{r \in \mathbb{N} \mid \prod_{i=0}^{r-1}\operatorname{Hom}_H(\tau^i[j]Z,\tau^{i+1}[j]Z) = 0\}$ if this set in nonempty, and $c([j]Z) = \infty$ otherwise.

For $j \le t$ one has $c([j]Z) = 1$, whereas $c([t+1]Z) \ge 2$. We will first show that

$$c([t+i-1]Z) \le c([t+i]Z) \le \min\{i+1, 1 + c([t+i-1]Z)\}$$

for $i \ge 0$, where we define $c([0]Z) = 1$. Since $\operatorname{Hom}_H(\tau^m Z,\tau^{m+1}Z) = 0$ for all integers m, Lemma 2.4 implies that $\beta\pi_{r-1}^1 \dots \pi_1^1 = 0 = \varepsilon_1^{r-1} \dots \varepsilon_{r-1}^1\beta$ for any $r > 1$ and $\beta \in \operatorname{Hom}_H([r]Z,\tau[r]Z)$.

From this and Lemma 2.7, the proof of the first inequality follows.

The second inequality is shown by induction. It is clear for $i = 0$. Let $i > 0$ and assume inductively that

$$1 \le c = c([t+i]Z) \le \min\{i+1, 1 + c([t+i-1]Z)\}.$$

Take β_j: $\tau^{j-1}[t+i+1]Z \to \tau^j[t+i+1]Z$ for $1 \le j \le c+1$. By Lemma 2.7 we get $\prod_{j=1}^{c}\beta_j = \pi_{t+i} \dots \pi_1\gamma\varepsilon_1^{c+t+i} \dots \varepsilon_{t+i}^{c+1}$ with $\gamma \in \operatorname{Hom}_H(Z,\tau^{c+t+i}Z)$.

Consequently, $\prod_{j=1}^{c+1} \beta_j = \pi_{t+i} \ldots \pi_1 \gamma \varepsilon_1^{c+t+i} \ldots \varepsilon_{t+i}^{c+1} \beta_{c+1} = 0$, which shows the inductive step.

From $\operatorname{rad}\operatorname{End}_H([t+1]Z) = 0$ one gets $\operatorname{rad}^r \operatorname{End}_H([t+r]Z) = 0$, by Proposition 2.8. Let $1 \leq r \leq i$. Proposition 2.6 implies $\operatorname{rad}^r \operatorname{End}_H([t+i]Z) = 0$ if and only if $c([t+i-r]Z) \leq r$. Consequently,

$$e(\operatorname{rad}\operatorname{End}_H([t+i]Z)) = \min\{r \mid c([t+i-r]Z) \leq r\}.$$

Since $c([t+i-r]Z) \leq \min\{1+i-r, 1+c([t+i-r-1]Z)\}$, the assertion follows. Indeed, if $e(\operatorname{rad}\operatorname{End}_H([t+i]Z)) = e'$, then $c([t+i-e'+1]Z) \in \{e', e'+1\}$. For $c([t+i-e'+1]Z) = e'$, one gets $e(\operatorname{rad}\operatorname{End}_H([t+i+1]Z)) = e'$, hence $e(\operatorname{rad}\operatorname{End}_H([t+i+2]Z)) \leq e'+1$. From $c([t+i-e'+1]Z) = e'+1$ we immediately deduce $e(\operatorname{rad}\operatorname{End}_H([t+i+2]Z)) = e(\operatorname{rad}\operatorname{End}_H([t+i+1]Z)) = e'+1$.

The condition $e(\operatorname{rad}\operatorname{End}_H([t+i]Z)) \leq [i/2]+1$ is shown similarly.

