COLLOQUIUM MATHEMATICUM

VOL. 97 2003 NO. 2

A SIMPLE-MINDED COMPUTATION OF HEAT KERNELS ON HEISENBERG GROUPS

BY

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Abstract. We compute the heat kernel on the classical and nonisotropic Heisenberg groups, and on the free step two nilpotent groups $N_{n,2}$, by an elementary method, in particular without using Laguerre calculus.

Introduction. The heat kernels $(p_t)_{t>0}$ associated with subelliptic Laplacians L on the Heisenberg groups G, or on the free step two nilpotent groups $G = N_{n,2}$, are well known. Their computation is usually rather elaborate, using either Brownian motion [G], [K], or the spherical Fourier transform [H], [C], [CT] (see also the book [FH]), i.e. the knowledge of the eigenvectors of L, through the representations of G, finally reducing the problem to the knowledge of the kernel of the harmonic oscillator on some $\mathbb{R}^d$ (Laguerre calculus). Let us also mention [B], where a simple direct computation (for the Heisenberg groups) relies on a guessed form of the solution.

The heat equation

$$(H) \qquad -\frac{1}{2}\,Lp_t = \frac{\partial}{\partial t}p_t, \qquad t > 0,$$

combined with the expression of p_t as a dilation of p_1, gives a partial differential equation (H_1) for p_1. We simply give sufficient conditions on a solution p of (H_1) which ensure that the corresponding p_t are the kernels of a strongly continuous semigroup of contractions on $L^2(G)$, whose generator is of course $-\frac{1}{2}L$, and we exhibit such a solution $p = p_1$, which is obviously unique.

It seems difficult to verify directly that the exhibited solution p_1 is a positive function on G. This is true a posteriori, because the existence of such a semigroup, and the positivity of the associated kernels, are well known facts for every stratified group (see e.g. [FS, Prop. 1.68, 1.70]).

More precisely, the paper is organized as follows. In the first part, we consider homogeneous groups G, and more particularly step two stratified groups, with dilations δ_t, $t > 0$, and (unbounded) positive operators L on

2000 *Mathematics Subject Classification*: 43A85, 35K05.

Key words and phrases: heat kernel, Heisenberg groups.

$L^2(G)$ which commute with left translations and satisfy $\delta_{t^{-1}} L \delta_t = t^2 L$. We give in Proposition 4 elementary sufficient conditions on p as mentioned above; a weaker sufficient condition is given in Proposition 6 when moreover $\frac{\partial}{\partial t} + \frac{L}{2}$ is hypoelliptic. Computations when L is the sublaplacian on a Heisenberg group or $G = N_{n,2}$ are done in the second part, where we express (H_1) explicitly as a partial differential equation on $\mathbb{R}^d$. Using ordinary Fourier transform, we exhibit a solution of (H_1) and check that the conditions of Proposition 4 are satisfied in the case of Heisenberg groups, while those of Proposition 6 are satisfied in the case of $N_{n,2}$.

Acknowledgments. This paper is an (unexpected) byproduct of a research initiated with J. L. Torrea during a stay of the author at the Universidad Autónoma (Madrid), which was supported by the European network in Harmonic Analysis. We also thank T. Coulhon and V. Georgescu for useful discussions, and Z. Rzeszotnik for providing references.

Notation. G will denote a homogeneous Lie group, as defined in [FS, Chapter I A]. The group operation is denoted multiplicatively, unless otherwise specified. Let us recall the main properties of these groups ([FS, Chapter I A,B,C,D]). G is a connected and simply connected nilpotent Lie group, its Haar measure dg is left and right invariant. G is equipped with a group of dilations δ_t, $t > 0$, which are automorphisms of G, i.e. $\delta_t(gg') = \delta_t(g)\delta_t(g')$ for $g, g' \in G$ and $\delta_{st}(g) = \delta_s \delta_t(g)$. More precisely, the Lie algebra $\mathcal{G}$ of G is real, and has finite dimension d, $\delta_t = \exp(A \log t)$, where A is a diagonalizable linear operator on $\mathcal{G}$, with strictly positive eigenvalues d_j, $1 \le j \le d$. Let $(Z_j)_{j=1}^d$ be a linear basis of $\mathcal{G}$ given by eigenvectors of A, corresponding to the eigenvalues $(d_j)_{j=1}^d$; since the exponential mapping $\mathcal{G} \to G$ is a diffeomorphism, every $g \in G$ has the form $\exp(z_1 Z_1 + \ldots + z_d Z_d)$ for a unique $(z_1, \ldots, z_d) \in \mathbb{R}^d$; we write $g = (z_1, \ldots, z_d)$, hence

$$\delta_t(g) = (t^{d_1} z_1, \ldots, t^{d_d} z_d).$$

We denote by $N = \sum_{j=1}^d d_j$ the homogeneous dimension of G, which implies

$$d(\delta_t(g)) = t^N dg \quad \text{and} \quad t^{-N} \|f \circ \delta_{t^{-1}}\|_{L^1(G)} = \|f\|_{L^1(G)}.$$

Setting $\delta_t(f) = f \circ \delta_t$ for a function f on G, we obtain

$$Z_j(\delta_t(f)) = t^{d_j} \delta_t(Z_j f).$$

$\mathcal{K}(G)$ denotes the space of complex-valued, continuous, compactly supported functions on G, $\mathcal{D}(G)$ denotes $\mathcal{K}(G) \cap \mathcal{C}^\infty(G)$, and $\mathcal{D}'(G)$ denotes the distributions on G. The space $\mathcal{S}(G)$ and the space $\mathcal{S}'(G)$ of tempered distributions are defined as in [FS, Chapter I D].

The right translate f_g of a function f on G is $\gamma \mapsto f(\gamma g)$, the left one is ${}_\gamma f : g \mapsto f(\gamma g)$. The convolution of two functions $f, p \in \mathcal{K}(G)$ is defined by

$$f * p(\gamma) = \int_G f(\gamma g^{-1}) p(g)\, dg = \int_G f(g) p(g^{-1}\gamma)\, dg,$$

and for every left invariant vector field $X \in \mathcal{G}$,

$$X(f * p) = f * X(p).$$

In particular, if G is a step two stratified Lie group, as defined in [FS, p. 5], we denote by $X_1, \ldots, X_n$ a linear basis of the first layer of $\mathcal{G}$, which spans $\mathcal{G}$ as a Lie algebra, and corresponds to $d_1 = 1$, and by $U_1, \ldots, U_m$ a linear basis of the (second) central layer, which corresponds to $d_2 = 2$. Hence

$$X_k(\delta_t(f)) = t\delta_t(X_k f), \qquad U_j(\delta_t(f)) = t^2 \delta_t(U_j f).$$

This basis corresponds to coordinates $g = (x, u) = (x_1, \ldots, x_n, u_1, \ldots, u_m)$ and $\delta_t(g) = (tx_1, \ldots, tx_n, t^2u_1, \ldots, t^2u_m)$. Here $N = n + 2m$.

