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INFINITELY MANY POSITIVE SOLUTIONS FOR THE NEUMANN PROBLEM INVOLVING THE p-LAPLACIAN

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Abstract. We present two results on existence of infinitely many positive solutions to the Neumann problem

$$\begin{cases} -\Delta_p u + \lambda(x)|u|^{p-2}u = \mu f(x,u) & \text{in } \Omega, \\ \partial u/\partial \nu = 0 & \text{on } \partial\Omega, \end{cases}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded open set with sufficiently smooth boundary $\partial\Omega$, ν is the outer unit normal vector to $\partial\Omega$, $p > 1$, $\mu > 0$, $\lambda \in L^\infty(\Omega)$ with $\operatorname{ess\,inf}_{x\in\Omega} \lambda(x) > 0$ and $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function. Our results ensure the existence of a sequence of nonzero and nonnegative weak solutions to the above problem.

1. Introduction. In this paper, we consider the problem

$$(\mathrm{P}_\mu) \qquad \begin{cases} -\Delta_p u + \lambda(x)|u|^{p-2}u = \mu f(x,u) & \text{in } \Omega, \\ \partial u/\partial \nu = 0 & \text{on } \partial\Omega, \end{cases}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded open set with sufficiently smooth boundary $\partial\Omega$, ν is the outer unit normal vector to $\partial\Omega$, $p > 1$, Δ_p is the p-Laplacian operator, that is, $\Delta_p u = \operatorname{div}(|\nabla u|^{p-2}\nabla u)$, $\mu > 0$, $\lambda \in L^\infty(\Omega)$ with $\operatorname{ess\,inf}_{x\in\Omega} \lambda(x) > 0$ and $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function. We are interested in the existence of a sequence of nonzero and nonnegative weak solutions of (P_μ) in $W^{1,p}(\Omega)$. The space $W^{1,p}(\Omega)$ is endowed with the norm

$$\|u\| = \Big(\int_\Omega \lambda(x)|u|^p\,dx + \int_\Omega |\nabla u|^p\,dx\Big)^{1/p}$$

equivalent to the usual one.

A weak solution of (P_μ) is any $u \in W^{1,p}(\Omega)$ such that

$$\int_\Omega \lambda(x)|u(x)|^{p-2}u(x)v(x)\,dx + \int_\Omega |\nabla u(x)|^{p-2}\nabla u(x)\nabla v(x)\,dx - \mu \int_\Omega f(x,u(x))v(x)\,dx = 0,$$

for each $v \in W^{1,p}(\Omega)$.

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To obtain our multiplicity results, we employ the same methods that allowed us to find infinitely many small positive solution for the analogous Dirichlet problem [2]. We follow the general approach applied by Ricceri in [10], that is, to look for solutions to problem (P_μ) as local minima of the underlying energy functional.

While for the Dirichlet problem the existence of infinitely many solutions has been widely studied (see for instance [2, 3, 4, 6–9, 12]), actually, the only paper that deals with the existence of infinitely many solutions to the Neumann problem is [11]. There Ricceri applies the variational principle of [10].

For the reader's convenience, we quote below his result about the existence of a sequence of small weak solutions.

Theorem. A ([11, Theorem 2]). *Let* $f, g : \mathbb{R} \to \mathbb{R}$ *be two continuous functions with* $\sup_{\xi\in\mathbb{R}} \int_0^\xi g(t)\,dt \le 0$, *let* $\alpha, \beta \in L^1(\Omega)$ *with* $\min\{\alpha(x), \beta(x)\} \ge 0$ *a.e. in* Ω, *let* $\lambda \in L^\infty(\Omega)$ *with* $\operatorname{ess\,inf}_{x\in\Omega} \lambda(x) > 0$, *and let* $p > N$. *Moreover, assume that there are sequences* $\{r_n\}$ *in* $\mathbb{R}_+$ *with* $\lim_{n\to\infty} r_n = 0$ *and* $\{\xi_n\}$ *in* $\mathbb{R}$ *such that, for each* $n \in \mathbb{N}$,

$$\frac{\int_\Omega \lambda(x)\,dx}{p}\,|\xi_n|^p - \int_\Omega \beta(x) \int_0^{\xi_n} g(t)\,dt\,dx < r_n \tag{1.1}$$

and

$$\int_0^{\xi_n} f(t)\,dt = \sup_{|\xi|\le c(pr_n)^{1/p}} \int_0^{\xi} f(t)\,dt, \tag{1.2}$$

where

$$c = \sup_{u\in W^{1,p}(\Omega)\setminus\{0\}} \frac{\sup_{x\in\Omega} |u(x)|}{(\int_\Omega |\nabla u(x)|^p\,dx + \int_\Omega \lambda(x)|u(x)|^p\,dx)^{1/p}}.$$