(c) From $\operatorname{rad}\operatorname{End}_H([r]Z) = \pi_{r-1}\operatorname{Hom}_H([r-1]Z, \tau[r-1]Z)\varepsilon_{r-1}^1$ it follows that the left socle of $\operatorname{End}_H([r]Z)$, which is

$$\{f \in \operatorname{End}_H([r]Z) \mid \operatorname{rad}\operatorname{End}_H([r]Z)f = 0\},$$

contains $\pi_{r-1} \ldots \pi_1 \operatorname{Hom}_H(Z, \tau[r-1]Z)\varepsilon_{r-1}^1 = \operatorname{Ker}\widehat{\varepsilon}_{r-1}^1$.

Since $\dim\operatorname{Hom}_H(Z, \tau^{r-1}Z)$ and $\dim_K[r]Z$ grow exponentially with r, so does $\dim\operatorname{Hom}_H(Z, \tau[r-1]Z)$ and hence $\dim\operatorname{soc}_{\operatorname{End}_H([r]Z)}\operatorname{End}_H([r]Z)$. The conclusion for the right socle is obtained analogously.

4. Elementary modules. A quasi-simple regular H-module Z is called *elementary* if it satisfies the following equivalent conditions [9]:

(a) $\tau^m Z$ has no nontrivial regular factor modules for $m \gg 0$.
(b) $\tau^{-m} Z$ has no nontrivial regular submodule for $m \gg 0$.
(c) If R is regular and $f\colon Z \to R$ is nonzero, then $\operatorname{Ker} f$ is preprojective.
(d) If R is regular and $g\colon R \to Z$ is nonzero, then $\operatorname{Coker} g$ is preinjective.

It follows from the definition that elementary modules are bricks. Therefore the Theorem holds for elementary modules. For elementary modules with self-extensions one gets:

PROPOSITION 4.1. *Let Z be an elementary H-module with self-extensions. Then, for all $r \geq 1$:*

(a) $e(\operatorname{rad}\operatorname{End}_H([r]Z)) = r$.
(b) $\operatorname{soc}_{\operatorname{End}_H([r]Z)}\operatorname{End}_H([r]Z) = \operatorname{Ker}\widehat{\varepsilon}_{r-1}^1$ *and* $\operatorname{soc}\operatorname{End}_H([r]Z)_{\operatorname{End}_H([r]Z)} = \operatorname{Ker}\widehat{\pi}_{r-1}$.

Proof. (a) Take $0 \neq \alpha_i \in \operatorname{Hom}_H(\tau^{i-1}Z, \tau^i Z)$ for $1 \leq i \leq r-1$. By (c) and (d) above we get $\prod_{i=1}^{r-1}\alpha_i \neq 0$, and consequently $\operatorname{rad}^{r-1}\operatorname{End}_H([r]Z) \neq 0$, by Proposition 2.6. Hence $r-1 < e(\operatorname{rad}\operatorname{End}_H([r]Z)) \leq r$.

(b) It was already shown in Section 3 that $\operatorname{soc}_{\operatorname{End}_H([r]Z)} \operatorname{End}_H([r]Z) \supset \operatorname{Ker} \widehat{\varepsilon}_{r-1}^1$.

For $r > 1$, take $g \in \operatorname{soc}_{\operatorname{End}_H([r]Z)} \operatorname{End}_H([r]Z) \subset \operatorname{rad} \operatorname{End}_H([r]Z)$. Then $\widehat{\varepsilon}_{r-1}^1(g) = \pi_{r-2}^1 \gamma \varepsilon_{r-2}^2$ for some $\gamma\colon \tau[r-2]Z \to \tau^2[r-2]Z$, by Lemma 2.3(c). By Proposition 2.6, $fg = 0$ for all $f \in \operatorname{rad} \operatorname{End}_H([r]Z)$ is equivalent to $\beta\gamma = 0$ for all morphisms $\beta \in \operatorname{Hom}_H([r-2]Z, \tau[r-2]Z)$.