The Fourier transform of a function $f \in \mathcal{D}(\mathbb{R}^d)$ is defined by $\widehat{f}(\xi) = \int_{\mathbb{R}^d} e^{-i\langle \xi, x\rangle} f(x)\, dx$.

I. Remarks on some kernels on homogeneous Lie groups. The results gathered in the next three lemmas are classical.

LEMMA 1. *Let G be a homogeneous Lie group. Let L be a linear operator $\mathcal{D}'(G) \to \mathcal{D}'(G)$ and $\mathcal{D}(G) \subset \mathrm{dom}_{L^2} L \to L^2(G)$ which preserves real-valued functions, such that $\langle L(w), w\rangle \geq 0$ for every $w \in \mathrm{dom}_{L^2} L$. Assume that for every real-valued $f \in \mathcal{D}(G)$ and $t > 0$, there exists a real-valued function $u(t)$ such that*

(i) *u is differentiable $]0, \infty[\to L^2(G)$,*
(ii) *$\|u(t) - f\|_{L^2(G)} \to 0$ as $t \to 0^+$,*
(iii) *for every $t > 0$, in $\mathcal{D}'(G)$,*

$$-\frac{1}{2} L(u(t)) = u'(t). \tag{E}$$

Then such a u is unique, $-L$ is the generator of a strongly continuous semigroup of contractions on $L^2(G)$ and $u(t) = e^{-tL/2}(f)$.

Proof. Since (E) holds in $\mathcal{D}'(G)$ and $u'(t) \in L^2(G)$, $L(u(t))$ also belongs to $L^2(G)$, i.e. $u(t) \in \mathrm{dom}_{L^2} L$. Let u, v be two functions satisfying (i)–(iii); by (i) and (iii) the derivative of $h(t) = \|u(t) - v(t)\|^2_{L^2(G)}$ is

$$2\langle u' - v', u - v\rangle = -\langle L(u - v), u - v\rangle \leq 0,$$

hence h is a decreasing function on $]0, \infty[$, which by (ii) tends to 0 as $t \to 0^+$, hence $h = 0$. In the same way, $\|u(t)\|^2_{L^2(G)}$ is decreasing on $]0, \infty[$, hence less than $\|f\|^2_{L^2(G)}$. The last assertion follows by definition. ∎

LEMMA 2. *Let G be a homogeneous Lie group. Let L be a linear operator $\mathcal{D}'(G) \to \mathcal{D}'(G)$ and $\mathcal{D}(G) \subset \operatorname{dom}_{L^2} L \to L^2(G)$ such that*

(a) *L commutes with left translations*,
(b) *$\delta_{t^{-1}} L \delta_t = t^2 L$.*

Let $p \in L^1(G)$ and $p_t = t^{-N/2}\delta_{t^{-1/2}}(p)$. Then the following assertions are equivalent:

(1) *For every $f \in \mathcal{D}(G)$, the function $f * p_t$ satisfies equation (E) in $\mathcal{D}'(G \times]0,\infty[)$.*
(2) *The function p_t satisfies in $\mathcal{D}'(G \times]0,\infty[)$ the equation*

$$(H) \qquad -\frac{1}{2} L p_t = \frac{\partial p_t}{\partial t}.$$

(3) *The function p satisfies in $\mathcal{D}'(G)$ the equation*

$$(H_1) \qquad (N \operatorname{Id} - L)p = t\delta_t \frac{d}{dt}(\delta_{t^{-1}}(p)).$$

Proof. (1)$\Leftrightarrow$(2). By (a), $L(f * p_t) = f * Lp_t$ in $\mathcal{D}'(G)$ for $t > 0$, and $\frac{\partial (f*p_t)}{\partial t} = f * \frac{\partial p_t}{\partial t}$ in $\mathcal{D}'(G \times]0,\infty[)$.

(2)$\Leftrightarrow$(3). (H) is equivalent to

$$-tLp_{t^2} = \frac{\partial}{\partial t} p_{t^2}.$$

By (b), $Lp_{t^2} = t^{-N-2}\delta_{t^{-1}}(Lp)$ and the above equation can be rewritten as

$$-t^{-N-1}\delta_{t^{-1}}(Lp) = -Nt^{-N-1}\delta_{t^{-1}}(p) + t^{-N}\frac{d}{dt}\delta_{t^{-1}}(p),$$

which is (H_1). ∎

We now collect some properties of the functions $f * p_{t^2}$ (see e.g. [FS, Prop. 1.20] for (a), (b)). The use of the expression $\int_G f(\gamma\delta_t(g^{-1}))p(g)\,dg$ is also reminiscent of [CMZ].

LEMMA 3. *Let $1 \le q < \infty$ and let G be a homogeneous Lie group. Let $p \in L^1(G)$ and $p_{t^2} = t^{-N}\delta_{t^{-1}}(p)$. For $f \in \mathcal{K}(G)$, let $F_0 = f\int_G p(g)\,dg$, and for $t > 0$,*

$$F_t = f * p_{t^2}.$$

Then

(a) *For $\gamma \in G$,*

$$F_t(\gamma) = \int_G f(\gamma\delta_t(g^{-1}))p(g)\,dg = (\delta_{t^{-1}}[(f \circ \delta_t) * p])(\gamma).$$

In particular,

$$\|F_t\|_{L^q(G)} \le \|f\|_{L^q(G)}\|p\|_{L^1(G)}.$$

(b) *For* $t \geq 0$,

$$\|F_s - F_t\|_{L^q(G)} \underset{s\to t}{\longrightarrow} 0.$$

(c) *Assume that* G *is a step two stratified group. Let* $(X_1, \dots, X_n, U_1, \dots, U_m)$ *be a linear basis of* $\mathcal{G}$ *as defined in the notation. Then*, *for* $f \in \mathcal{D}(G)$ *and* $t > 0$, $g, \gamma \in G$, $g = (x_1, \dots, x_n, u_1, \dots, u_m)$,

$$\frac{\partial}{\partial t} f(\gamma\delta_t(g)) = \sum_{i=1}^{n} x_i (X_i f)(\gamma\delta_t(g)) + 2t \sum_{j=1}^{m} u_j (U_j f)(\gamma\delta_t(g)).$$

(d) *With the notation of* (c), *the function* $t \mapsto F_t$ *is differentiable* $]0,\infty[\ \to L^q(G)$ *whenever all* $x_i p$, $u_j p$, $1 \leq i \leq n$, $1 \leq j \leq m$, *belong to* $L^1(G)$.

Proof. (a) comes from the change of variable $g = \delta_{t^{-1}}(g')$.