Finally, assume that

$$\limsup_{\xi\to 0} \frac{\int_\Omega \alpha(x)\,dx \int_0^\xi f(t)\,dt + \int_\Omega \beta(x)\,dx \int_0^\xi g(t)\,dt}{|\xi|^p} > \frac{\int_\Omega \lambda(x)\,dx}{p}. \tag{1.3}$$

Then the problem

$$\begin{cases} -\Delta_p u + \lambda(x)|u|^{p-2}u = \alpha(x)f(u) + \beta(x)g(u) & \text{in } \Omega, \\ \partial u/\partial\nu = 0 & \text{on } \partial\Omega, \end{cases} \tag{P}$$

admits a sequence of nonzero weak solutions which strongly converges to zero in $W^{1,p}(\Omega)$.

From (1.1) it follows $|\xi_n| < c(pr_n)^{1/p}$. In Remark 2 of [11], Ricceri asked if the conclusion of Theorem A would hold when, instead of (1.1) and (1.2), it is supposed that there is a sequence $\{b_n\}$ in $\mathbb{R}_+$, convergent to zero, such that for each $n \in \mathbb{N}$,

$$\int_0^{\xi_n} f(t)\,dt = \sup_{|\xi|\le b_n} \int_0^{\xi} f(t)\,dt$$

for some ξ_n with $|\xi_n| < b_n$.

In Section 4, we show that the answer is positive when $g(t) = 0$ but the sequence $\{\xi_n\}$ is in $\mathbb{R}_+$ and (1.3) is replaced by the following stronger condition:

$$\limsup_{\xi\to 0^+} \frac{\int_\Omega \alpha(x)\,dx \int_0^\xi f(t)\,dt}{\xi^p} > \frac{\int_\Omega \lambda(x)\,dx}{p}.$$

Under our assumptions, the weak solutions are almost everywhere nonnegative in Ω.

2. Unbounded sequence of solutions. In this section, we establish the existence of $\mu^* \ge 0$ such that for any $\mu > \mu^*$ problem (P_μ) admits an unbounded sequence of nonzero and nonnegative weak solutions.

Throughout this section, we assume that when $1 < p \le N$, there exist $a \in \mathbb{R}_+$ and $q > p-1$, with $q < \frac{(p-1)N+p}{N-p}$ if $p < N$, such that

$$|f(x,t)| \le a(1+|t|^q) \tag{2.1}$$

for a.e. $x \in \Omega$ and $t \in \mathbb{R}$. In the case $p > N$, we assume that for every $r > 0$,

$$\sup_{|t|\le r} |f(\cdot,t)| \in L^1(\Omega). \tag{2.2}$$

THEOREM 2.1. *Suppose that the function f satisfies the following conditions*:

(i) $f(x,0) \ge 0$ *for a.e.* $x \in \Omega$.

(ii) *There exist two sequences* $\{\xi_n\}, \{\xi'_n\}$ *in* $\mathbb{R}$ *with* $\lim_{n\to+\infty} \xi_n = +\infty$ *such that* $0 \le \xi_n < \xi'_n$ *and*

$$\int_0^{\xi_n} f(x,s)\,ds = \sup_{t\in[\xi_n,\xi'_n]} \int_0^t f(x,s)\,ds$$

for each $n \in \mathbb{N}$ *and a.e.* $x \in \Omega$.