So assume $\beta\gamma = 0$ for all $\beta \in \operatorname{Hom}_H([r-2]Z, \tau[r-2]Z)$. Let $i \le r-2$ be maximal with $(\tau^{r-i-1}[i]Z)\varepsilon_i^{r-i-1} \ldots \varepsilon_{r-3}^2 \subset \operatorname{Ker} \gamma$ and suppose $i < r-2$. Then $\varepsilon_{i+1}^{r-i-2} \ldots \varepsilon_{r-3}^2 \gamma$ is nonzero and has kernel $(\tau^{r-i-1}[i]Z)\varepsilon_i^{r-i-1} \oplus P$, where P is preprojective, since

$$\tau^{r-i-2} Z \cong \tau^{r-i-2}[i+1]Z/(\tau^{r-i-1}[i]Z)\varepsilon_i^{r-i-1}$$

is elementary and ε_i^{r-i-1} is irreducible.

Take any nonzero morphism $h'\colon Z \to \tau^{r-i-2}Z$. Consider the composition $0 \neq h^* = \pi_{r-3} \ldots \pi_1 h'\colon [r-2]Z \to \tau^{r-i-2}Z$ and let

$$h\colon [r-2]Z \to \tau^{r-i-2}[i+1]Z$$

be the lifting of h^*, due to Lemma 1.1. Since the image of h is regular and not contained in $(\tau^{r-i-1}[i]Z)\varepsilon_i^{r-i-1}$, we get $(h\varepsilon_{i+1}^{r-i-2} \ldots \varepsilon_{r-3}^1)\gamma \neq 0$, a contradiction.

Consequently, $i = r-2$, which means $\gamma = 0$, hence $\widehat{\varepsilon}_{r-1}^1(g) = 0$. Therefore

$$\operatorname{soc}_{\operatorname{End}_H([r]Z)} \operatorname{End}_H([r]Z) = \operatorname{Ker} \widehat{\varepsilon}_{r-1}^1.$$

Analogously one proves the assertion for the right socle of $\operatorname{End}_H([r]Z)$.

Remark. Since

$$\begin{aligned} \operatorname{End}_H([i+1]Z)/\operatorname{Ker} \widehat{\varepsilon}_i^1 &\cong \operatorname{End}_H(\tau[i]Z), \\ \operatorname{End}_H([i+1]Z)/\operatorname{Ker} \widehat{\pi}_i &\cong \operatorname{End}_H([i]Z), \\ \operatorname{End}_H([i]Z) &\cong \operatorname{End}_H(\tau[i]Z) \end{aligned}$$

for all i, we see that the algebras $\operatorname{End}_H([r]Z)/\operatorname{soc}_{\operatorname{End}_H([r]Z)} \operatorname{End}_H([r]Z)$ and $\operatorname{End}_H([r]Z)/\operatorname{soc} \operatorname{End}_H([r]Z)_{\operatorname{End}_H([r]Z)}$ are isomorphic, for an elementary module with self-extensions.

Examples. Consider the following quivers:

$$\begin{tikzcd} \underset{1}{\bullet} & \underset{2}{\bullet} \arrow[l, shift right=2] \arrow[l] \arrow[l, shift left=2] \end{tikzcd} \qquad\qquad \begin{tikzcd} \underset{1}{\bullet} & \underset{2}{\bullet} \arrow[l, shift right=1] \arrow[l, shift left=1] & \underset{3}{\bullet} \arrow[l] \end{tikzcd}$$

$$\mathcal{Q} \qquad\qquad\qquad \mathcal{Q}'$$

For numerical data about modules in $K\mathcal{Q}'$-mod see [12].

(a) Let $H' = K\mathcal{Q}'$ and Z the indecomposable regular H'-module with $\underline{\dim}\, Z = (1, 1, 0)$. Clearly Z is elementary with $\dim \operatorname{Ext}_{H'}^1(Z, Z) = 1$, hence $e(\operatorname{rad} \operatorname{End}_{H'}([r]Z)) = r$ for $r > 1$, by Proposition 4.1.