(b) For every $g \in G$, $\|f_{\delta_s(g^{-1})} - f_{\delta_t(g^{-1})}\|_{L^q(G)} \to 0$ as $s \to t$; by (a) and the dominated convergence theorem,

$$\begin{aligned}\|F_s - F_t\|_{L^q(G)} &= \Big\| \int_G [f(\gamma\delta_s(g^{-1})) - f(\gamma\delta_t(g^{-1}))] p(g)\, dg \Big\|_{L^q(G)} \\ &\leq \int_G \|f_{\delta_s(g^{-1})} - f_{\delta_t(g^{-1})}\|_{L^q(G)} |p(g)|\, dg \underset{s\to t}{\longrightarrow} 0.\end{aligned}$$

(c) Let $g = \exp(X+U) \in G$, where $X = \sum_{i=1}^n x_i X_i$ and $U = \sum_{j=1}^m u_j U_j$. Since $tX + t^2 U$ commutes with $sX + (2st + s^2)U$,

$$\delta_{t+s}(g) = \exp((t+s)X + (t+s)^2 U) = \delta_t(g) \exp(sX + (2st + s^2)U).$$

For $h \in \mathcal{D}(G)$ and $t > 0$,

$$\frac{\partial}{\partial t} h \circ \delta_t(g) = \frac{\partial}{\partial s} h(\delta_{t+s}(g)) \Big|_{s=0} = (X + 2tU)(h)(\delta_t(g)).$$

Taking $h = {}_\gamma f$ yields the claimed formula by the left invariance of the X_i's and U_j's.

(d) Defining

$$\phi_t(\gamma, g) = \frac{\partial}{\partial t} f(\gamma\delta_t(g^{-1}))$$

and writing

$$f(\gamma\delta_{t+h}(g^{-1})) - f(\gamma\delta_t(g^{-1})) = h \int_0^1 \phi_{t+\varrho h}(\gamma, g)\, d\varrho$$

we get

$$\begin{aligned}B(t,h) &= \frac{1}{|h|} \Big\| F_{t+h} - F_t - h \int_G \phi_t(\cdot, g) p(g)\, dg \Big\|_q \\ &= \Big\| \int_G \Big[\int_0^1 \phi_{t+\varrho h}(\cdot, g)\, d\varrho - \phi_t(\cdot, g) \Big] p(g)\, dg \Big\|_q\end{aligned}$$

$$\le \Big\| \int_0^1 \int_G |\phi_{t+\varrho h}(\cdot, g) - \phi_t(\cdot, g)|\, |p(g)|\, dg\, d\varrho \Big\|_q$$

$$\le \int_0^1 \int_G \|\phi_{t+\varrho h}(\cdot, g) - \phi_t(\cdot, g)\|_q |p(g)|\, dg\, d\varrho.$$

By (c), for $g = (x_1, \ldots, x_n, u_1, \ldots, u_m) \in G$,

$$\|\phi_s(\cdot, g)\|_{L^q(G,d\gamma)} \le \sum_{i=1}^{n} |x_i|\, \|X_i f\|_{L^q(G)} + 2s \sum_{j=1}^{m} |u_j|\, \|U_j f\|_{L^q(G)},$$

and $\|\phi_s(\cdot, g) - \phi_t(\cdot, g)\|_{L^q(G)} \to 0$ as $s \to t$, as in the proof of (b); hence the assumption and the dominated convergence theorem imply $B(t, h) \to 0$ as $h \to 0$, which proves the differentiability of $t \mapsto F_t$ on $]0, \infty[$. ■

Putting together Lemmas 1–3, we get:

PROPOSITION 4. *Let G be a step two stratified Lie group. Let L be a linear operator $\mathcal{D}'(G) \to \mathcal{D}'(G)$ and $\mathcal{D}(G) \subset \operatorname{dom}_{L^2} L \to L^2(G)$ which preserves real-valued functions and satisfies*:

(a) *L commutes with left translations*,

(b) $\delta_{t^{-1}} L \delta_t = t^2 L$,

(c) $\langle L(w), w\rangle \ge 0$ *for* $w \in \operatorname{dom}_{L^2} L$.

Then there exists at most one real-valued function p such that

(i) $p \in L^1(G)$ *with* $\int_G p(g)\, dg = 1$ *and* $x_k p, u_j p \in L^1(G)$, $1 \le k \le n$, $1 \le j \le m$,

(ii) *p satisfies in $\mathcal{D}'(G)$ the equation*

$$(H_1) \qquad (N \operatorname{Id} - L)p + \sum_{k=1}^{n} x_k X_k p + 2 \sum_{j=1}^{m} u_j U_j p = 0.$$

If such a p exists, then $-L$ is the generator of a strongly continuous semigroup of contractions of $L^2(G)$, and for $f \in \mathcal{D}(G)$ and $t > 0$, $e^{-\frac{t}{2}L} f = f * p_t$.

Proof. If G is step two stratified, and $g = (x, u)$, then by Lemma 3,

$$-\frac{d}{dt} p(\delta_{t^{-1}}(g)) = \frac{1}{t^2} \sum_{k=1}^{n} x_k (X_k p)(\delta_{t^{-1}}(g)) + \frac{2}{t^3} \sum_{j=1}^{m} u_j (U_j p)(\delta_{t^{-1}}(g))$$

and

$$-t\delta_t \frac{d}{dt}(p \circ \delta_{t^{-1}}) = \sum_{k=1}^{n} x_k X_k p + 2 \sum_{j=1}^{m} u_j U_j p,$$

hence (ii) specifies equation (H_1) from Lemma 2 in this setting.

If there exists a p satisfying (i), (ii), then for every $f \in \mathcal{D}(G)$, $u(t) = f * p_t$ satisfies all assumptions of Lemma 1, owing to Lemmas 2 and 3. ∎

We will now weaken condition (i) in the above proposition. We have not found the statement of the next lemma in the literature, though it probably belongs to the folklore, and the implication (1)⇒(2),(3) is known ([FS, Props. 1.71, 1.72, 1.74]). We denote by $\langle S, f\rangle$ the action of a distribution S on a function f. We recall that for every compact set $K \subset \mathbb{R}^d$, the topological space $\mathcal{D}_K(\mathbb{R}^d)$ of K-supported functions in $\mathcal{D}(\mathbb{R}^d)$ embeds continuously in the space $A(\mathbb{R}^d)$ of functions on $\mathbb{R}^d$ whose Fourier transforms lie in $L^1(\mathbb{R}^d)$, equipped with the norm $\|f\|_{A(\mathbb{R}^d)} = \|\widehat{f}\|_{L^1(\mathbb{R}^d)}$.