(iii) *One has*

$$\limsup_{t\to+\infty} \frac{\int_\Omega \int_0^t f(x,s)\,ds\,dx}{t^p} > 0.$$

Set

$$\mu^* = \frac{\int_\Omega \lambda(x)\,dx}{p} \liminf_{t\to+\infty} \frac{t^p}{\int_\Omega \int_0^t f(x,s)\,ds\,dx}.$$

Then, for every $\mu > \mu^*$, problem (P_μ) *admits an unbounded sequence* $\{u_n\}$ *of nonnegative weak solutions.*

Proof. Define

$$g(x,t) = \begin{cases} f(x,t) & \text{if } t \geq 0, \\ f(x,0) & \text{if } t < 0. \end{cases}$$

Consider the problem

$$(\mathrm{P}_{\mu,g}) \qquad \begin{cases} -\Delta_p u + \lambda(x)|u|^{p-2}u = \mu g(x,u) & \text{in } \Omega, \\ \partial u/\partial \nu = 0 & \text{on } \partial\Omega. \end{cases}$$

The weak solutions of $(P_{\mu,g})$ are the critical points of the functional

$$\Phi_\mu(u) = \frac{1}{p\mu}\Big(\int_\Omega \lambda(x)|u|^p\,dx + \int_\Omega |\nabla u|^p\,dx\Big) - \int_\Omega\Big(\int_0^{u(x)} g(x,t)\,dt\Big)\,dx$$

in $W^{1,p}(\Omega)$. Owing to (2.1) and the compact embedding of $W^{1,p}(\Omega)$ into $L^{q+1}(\Omega)$, Φ_μ is well defined, weakly sequentially lower semicontinuous and Gateaux differentiable in $W^{1,p}(\Omega)$.

Fix $n \in \mathbb{N}$. We set

$$E_n = \{u \in W^{1,p}(\Omega) : 0 \leq u(x) \leq \xi_n' \text{ a.e. in } \Omega\}, \qquad \alpha_n = \inf_{E_n} \Phi_\mu.$$

Following the arguments used in [2], we can prove that there exists $u_n \in E_n$ such that

$$\Phi_\mu(u_n) = \alpha_n.$$

Moreover, $u_n(x) \in [0, \xi_n]$ a.e. in Ω.

Define $h : \mathbb{R} \to \mathbb{R}$ as follows:

$$h(t) = \begin{cases} \xi_n, & t > \xi_n, \\ t, & 0 \leq t \leq \xi_n, \\ 0, & t < 0, \end{cases}$$

and consider the continuous superposition operator $T : W^{1,p}(\Omega) \to E_n$,

$$Tu(x) = h(u(x)) \qquad (x \in \Omega).$$

We put $v^* = Tu_n$ and $X = \{x \in \Omega : u_n(x) \notin [0, \xi_n]\}$. For a.e. $x \in X$, one has $\xi_n < u_n(x) \leq \xi_n'$, hence

$$\int_0^{u_n(x)} g(x,t)\,dt \leq \int_0^{v^*(x)} g(x,t)\,dt$$

and $|\nabla v^*| = 0$. We have

$$\begin{aligned}\|v^*\|^p - \|u_n\|^p &= \int_\Omega \lambda(x)(|v^*|^p - |u_n|^p)\,dx + \int_\Omega (|\nabla v^*|^p - |\nabla u_n|^p)\,dx\\ &= \int_X \lambda(x)(\xi_n^p - (u_n(x))^p)\,dx - \int_X |\nabla u_n|^p\,dx\\ &\le -\int_X \lambda(x)(u_n(x)-\xi_n)^p\,dx - \int_X |\nabla v^* - \nabla u_n|^p\,dx\\ &= -\int_\Omega \lambda(x)|v^* - u_n|^p\,dx - \int_\Omega |\nabla v^* - \nabla u_n|^p\,dx = -\|v^*-u_n\|^p.\end{aligned}$$

Hence

$$\begin{aligned}\Phi_\mu(v^*) - \Phi_\mu(u_n) &= \frac{1}{p\mu}(\|v^*\|^p - \|u_n\|^p) - \int_\Omega \Big(\int_{u_n(x)}^{v^*(x)} g(x,t)\,dt\Big)\,dx\\ &\le -\frac{1}{p\mu}\|v^*-u_n\|^p - \int_X \Big(\int_{u_n(x)}^{v^*(x)} g(x,t)\,dt\Big)\,dx\\ &\le -\frac{1}{p\mu}\|v^*-u_n\|^p.\end{aligned}$$

Since $v^* \in E_n$, it follows that $\Phi_\mu(v^*) - \Phi_\mu(u_n) \ge 0$. Then $\|v^*-u_n\|^p = 0$, which entails that $u_n(x) = v^*(x) \in [0,\xi_n]$ a.e. in Ω.