Since $\dim \operatorname{Hom}_{H'}(Z, \tau Z) = 1$, and each nonzero morphism $f\colon Z \to \tau Z$ is a monomorphism, Proposition 2.6 implies

$$\dim \operatorname{rad}^{r-1} \operatorname{End}_{H'}([r]Z) = 1.$$

(b) Let X be the quasi-simple regular H'-module with $\underline{\dim}\, X = (15, 10, 4)$. The projective generator in the right perpendicular category $X^{\perp}$ is $P(3) \oplus \tau^{-2}P(3)$, and $X^{\perp} \cong H$-mod, where $H = K\mathcal{Q}$. The H'-module $[2]X = Z'$ is in $X^{\perp}$, and $\underline{\dim}_H\, Z' = (4, 2)$. The module $T = P(3) \oplus \tau^{-2}P(3) \oplus X$ is a tilting module in H'-mod and $\operatorname{End}_{H'}(T) \cong H[Z']$, where $H[Z']$ denotes the one-point extension of H by the module Z'. As an H-module, Z' is elementary [10, 3.1]. It even satisfies a stronger condition of being *orbital elementary* in the sense of [8]. Consequently, by [8, Section 5] for all $r \geq 1$ we get

$$\dim \Big(\prod_{i=1}^{r-1} \operatorname{Hom}_H(\tau^{i-1} Z', \tau^i Z') \Big) = (\dim \operatorname{Hom}_H(Z', \tau Z'))^{r-1}.$$

Since $\dim \operatorname{Hom}_H(Z', \tau Z') = 1 - q_H(\underline{\dim}\, Z') = 5$, where q_H denotes the Tits form of the algebra H, we have $\dim(\prod_{i=1}^{r-1} \operatorname{Hom}_H(\tau^{i-1} Z', \tau^i Z')) = 5^{r-1}$. Consequenty, in H-mod we have

$$\dim \operatorname{rad}^{r-1} \operatorname{End}_H([r]Z') = 5^{r-1}.$$

Clearly, again $e(\operatorname{rad} \operatorname{End}_H([r]Z')) = r$.

Since $\underline{\dim}\, \tau_{H'} X = (5, 4, 0)$, all proper submodules of $\tau_{H'} X$ are preprojective in H'-mod. Consequently, $\tau_{H'} X$ and hence all $\tau^i_{H'} X$ are elementary H'-modules without self-extensions. The module $[2]X$ is a brick with self-extensions.

If Z is an elementary H-module without self-extensions and $t \leq n-2$ is maximal such that $\operatorname{Ext}^1_H([t]Z, [t]Z) = 0$, then $\operatorname{Hom}_H(Z, \tau^{t+1} Z) \neq 0$.

Let $0 \neq \alpha_i\colon \tau^{(i-1)(t+1)} Z \to \tau^{i(t+1)} Z$ for $i \geq 1$. Since Z is elementary, for all $r \geq 1$ the composition $\alpha_1 \dots \alpha_r\colon Z \to \tau^{r(t+1)} Z$ is also nonzero. It follows from Lemma 1.2 that then $e(\operatorname{rad} \operatorname{End}_H([1 + r(t+1)]Z)) \geq r + 1$. In the special case $t = 1$ we therefore get

COROLLARY 4.2 *Let Z be an elementary H-module with* $\operatorname{Ext}^1_H(Z, Z) = 0$ *but* $\operatorname{Ext}^1_H([2]Z, [2]Z) \neq 0$. *Then* $e(\operatorname{rad} \operatorname{End}_H([1+i]Z)) = [i/2] + 1$.

Proof. Let first $i = 2r$ be even. From the formula above and the Theorem we get $r+1 \leq e(\operatorname{rad} \operatorname{End}_H([1+2r]Z)) \leq r+1$, hence $e(\operatorname{rad} \operatorname{End}_H([1+i]Z)) = [i/2]+1$ for i even. If $i = 2r+1$ is odd, then $r+1 = e(\operatorname{rad} \operatorname{End}_H([1+2r]Z)) \leq e(\operatorname{rad} \operatorname{End}_H([1+2r+1]Z)) \leq [(2r+1)/2]+1 = r+1$, hence the claim follows.

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