LEMMA 5. *Let $p \in \mathcal{S}'(\mathbb{R}^d)$ be the inverse Fourier transform of a bounded continuous function $\widehat{p}$ on $\mathbb{R}^d$ satisfying $\widehat{p}(0) = 1$. Let p_t be the inverse Fourier transform of $\widehat{p}(t^{d_1/2}\xi_1, t^{d_2/2}\xi_2, \ldots, t^{d_d/2}\xi_d)$, where $t, d_j > 0$. Then*

(1) (a) *$p_t \to \delta_0$ in $\mathcal{D}'(\mathbb{R}^d)$ as $t \to 0^+$,*

(b) *$P_\varepsilon = 1_{[\varepsilon,\infty[}(t)p_t \to P = 1_{]0,\infty[}(t)p_t$ in $\mathcal{D}'(\mathbb{R}^{d+1})$ as $\varepsilon \to 0^+$,*

(c) *p_t satisfies the following homogeneity condition, with $N = \sum_{j=1}^d d_j$:*

$$p_{r^2 t}(r^{d_1}y_1, \ldots, r^{d_d}y_d) = r^{-N} p_t(y_1, \ldots, y_d).$$

(2) *If p_t satisfies the equation (H): $-\frac{1}{2}Lp_t = \frac{\partial p_t}{\partial t}$ in $\mathcal{S}'(\mathbb{R}^d \times]0,\infty[)$, where L is a differential operator on $\mathbb{R}^d$, then*

$$\left(\frac{\partial}{\partial t} + \frac{1}{2}L\right)P = \delta_{(0,0)}.$$

(3) *Assume moreover that $\frac{\partial}{\partial t} + \frac{1}{2}L$ is hypoelliptic. Then $p \in \mathcal{S}(\mathbb{R}^d)$.*

Proof. (1)(a) Since $\widehat{p}$ is bounded and continuous, the dominated convergence theorem implies, for $f \in \mathcal{D}(\mathbb{R}^d)$,

$$\int_{\mathbb{R}^d} \overline{\widehat{p}}(t^{d_1/2}\xi_1, t^{d_2/2}\xi_2, \ldots, t^{d_d/2}\xi_d)\widehat{f}(\xi)\,d\xi \underset{t\to 0^+}{\longrightarrow} \int_{\mathbb{R}^d} \overline{\widehat{p}}(0)\widehat{f}(\xi)\,d\xi = 2\pi f(0).$$

(b) Let $f \in \mathcal{D}(\mathbb{R}^{d+1})$ be supported on $\mathbb{R}^d \times [-T, T]$, $T > 0$, and let $\widehat{f}(\cdot, t)$ denote the Fourier transform of $f(z,t)$ with respect to $z \in \mathbb{R}^d$. Then

$$\begin{aligned}|\langle P, f(z,t)\rangle| = \left|\int_0^\infty \langle p_t, f(\cdot,t)\rangle\,dt\right| &\le \left|\int_0^\infty \langle \overline{\widehat{p}}_t, \widehat{f}(\cdot,t)\rangle\,dt\right| \\ &\le T\|\widehat{p}\|_\infty \|f\|_{L^\infty(\mathbb{R}, A(\mathbb{R}^d))} \le T\|\widehat{p}\|_\infty \|f\|_{A(\mathbb{R}^{d+1})},\end{aligned}$$

hence P is well defined as a distribution on $\mathbb{R}^{d+1}$. In the same way, for $\varepsilon > 0$,

$$|\langle P - P_\varepsilon, f(z,t)\rangle| \le \left|\int_0^\varepsilon \langle \widehat{p}_t, \widehat{f}(\cdot,t)\rangle\,dt\right| \le \varepsilon \|\widehat{p}\|_\infty \|f\|_{A(\mathbb{R}^{d+1})}.$$

(c) This is obvious since $p_t(y_1, \ldots, y_d) = t^{-N/2} p(t^{-d_1/2} y_1, \ldots, t^{-d_d/2} y_d)$.

(2) By (1)(b) it is enough to show that, in $\mathcal{D}'(\mathbb{R}^{d+1})$,

$$\left(\frac{\partial}{\partial t} + \frac{1}{2} L\right) P_\varepsilon \underset{\varepsilon \to 0^+}{\longrightarrow} \delta_{(0,0)}.$$

By (H) and (1)(a),

$$\begin{aligned}\left(\frac{\partial}{\partial t} + \frac{L}{2}\right)(1_{[\varepsilon,\infty[}(t) p_t) &= 1_{[\varepsilon,\infty[}(t) \frac{L}{2}(p_t) + 1_{[\varepsilon,\infty[}(t) \frac{\partial}{\partial t}(p_t) + p_t \frac{\partial}{\partial t}(1_{[\varepsilon,\infty[}(t)) \\ &= p_t \frac{\partial}{\partial t}(1_{[\varepsilon,\infty[}(t)) = p_\varepsilon \otimes \delta_\varepsilon \\ &\underset{\varepsilon \to 0^+}{\longrightarrow} \delta_{(0,0)}.\end{aligned}$$

(3) Since $\frac{\partial}{\partial t} + \frac{L}{2}$ is hypoelliptic on $\mathbb{R}^{d+1}$, (2) implies that P is $\mathcal{C}^\infty$ on $\mathbb{R}^{d+1} \setminus (0,0)$. By (1)(c) the claim now follows from [FS, Props. 1.72, 1.74]. ∎

In the next proposition we identify the step two stratified group G with some $\mathbb{R}^d$, both as a manifold and a measure space; the Fourier transform is the usual one on $\mathbb{R}^d$.

PROPOSITION 6. *Let G be a step two stratified Lie group, and $L = -\sum_{k=1}^n X_k^2$, where $(X_1, \ldots, X_n)$ is a basis of the first layer of $\mathcal{G}$. Then there is at most one p satisfying both equation (H_1) in Proposition 4 and*

(i)′ $p \in \mathcal{S}'(G) = \mathcal{S}'(\mathbb{R}^d)$ is the inverse Fourier transform of a bounded continuous function $\widehat{p}$ on $\mathbb{R}^d$ satisfying $\widehat{p}(0) = 1$.

Such a p lies in $\mathcal{S}(G)$.

Proof. In the stratified setting, the sublaplacian L satisfies the assumptions (a)–(c) of Proposition 4; by a theorem of Hörmander, L and $\frac{\partial}{\partial t} + \frac{L}{2}$ are hypoelliptic. By Lemmas 2 and 5, the two assumptions of Proposition 6 imply $p \in \mathcal{S}(G)$, hence all $x_k p$ and $u_j p$ lie in $L^1(G)$, i.e. (i) of Proposition 4 is satisfied. ∎

For the next examples we follow the definitions of [G]. We present them in increasing order of difficulty. By [C], the heat kernels of all step two stratified Lie groups can theoretically be recovered from the $N_{n,2}$ heat kernel. Let us mention that the heat kernels of the generalized Heisenberg groups, which we do not treat here, can be explicitly deduced from those of Heisenberg groups [R].