Now we prove that u_n is a local minimum of Φ_μ. Let $u \in W^{1,p}(\Omega)$ and put $X = \{x \in \Omega : u(x) \notin [0,\xi_n]\}$. In case $p > N$, owing to the compact embedding of $W^{1,p}(\Omega)$ into $C^0(\overline{\Omega})$ and the fact that $u_n(x) \in [0,\xi_n]$ for each $x \in \Omega$, it follows that $u(x) \le \xi_n'$ for all $x \in \Omega$, provided that u is chosen in a suitable neighbourhood of u_n.

By definition of the operator T, one has $\int_{Tu(x)}^{u(x)} g(x,t)\,dt = 0$ for $x \in \Omega\backslash X$. Suppose $x \in X$. Then $\int_{Tu(x)}^{u(x)} g(x,t)\,dt \le 0$ whenever $u(x) \le \xi_n'$. In case $p \le N$ and $u(x) > \xi_n'$, we exploit (2.1), where without loss of generality we can suppose that $q > p-1$, and so

$$\begin{aligned}\int_{Tu(x)}^{u(x)} g(x,t)\,dt &= \int_{\xi_n}^{u(x)} g(x,t)\,dt \le \int_{\xi_n}^{u(x)} a(1+t^q)\,dt\\ &= a(u(x)-\xi_n) + \frac{a}{q+1}((u(x))^{q+1} - \xi_n^{q+1}).\end{aligned}$$

Define

$$C = \sup_{\xi \ge \xi_n'} \frac{a(\xi-\xi_n) + \frac{a}{q+1}(\xi^{q+1} - \xi_n^{q+1})}{(\xi-\xi_n)^{q+1}}.$$

It follows that for a.e. $x \in \Omega$,

$$\int_{Tu(x)}^{u(x)} g(x,t)\,dt \le C|u(x) - Tu(x)|^{q+1}$$

and so

$$\int_{\Omega}\Big(\int_{Tu(x)}^{u(x)} g(x,t)\,dt\Big)\,dx \le C\gamma^{q+1}\|u-Tu\|^{q+1},$$

where we have put

$$\gamma = \sup_{u\in W^{1,p}(\Omega)\setminus\{0\}} \frac{(\int_{\Omega}|u|^{q+1}\,dx)^{1/(q+1)}}{\|u\|},$$

which is finite because of the embedding theorem.

Then, since

$$\begin{aligned}\|u\|^p-\|Tu\|^p &= \int_{\Omega}\lambda(x)(|u|^p-|Tu|^p)\,dx+\int_{\Omega}(|\nabla u|^p-|\nabla(Tu)|^p)\,dx\\ &= \int_{\{x\in X\,:\,u(x)<0\}}\lambda(x)|u|^p\,dx\\ &\quad+\int_{\{x\in X\,:\,u(x)>0\}}\lambda(x)((u(x))^p-\xi_n^p)\,dx+\int_X|\nabla u|^p\,dx\\ &\ge \int_{\{x\in X\,:\,u(x)<0\}}\lambda(x)|u-Tu|^p\,dx\\ &\quad+\int_{\{x\in X\,:\,u(x)>0\}}\lambda(x)(u(x)-\xi_n)^p\,dx+\int_X|\nabla u-\nabla Tu|^p\,dx\\ &= \|u-Tu\|^p,\end{aligned}$$

we have

$$\begin{aligned}\Phi_\mu(u)-\Phi_\mu(Tu) &= \frac{1}{p\mu}\,(\|u\|^p-\|Tu\|^p)-\int_{\Omega}\Big(\int_{Tu(x)}^{u(x)} g(x,t)\,dt\Big)\,dx\\ &\ge \frac{1}{p\mu}\,\|u-Tu\|^p-\int_{\Omega}\Big(\int_{Tu(x)}^{u(x)} g(x,t)\,dt\Big)\,dx\\ &\ge \frac{1}{p\mu}\,\|u-Tu\|^p-C\gamma^{q+1}\|u-Tu\|^{q+1}.\end{aligned}$$