II.A. Computation of the heat kernel on the Heisenberg groups. Let $(a_j)_{j=1}^n$ be strictly positive numbers. We consider the isotropic (i.e. $a_j = 1$, $1 \le j \le n$) or nonisotropic Heisenberg group $G = H_n^a$. Via the exponential map, it may be identified with $\mathbb{R}^{2n+1}$, $n \ge 1$, equipped with Lebesgue measure and the group law

$$\gamma g = (x_1, y_1, \ldots, x_n, y_n, u)(x'_1, y'_1, \ldots, x'_n, y'_n, u')$$
$$= \Big(x_1 + x'_1, y_1 + y'_1, \ldots, x_n + x'_n, y_n + y'_n, u + u' + 2\sum_{j=1}^{n} a_j (y_j x'_j - x_j y'_j)\Big).$$

Here the first layer has dimension $2n$, the central one has dimension 1, and $N = 2n+2$. A basis of the top layer is

$$X_j = \frac{\partial}{\partial x_j} + 2a_j y_j \frac{\partial}{\partial u}, \qquad Y_j = \frac{\partial}{\partial y_j} - 2a_j x_j \frac{\partial}{\partial u},$$

and

$$[X_j, Y_j] = -4a_j \frac{\partial}{\partial u} = -4a_j U, \qquad 1 \le j \le n,$$

the other brackets being zero. Equation (H_1) in Proposition 4 reduces to

$$(2n+2)p - Lp = -\sum_{j=1}^{n}\left(x_j \frac{\partial p}{\partial x_j} + y_j \frac{\partial p}{\partial y_j}\right) - 2uUp.$$

The sublaplacian is given by

$$-L = \sum_{j=1}^{n}\left(\frac{\partial^2}{\partial x_j^2} + \frac{\partial^2}{\partial y_j^2} + 4a_j^2(x_j^2 + y_j^2)\frac{\partial^2}{\partial u^2} + 4a_j\left(y_j \frac{\partial^2}{\partial x_j \partial u} - x_j \frac{\partial^2}{\partial y_j \partial u}\right)\right).$$

Since $y_j \frac{\partial}{\partial x_j} - x_j \frac{\partial}{\partial y_j}$ is the generator of rotations in the x_j, y_j coordinates, L commutes with these rotations; hence, if a function p as in Proposition 4 exists, it must depend only on u and $r_j = (x_j^2 + y_j^2)^{1/2}$, $1 \le j \le n$; if we set $p(x, y, u) = h(r_1, \ldots, r_j, u)$, equation (H_1) reduces to

$$(*) \qquad (2n+2)h + \sum_{j=1}^{n}\left(\frac{\partial^2 h}{\partial r_j^2} + \frac{1}{r_j}\frac{\partial h}{\partial r_j} + 4a_j^2 r_j^2 \frac{\partial^2 h}{\partial u^2}\right) + \sum_{j=1}^{n} r_j \frac{\partial h}{\partial r_j} + 2u\frac{\partial h}{\partial u} = 0.$$

We first consider the *isotropic case* $a_j = 1$, $1 \le j \le n$, $G = H_n$. Here, L commutes with rotations on $\mathbb{R}^{2n}$, p must depend only on u and $r = (x_1^2 + y_1^2 + \ldots + x_n^2 + y_n^2)^{1/2}$. We set $p(x_1, y_1, \ldots, x_n, y_n, u) = h(r, u)$. Then

$$-Lh = \frac{\partial^2 h}{\partial r^2} + \frac{2n-1}{r}\frac{\partial h}{\partial r} + 4r^2 \frac{\partial^2 h}{\partial u^2}$$

and equation (H_1) reduces to

$$(*) \qquad \frac{\partial^2 h}{\partial r^2} + 4r^2 \frac{\partial^2 h}{\partial u^2} + \left(r + \frac{2n-1}{r}\right)\frac{\partial h}{\partial r} + 2u\frac{\partial h}{\partial u} + (2n+2)h = 0,$$

with $h \in L^1(\mathbb{R}^+ \times \mathbb{R}, r^{2n-1} dr du)$ and $\sigma_{2n}\int_{\mathbb{R}^+ \times \mathbb{R}} h(r,u) r^{2n-1}\, dr\, du = 1$, where σ_{2n} denotes the area of the unit sphere in $\mathbb{R}^{2n}$. We denote by $q(r, \lambda)$ the

Fourier transform of $h(r,u)$ with respect to u. Since the transform of $u\frac{\partial h}{\partial u}$ is $-q-\lambda\frac{\partial q}{\partial\lambda}$, $(*)$ becomes

$$(**)\qquad \frac{\partial^2 q}{\partial r^2}+\left(r+\frac{2n-1}{r}\right)\frac{\partial q}{\partial r}-2(2r^2\lambda^2-n)q-2\lambda\,\frac{\partial q}{\partial\lambda}=0,$$

with $\sigma_{2n}\int_{\mathbb{R}^+}q(r,0)r^{2n-1}\,dr=1$. In particular, $Q(r)=q(r,0)$ satisfies

$$Q''+\left(r+\frac{2n-1}{r}\right)Q'+2nQ=0,$$

whose solutions belong to a two-dimensional vector space; an obvious solution is $Q_1(r)=(2\pi)^{-n}e^{-r^2/2}\in L^1(\mathbb{R}^+,r^{2n-1}dr)$. Up to a multiplicative constant, it is the only solution lying in this space, which is proved either by computing another independent solution, or by Proposition 4 applied to $G=\mathbb{R}^{2n}$, or by noticing that we have indeed switched to the problem of recovering the usual heat kernel on $G=\mathbb{R}^{2n}$.

This is a motivation for looking for a solution of $(**)$ of the form

$$q(r,\lambda)=C(\lambda)e^{-\alpha(\lambda)r^2/2}$$

with $\alpha(0)=1$ and $C(0)=(2\pi)^{-n}$. We get

$$r^2(\alpha^2-\alpha-4\lambda^2+\lambda\alpha')q+\left(-2n\alpha+2n-2\lambda\,\frac{C'}{C}\right)q=0,$$

which splits into the two differential equations

$$\begin{aligned}&\alpha^2-\alpha-4\lambda^2+\lambda\alpha'=0,\qquad \alpha(0)=1,\\ &\frac{C'}{C}=n\,\frac{1-\alpha}{\lambda},\qquad C(0)=\frac{1}{(2\pi)^n}.\end{aligned}$$

The first one is a Riccati equation (as in [B]) with an obvious solution $\alpha_1(\lambda)=2\lambda$, so we look for a solution $\alpha(\lambda)=2\lambda+1/\beta(\lambda)$. We get

$$(4\lambda-1)\beta-\lambda\beta'=-1,\qquad \beta(0)=1,$$

hence $\beta(\lambda)=\frac{e^{4\lambda}-1}{4\lambda}$, and

$$\alpha(\lambda)=2\lambda\coth 2\lambda.$$

Then $\frac{C'}{C}=\frac{n}{\lambda}-2\coth 2\lambda$, hence

$$q(r,\lambda)=\left(\frac{2\lambda}{2\pi\sinh 2\lambda}\right)^n e^{-(\lambda\coth 2\lambda)r^2}.$$

Let h be the inverse Fourier transform of q with respect to λ and let p be the corresponding function on $\mathbb{R}^{2n+1}$. We will now show directly that p satisfies (i) of Proposition 4. (At this step, the positivity of p is not known.) It is also easy to verify that $\widehat{p}$ is continuous and bounded, and we may use Proposition 6, which is however less elementary than Lemma 7.

LEMMA 7. (a) *The function*

$$\phi(x,\lambda)=\sqrt{\frac{2\lambda}{2\pi\sinh 2\lambda}}\,e^{-2\lambda(\coth 2\lambda)x^2/2}$$

is the Fourier transform with respect to u *of a function* $\varphi(x,u)\in L^1(\mathbb{R}^2)$.

(b) $x\varphi(x,u)$ *and* $u\varphi(x,u)$ *also belong to* $L^1(\mathbb{R}^2)$.

(c) *The function*

$$2\pi p(x_1,y_1,\ldots,x_n,y_n,u)=\int_{\mathbb{R}} e^{i\lambda u}\left(\frac{2\lambda}{2\pi\sinh 2\lambda}\right)^n e^{-\lambda\coth 2\lambda\sum_{j=1}^n(x_j^2+y_j^2)}\,d\lambda$$

belongs to $L^1(\mathbb{R}^{2n+1})$, *and so do* up, x_jp, y_jp, $1\le j\le n$.

Proof. (a)&(b) By the Cauchy–Schwarz inequality,

$$\|\varphi(x,u)\|_1\le\|(1+|u|)\varphi(x,u)\|_{L^1(dx,L^2(du))}\|(1+|u|)^{-1}\|_{L^2(du)},$$

and by the Plancherel formula,

$$\|(1+|u|)\varphi(x,u)\|_{L^1(dx,L^2(du))}\le\|\phi(x,\lambda)\|_{L^1(dx,L^2(d\lambda))}+\left\|\frac{\partial}{\partial\lambda}\phi(x,\lambda)\right\|_{L^1(dx,L^2(d\lambda))}.$$

In the same way, $\|x\varphi(x,u)\|_1$ and $\|u\varphi(x,u)\|_1$ are controlled respectively by the $L^1(dx,L^2(d\lambda))$ norms of $x\phi$, $x\frac{\partial\phi}{\partial\lambda}$, and $\frac{\partial\phi}{\partial\lambda}$, $\frac{\partial^2\phi}{\partial\lambda^2}$.

Let $c>0$ be such that $\lambda\coth\lambda\ge c$ for every $\lambda\in\mathbb{R}$. Since

$$|\phi(x,\lambda)|\le\sqrt{\frac{2\lambda}{2\pi\sinh 2\lambda}}\,e^{-cx^2/2},$$

ϕ and $x\phi$ belong to $L^1(dx,L^2(d\lambda))$.

The first two derivatives of $\lambda\coth\lambda$ are bounded on $\mathbb{R}$, and the function $v(\lambda)=\sqrt{2\lambda/(\sinh 2\lambda)}$ is such that v'/v and v''/v are bounded on $\mathbb{R}$. Hence

$$\left|\frac{\partial\phi}{\partial\lambda}\right|\le C(1+x^2)\phi,\qquad\left|\frac{\partial^2\phi}{\partial\lambda^2}\right|\le C(1+x^2)^2\phi,$$

which implies that $x\frac{\partial\phi}{\partial\lambda},\frac{\partial\phi}{\partial\lambda},\frac{\partial^2\phi}{\partial\lambda^2}\in L^1(dx,L^2(d\lambda))$.

(c) Let $\widehat{\varphi}(\xi,\lambda)$ be the full Fourier transform of φ on $\mathbb{R}^2$. Since

$$\frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} e^{-i\xi x}e^{-(2\lambda\coth 2\lambda)x^2/2}\,dx=\sqrt{\frac{\tanh 2\lambda}{2\lambda}}\,e^{-\frac{\tanh 2\lambda}{2\lambda}\frac{\xi^2}{2}}$$

we get

$$\widehat{\varphi}(\xi,\lambda)=\frac{1}{\sqrt{\cosh 2\lambda}}\,e^{-\frac{\tanh 2\lambda}{2\lambda}\frac{\xi^2}{2}}.$$

The full Fourier transform $\widehat{p}$ of p on $\mathbb{R}^{2n+1}$ is

$$\widehat{p}(\xi_1,\eta_1,\ldots,\xi_n,\eta_n,\lambda)=\prod_{j=1}^{n}\frac{1}{\cosh 2\lambda}\,e^{-\frac{\tanh 2\lambda}{2\lambda}\,\frac{\xi_j^2+\eta_j^2}{2}}=\prod_{j=1}^{n}\widehat{\varphi}(\xi_j,\lambda)\widehat{\varphi}(\eta_j,\lambda);$$

it is the Fourier transform of a bounded measure, because $\widehat{\varphi}(\xi_j,\lambda)$ is the transform on $\mathbb{R}^{2n+1}$ of the bounded measure

$$\delta_0(x_1)\otimes\ldots\otimes\varphi(x_j,u)dx_jdu\otimes\delta_0(x_{j+1})\otimes\ldots\otimes\delta_0(y_n).$$

On the other hand, by the Plancherel formula, $p\in L^2(\mathbb{R}^{2n+1})$: indeed,

$$\begin{aligned}\|\widehat{p}\|^2_{L^2(\mathbb{R}^{2n+1})}&=\int_{\mathbb{R}}\frac{1}{(\cosh 2\lambda)^{2n}}\|e^{-\frac{\tanh 2\lambda}{2\lambda}\,\frac{t^2}{2}}\|^{4n}_{L^2(dt)}\,d\lambda\\&=c_n\int_{\mathbb{R}}\frac{1}{(\cosh 2\lambda)^{2n}}\left(\frac{2\lambda}{\tanh 2\lambda}\right)^{2n}d\lambda<\infty.\end{aligned}$$

Hence $p\in L^1(\mathbb{R}^{2n+1})$. In the same way, $\widehat{x_jp}=i\frac{\partial}{\partial\xi_j}\widehat{p}$ and $\widehat{up}=i\frac{\partial}{\partial\lambda}\widehat{p}$ are Fourier transforms of bounded measures, because so are, by (b), $i\frac{\partial}{\partial\xi}\widehat{\varphi}=\widehat{x\varphi}$ and $\frac{\partial}{\partial\lambda}\widehat{\varphi}=\widehat{u\varphi}$. ■

Proposition 4 implies:

THEOREM 8 ([G], [H]). *The heat kernel on H_n is $p_t=t^{-(n+1)}\delta_{t^{-1/2}}(p)$, i.e.*

$$\begin{aligned}&2\pi p_t(x_1,y_1,\ldots,x_n,y_n,u)\\&\quad=t^{-(n+1)}\int_{\mathbb{R}}e^{i\lambda u/t}\left(\frac{2\lambda}{2\pi\sinh 2\lambda}\right)^{n}e^{-(\lambda\coth 2\lambda)t^{-1}\sum_{j=1}^n(x_j^2+y_j^2)}\,d\lambda.\end{aligned}$$