From $Tu\in E_n$, it follows that $\Phi_\mu(Tu)\ge\Phi_\mu(u_n)$. Thus, we have

$$\Phi_\mu(u)\ge\Phi_\mu(u_n)+\|u-Tu\|^p\Big(\frac{1}{p\mu}-C\gamma^{q+1}\|u-Tu\|^{q+1-p}\Big).$$

Since T is continuous, $u_n=Tu_n$, $q+1-p>0$ and

$$\|u-Tu\|\le\|u-u_n\|+\|u_n-Tu\|=\|u-u_n\|+\|Tu_n-Tu\|,$$

there exists $\beta>0$ such that for every $u\in W^{1,p}(\Omega)$ with $\|u-u_n\|<\beta$, one has $\|u-Tu\|^{q+1-p}\le 1/(2\mu p\,C\gamma^{q+1})$. Consequently, if $\|u-u_n\|<\beta$, it turns out that

$$\Phi_\mu(u) \geq \Phi_\mu(u_n) + \frac{1}{2p\mu}\|u - Tu\|^p \geq \Phi_\mu(u_n).$$

Fix $\mu > \mu^*$. Then

$$\frac{1}{p\mu} < \frac{1}{\int_\Omega \lambda(x)\,dx} \limsup_{t\to+\infty} \frac{\int_\Omega \int_0^t f(x,s)\,ds\,dx}{t^p}.$$

Now, we prove that, for this μ, one has $\liminf_{n\to+\infty} \alpha_n = -\infty$.

Let $L \in \mathbb{R}$ be such that

$$\frac{1}{p\mu} < L < \frac{1}{\int_\Omega \lambda(x)\,dx} \limsup_{t\to+\infty} \frac{\int_\Omega \int_0^t f(x,s)\,ds\,dx}{t^p}.$$

Then there exists a sequence $\{t_k\}$ of positive numbers, diverging to $+\infty$, which satisfies

$$\frac{\int_\Omega \int_0^{t_k} f(x,s)\,ds\,dx}{t_k^p} > L \int_\Omega \lambda(x)\,dx$$

for every $k \in \mathbb{N}$. We can choose a subsequence $\{\xi'_{n_k}\}$ such that $\xi'_{n_k} > t_k$. Thus the constant function t_k belongs to E_{n_k}. This implies that for every $k \in \mathbb{N}$,

$$\begin{aligned} \alpha_{n_k} \leq \Phi_\mu(t_k) &= \frac{1}{p\mu} t_k^p \int_\Omega \lambda(x)\,dx - \int_\Omega \int_0^{t_k} f(x,s)\,ds\,dx \\ &< t_k^p \int_\Omega \lambda(x)\,dx \left(\frac{1}{p\mu} - L\right), \end{aligned}$$

hence $\lim_{k\to+\infty} \alpha_{n_k} = -\infty$. At this point, we can prove that the sequence of local minima u_{n_k} must be unbounded. In fact, if it were bounded, there would be a subsequence, denoted by $\{u_{n_k}\}$ again, weakly convergent to some $\overline{u} \in W^{1,p}(\Omega)$. Then we have the contradiction

$$\Phi_\mu(\overline{u}) \leq \liminf_{k\to+\infty} \Phi_\mu(u_{n_k}) = -\infty,$$

and the assertion is completely proved. ∎

3. Many small solutions. In this section, we consider the existence of infinitely many arbitrarily small positive solution to problem (P_μ). In this case we only require that $p > 1$. Since the proof is based on arguments similar to those used to prove Theorem 2.1, some details are omitted.

THEOREM 3.1. *Suppose that the function f satisfies the following conditions*:

(i′) $f(x,0) = 0$ *for a.e.* $x \in \Omega$.