We now come back to the *nonisotropic Heisenberg group H_n^a*. We recall that we must solve

$$(*)\qquad 2u\frac{\partial h}{\partial u}+(2n+2)h+\sum_{j=1}^{n}\left(\frac{\partial^2h}{\partial r_j^2}+4a_j^2r_j^2\frac{\partial^2h}{\partial u^2}+\left(r_j+\frac{1}{r_j}\right)\frac{\partial h}{\partial r_j}\right)=0$$

with $h\in L^1(\mathbb{R}_+^n\times\mathbb{R},(\prod_{j=1}^n r_jdr_j)du)$. By Fourier transform with respect to u we get

$$(**)\qquad -2\lambda\frac{\partial q}{\partial\lambda}+2nq+\sum_{j=1}^{n}\left(\frac{\partial^2q}{\partial r_j^2}+\left(r_j+\frac{1}{r_j}\right)\frac{\partial q}{\partial r_j}-4a_j^2r_j^2\lambda^2q\right)=0.$$

We look for a solution of $(**)$ of the form

$$q(r_1,\ldots,r_n,\lambda)=C(\lambda)\prod_{j=1}^{n}e^{-\alpha_j(\lambda)r_j^2/2}.$$

We get the differential equations

$$\alpha_j^2 - \alpha_j - 4\lambda^2 a_j^2 + \lambda\alpha_j' = 0, \qquad \frac{C'}{C} = \frac{n - \sum_{j=1}^n \alpha_j}{\lambda}, \qquad \alpha_j(0) = 1$$

with $\alpha_j(0) = 1$, $(2\pi)^n C(0) = 1$, whose solutions are, as before,

$$\alpha_j(\lambda) = 2\lambda a_j \coth 2\lambda a_j, \qquad C(\lambda) = \prod_{j=1}^n \frac{2\lambda a_j}{2\pi \sinh 2\lambda a_j}.$$

Hence

$$\widehat{p}(\xi_1, \dots, \xi_n, \eta_1, \dots, \eta_n, \lambda) = \prod_{j=1}^n \widehat{\varphi}(\xi_j, \lambda a_j)\widehat{\varphi}(\eta_j, \lambda a_j).$$

Using Lemma 7 and Proposition 4, we get, as in the isotropic case:

THEOREM 9 ([CT]). *The heat kernel on the nonisotropic Heisenberg group H_n^a is $p_t = t^{-(n+1)}\delta_{t^{-1/2}}(p)$, i.e.*

$$\begin{aligned} &2\pi p_t(x_1, y_1, \dots, x_n, y_n, u) \\ &= t^{-(n+1)} \int_{\mathbb{R}} e^{i\lambda u/t} \prod_{j=1}^n \frac{2\lambda a_j}{2\pi \sinh 2\lambda a_j} \exp\left(-\frac{1}{t}\sum_{j=1}^n (x_j^2 + y_j^2)\lambda a_j \coth 2\lambda a_j\right) d\lambda. \end{aligned}$$

II.B. Computation of the heat kernel on the group $N_{n,2}$. We consider the group $N_{n,2}$, whose Lie algebra is the step two nilpotent free algebra with n generators, i.e. $N_{n,2}$ is $\mathbb{R}^{n+C_n^2}$, $n \geq 1$, equipped with Lebesgue measure, and the group law

$$\begin{aligned} \gamma g &= (x_1, \dots, x_n, (u_{kl})_{1\leq k<l\leq n})(x_1', \dots, x_n', (u_{kl}')_{1\leq k<l\leq n}) \\ &= \left(x_1 + x_1', \dots, x_n + x_n', \left(u_{kl} + u_{kl}' + \frac{1}{2}(x_k x_l' - x_l x_k')\right)_{1\leq k<l\leq n}\right). \end{aligned}$$

The first layer has dimension n, the central one has dimension C_n^2, hence $N = n + 2C_n^2 = n^2$. A basis of the top layer is

$$X_j = \frac{\partial}{\partial x_j} + \frac{1}{2}\left(\sum_{1\leq k<j} x_k \frac{\partial}{\partial u_{kj}} - \sum_{j<l\leq n} x_l \frac{\partial}{\partial u_{jl}}\right), \qquad 1 \leq j \leq n,$$

and

$$[X_j, X_k] = \frac{\partial}{\partial u_{jk}} = U_{jk}, \qquad 1 \leq j < k \leq n.$$

On the other hand, L satisfies (see [G, p. 121])

$$-L - \sum_{j=1}^n \frac{\partial^2}{\partial x_j^2} + \sum_{j=1}^n \left(\sum_{k<j} x_k \frac{\partial}{\partial u_{kj}} - \sum_{j<l} x_l \frac{\partial}{\partial u_{jl}}\right)\frac{\partial}{\partial x_j} = \frac{1}{4}\sum_{j=1}^n R_j,$$

where

$$R_j = \sum_{k,l<j} x_k x_l \frac{\partial}{\partial u_{kj}} \frac{\partial}{\partial u_{lj}} + \sum_{k,l>j} x_k x_l \frac{\partial}{\partial u_{jk}} \frac{\partial}{\partial u_{jl}} - 2 \sum_{k<j<l} x_k x_l \frac{\partial}{\partial u_{kj}} \frac{\partial}{\partial u_{jl}}$$

and equation (H_1) reduces to

$$(*) \qquad -Lp + n^2 p + \sum_{j=1}^{n} x_j \frac{\partial p}{\partial x_j} + 2 \sum_{1\le k<l\le n} u_{kl} \frac{\partial p}{\partial u_{kl}} = 0.$$

Since L is formally self-adjoint, we must have $p(g) = p(g^{-1})$, which means that p is even on $\mathbb{R}^{n+C_n^2}$, and implies

$$\sum_{j=1}^{n} \left(\sum_{k<j} x_k \frac{\partial}{\partial u_{kj}} - \sum_{j<l} x_l \frac{\partial}{\partial u_{jl}} \right) \frac{\partial p}{\partial x_j} = 0.$$

We denote by $q(x_1, \ldots, x_n, (\lambda_{kl})_{1\le k<l\le n})$ the Fourier transform of p with respect to all u_{kl}. The Fourier transform of $u_{kl} \frac{\partial p}{\partial u_{kl}}$ is $-q - \lambda_{kl} \frac{\partial q}{\partial \lambda_{kl}}$, and the transform of R_j is $-(\Lambda X)_j^2 q$, where X is the column matrix of the coordinates $x_1, \ldots, x_n$, Λ is the $n \times n$ antisymmetric matrix with λ_{kl} above the diagonal and $(\Lambda X)_j$ denotes the jth coordinate of the column matrix ΛX. This gives

$$(**) \qquad nq - 2 \sum_{1\le k<l\le n} \lambda_{kl} \frac{\partial q}{\partial \lambda_{kl}} + \sum_{j=1}^{n} \frac{\partial^2 q}{\partial x_j^2} + \sum_{j=1}^{n} x_j \frac{\partial q}{\partial x_j} - \frac{1}{4} \|\Lambda X\|^2 q = 0.$$