(ii′) *There exists* $\overline{t} > 0$ *such that*

$$\sup_{t\in[0,\overline{t}]} |f(\cdot,t)| \in L^\infty(\Omega).$$

(iii′) *There exist two sequences* $\{\xi_n\}, \{\xi'_n\}$ *in* $\mathbb{R}$, *with* $\lim_{n\to+\infty} \xi'_n = 0$, *such that* $0 \le \xi_n < \xi'_n$ *and*

$$\int_0^{\xi_n} f(x,s)\,ds = \sup_{t\in[\xi_n,\xi'_n]} \int_0^t f(x,s)\,ds$$

for every $n \in \mathbb{N}$ *and a.e.* $x \in \Omega$.

(iv′) *One has*

$$\limsup_{t\to 0^+} \frac{\int_\Omega \int_0^t f(x,s)\,ds\,dx}{t^p} > 0.$$

Set

$$\mu^* = \frac{\int_\Omega \lambda(x)\,dx}{p} \liminf_{t\to 0^+} \frac{t^p}{\int_\Omega \int_0^t f(x,s)\,ds\,dx}.$$

Then, for every $\mu > \mu^*$, *problem* (P_μ) *admits a sequence* $\{u_n\}$ *of almost everywhere positive weak solutions strongly convergent to zero such that* $\lim_{n\to+\infty} \sup_\Omega u_n = 0$.

Proof. We choose $q \in \left]p-1, \frac{(p-1)N+p}{N-p}\right[$ if $p < N$. In the other cases it is enough to choose $q > p-1$. From (ii′), it follows that there exists $a > 0$ such that for every $0 \le t \le \overline{t}$ and a.e. $x \in \Omega$, one has

$$|f(x,t)| \le a.$$

Without loss of generality, we suppose that $\xi'_n \le \overline{t}$ for every $n \in \mathbb{N}$. Let $\mu > \mu^*$. Then we define $g : \Omega \times \mathbb{R} \to \mathbb{R}$ as follows:

$$g(x,t) = \begin{cases} f(x,\overline{t}) & \text{if } t > \overline{t}, \\ f(x,t) & \text{if } 0 \le t \le \overline{t}, \\ 0 & \text{if } t < 0. \end{cases}$$

Hence, for a.e. $x \in \Omega$ and $t \in \mathbb{R}$,

$$|g(x,t)| \le a. \tag{3.1}$$

Now, we consider the problem

$$(\mathrm{P}_{\mu,g}) \qquad \begin{cases} -\Delta_p u + \lambda(x)|u|^{p-2}u = \mu g(x,u) & \text{in } \Omega, \\ \partial u/\partial\nu = 0 & \text{on } \partial\Omega. \end{cases}$$

The weak solutions of $(\mathrm{P}_{\mu,g})$ are the critical points of the functional

$$\Phi_\mu(u) = \frac{1}{p\mu}\|u\|^p - \int_\Omega \Big(\int_0^{u(x)} g(x,t)\,dt \Big)\,dx \qquad (u \in W^{1,p}(\Omega)). \tag{3.2}$$

Owing to (3.1) and the compact embedding of $W^{1,p}(\Omega)$ into $L^{q+1}(\Omega)$ (resp. into $C^0(\overline{\Omega})$ if $p > N$), Φ_μ is well defined, weakly sequentially lower semicontinuous and Gateaux differentiable in $W^{1,p}(\Omega)$.

Taking into account (3.1) and condition (iii′) and using the same methods applied in the proof of Theorem 2.1, one can prove that for every $n \in \mathbb{N}$, Φ_μ admits a local minimum u_n that belongs to $E_n = \{u \in W^{1,p}(\Omega) : 0 \le u(x) \le \xi'_n\}$. More precisely, every u_n assumes its values in the interval $[0, \xi_n]$ except for a null measure subset of Ω.

For every $n \in \mathbb{N}$ and $u \in E_n$, one has

$$\Phi_\mu(u) \ge -am(\Omega)\xi'_n.$$

Then, since $-am(\Omega)\xi'_n \le \Phi_\mu(u_n) \le 0$, it follows that

$$\lim_{n\to+\infty} \Phi_\mu(u_n) = 0.$$

From $u_n \in E_n$, it follows that

$$\|u_n\|^p = p\mu\Big(\int_\Omega\Big(\int_0^{u_n(x)} g(x,t)\,dt\Big)\,dx + \Phi_\mu(u_n)\Big) \le p\mu(am(\Omega)\xi'_n + \Phi_\mu(u_n)).$$

Hence $\lim_{n\to+\infty}\int_\Omega |\nabla u_n|^p\,dx = 0$.

To obtain the conclusion, it is enough to prove that such local minima are pairwise distinct. We exploit the fact that for every $n \in \mathbb{N}$,

$$\Phi_\mu(u_n) = \inf_{u\in E_n} \Phi_\mu(u).$$

Fix $n \in \mathbb{N}$. Since

$$\frac{1}{p\mu} < \frac{1}{\int_\Omega \lambda(x)\,dx} \limsup_{t\to 0^+} \frac{\int_\Omega \int_0^t f(x,s)\,ds\,dx}{t^p},$$

there exists a sequence of positive numbers $t_k \searrow 0$ such that for every $k \in \mathbb{N}$,

$$\frac{\int_\Omega \int_0^{t_k} f(x,s)\,ds\,dx}{t_k^p} > \frac{1}{p\mu}\int_\Omega \lambda(x)\,dx.$$

Then there exists $\overline{k} \in \mathbb{N}$ such that $t_{\overline{k}} < \xi'_n$. Hence, the constant function on Ω, $v(x) \equiv t_{\overline{k}}$, belongs to E_n and this implies that

$$\Phi_\mu(u_n) \le \Phi_\mu(v).$$

Moreover, we have

$$-\frac{\int_\Omega \int_0^{t_{\overline{k}}} f(x,s)\,ds\,dx}{\|t_{\overline{k}}\|^p} < -\frac{1}{p\mu}.$$

Hence, $\Phi_\mu(u_n) < 0$. It is easily seen that since $\Phi_\mu(u_n) < 0$ for every $n \in \mathbb{N}$, there exists a subsequence of $\{u_n\}$ with pairwise distinct elements. ∎

REMARK 3.1. Condition (ii′) of Theorem 3.1 can be weakened when $p > N$. In that case, (ii′) can be replaced by the following assumption: There exists $\overline{t} > 0$ such that for $0 \leq t \leq \overline{t}$ and a.e. $x \in \Omega$,

$$|f(x,t)| \leq \alpha(x),$$

with $\alpha \in L^1(\Omega)$ almost everywhere nonnegative in Ω.

4. Comparison with existing results. This section is dedicated to the question asked by Ricceri and recalled in the first section.

THEOREM 4.1. *Let* $f : \mathbb{R} \to \mathbb{R}$ *be a continuous function*, $\alpha \in L^1(\Omega)$ *with* $\alpha(x) \geq 0$ *a.e. in* Ω, *and* $p > N$. *Assume that there are sequences* $\{b_n\}$ *and* $\{\xi_n\}$ *in* $\mathbb{R}_+$ *with* $\xi_n < b_n$ *and* $\lim_{n\to\infty} b_n = 0$ *such that for each* $n \in \mathbb{N}$,

$$\int_0^{\xi_n} f(t)\,dt = \sup_{|\xi|\leq b_n} \int_0^{\xi} f(t)\,dt. \tag{4.1}$$

Moreover, *assume that*

$$\limsup_{\xi\to 0^+} \frac{\int_\Omega \alpha(x)\,dx \int_0^\xi f(t)\,dt}{\xi^p} > \frac{\int_\Omega \lambda(x)\,dx}{p}. \tag{4.2}$$

Then the problem

$$\begin{cases} -\Delta_p u + \lambda(x)|u|^{p-2}u = \alpha(x)f(u) & \text{in } \Omega, \\ \partial u/\partial \nu = 0 & \text{on } \partial\Omega, \end{cases}$$

admits a sequence of weak solutions, *a.e. positive in* Ω, *which strongly converges to zero in* $W^{1,p}(\Omega)$.

Proof. By (4.1) it follows that $f(\xi_n) = 0$ for each $n \in \mathbb{N}$, and so $f(0) = 0$ because of the continuity of f. Hence, taking into account Remark 3.1, the statement follows by Theorem 3.1. ∎

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