Let $\Omega \in \mathrm{SO}(n)$ be an orthogonal matrix, and $X' = \Omega^* X$, $\Lambda' = \Omega^* \Lambda \Omega$. Obviously $\sum_{j=1}^{n} x_j \frac{\partial q}{\partial x_j}$, $\sum_{j=1}^{n} \frac{\partial^2 q}{\partial x_j^2}$, $\|\Lambda X\|$ are invariant under such a change of variable, and so is $\sum_{1\le k<l\le n} \lambda_{kl} \frac{\partial q}{\partial \lambda_{kl}}$, because if $(\lambda_{kl})_{1\le k<l\le n}$ and the overdiagonal entries $(\lambda'_{kl})_{1\le k<l\le n}$ of Λ' are identified with vectors in $\mathbb{R}^{n(n-1)/2}$, the mapping $\Lambda \mapsto \Lambda'$ defines a linear isometry T_Ω of $\mathbb{R}^{n(n-1)/2}$ equipped with the euclidean norm. Hence, by unicity of $L^1(G)$ solutions of $(*)$ with integral 1,

$$q(X, (\lambda_{kl})_{k<l}) = q(\Omega^* X, T_\Omega (\lambda_{kl})_{k<l}).$$

It follows, by inverse Fourier transform, that

$$p(X, (u_{kl})_{1\le k<l\le n}) = p(\Omega X, T_\Omega (u_{kl})_{1\le k<l\le n}).$$

We now use the same trick as in [G]. There exists an orthogonal matrix $\Omega \in \mathrm{SO}(n)$, which depends on Λ, such that

$$\Lambda = \Omega M \Omega^*,$$

where M is block diagonal: if n is even, each block of M is a 2×2 matrix

$$\mu_h \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \qquad 1 \le h \le n/2;$$

if n is odd, there are $[n/2]$ such blocks and the last one is the 1×1 zero matrix.

Let $Y=\Omega^* X$. We define

$$q(x_1,\dots,x_n,(\lambda_{kl})_{1\le k<l\le n})=Q(y_1,\dots,y_n,\mu_1,\dots,\mu_{[n/2]})$$

and get

$$nQ+\sum_{h=1}^{[n/2]}\left(-2\mu_h\,\frac{\partial Q}{\partial\mu_h}-\frac14\,\mu_h^2(y_{2h-1}^2+y_{2h}^2)Q\right)+\sum_{j=1}^{n}\left(\frac{\partial^2 Q}{\partial y_j^2}+y_j\,\frac{\partial Q}{\partial y_j}\right)=0.$$

Using polar coordinates r_h,θ_h instead of y_{2h-1},y_{2h}, $1\le h\le[n/2]$, denoting by R the function obtained from Q by this change of variable, we rewrite $(**)$ if n is even as

$$(***)\qquad nR+\sum_{h=1}^{n/2}\frac{\partial^2 R}{\partial r_h^2}+\left(\frac{1}{r_h}+r_h\right)\frac{\partial R}{\partial r_h}-\frac14\,\mu_h^2 r_h^2 R-2\mu_h\,\frac{\partial R}{\partial\mu_h}=0.$$

Since this looks like equation $(**)$ for the nonisotropic case, we look for a solution

$$R(r_1,\dots,r_{n/2},\mu_1,\dots,\mu_{n/2})=\prod_{h=1}^{n/2}C_h(\mu_h)e^{-\alpha_h(\mu_h)r_h^2/2},$$

which yields the differential equations

$$\alpha_h^2-\alpha_h-\frac14\,\mu_h^2+\mu_h\alpha_h'=0,\qquad \frac{C_h'}{C_h}=\frac{1-\alpha_h}{\mu_h}$$

with $\alpha_h(0)=1, 2\pi C_h(0)=1$. We get as in the previous cases

$$Q(y_1,\dots,y_n,\mu_1,\dots,\mu_{n/2})=\prod_{h=1}^{n/2}\frac{\frac12\mu_h}{2\pi\sinh\frac12\mu_h}\,e^{-(\frac12\mu_h\coth\frac12\mu_h)\frac12(y_{2h-1}^2+y_{2h}^2)}.$$

If n is odd, we get in the same way

$$Q(y_1,\dots,y_n,\mu_1,\dots,\mu_{[n]/2})$$
$$=\frac{1}{\sqrt{2\pi}}\,e^{-y_n^2/2}\prod_{h=1}^{[n]/2}\frac{\frac12\mu_h}{2\pi\sinh\frac12\mu_h}\,e^{-(\frac12\mu_h\coth\frac12\mu_h)\frac12(y_{2h-1}^2+y_{2h}^2)}.$$

THEOREM 10 ([G], [C]). *The heat kernel on $N_{n,2}$ is $p_t=t^{-n^2/2}\delta_{t^{-1/2}}(p)$. Writing $\Lambda=\Omega^*M\Omega$ as explained above, let*

$$P_h(t,Y,M)=\frac{\frac12\mu_h}{2\pi\sinh\frac12\mu_h}\,e^{-\frac{1}{2t}(y_{2h-1}^2+y_{2h}^2)\frac12\mu_h\coth\frac12\mu_h},\qquad 1\le h\le\frac{[n]}{2}.$$

Then $p_t(x_1,\ldots,x_n,(u_{kl})_{k<l})$ *is given, for even* n, *by*

$$(2\pi)^{-n(n-1)/2}t^{-n^2/2}\int\limits_{\mathbb{R}^{n(n-1)/2}} e^{\frac{i}{t}\sum_{1\le k<l\le n}\lambda_{kl}u_{kl}}\prod_{h=1}^{n/2}P_h(t,\Omega^*X,M)\prod_{1\le k<l\le n}d\lambda_{kl},$$

and for odd n, *by*

$$(2\pi)^{-n(n-1)/2}t^{-n^2/2}$$
$$\times\int\limits_{\mathbb{R}^{n(n-1)/2}} e^{\frac{i}{t}\sum_{1\le k<l\le n}\lambda_{kl}u_{kl}}\frac{1}{\sqrt{2\pi}}\,e^{-\frac{(\Omega^*X)_n^2}{2t}}\prod_{h=1}^{[n/2]}P_h(t,\Omega^*X,M)\prod_{1\le k<l\le n}d\lambda_{kl}.$$

Proof. We have not been able to prove directly that p, x_jp, $u_{kl}p$ belong to $L^1(\mathbb{R}^{n+C_n^2})$, so we will use Proposition 6. If n is even, let $\widehat{p}$ be the Fourier transform of q with respect to $x_1,\ldots,x_n$:

$$\begin{aligned}\widehat{p}(\xi_1,\ldots,\xi_n,(\lambda_{kl})_{k<l})&\\ &=\int\limits_{\mathbb{R}^n}e^{-i\sum_{j=1}^n(\Omega^*\xi)_jy_j}Q(y_1,\ldots,y_n,\mu_1,\ldots,\mu_{n/2})\,dy_1\ldots dy_n\\ &=\prod_{h=1}^{n/2}\widehat{\varphi}\bigg((\Omega^*\xi)_{2h-1},\frac{1}{4}\,\mu_h\bigg)\widehat{\varphi}\bigg((\Omega^*\xi)_{2h},\frac{1}{4}\,\mu_h\bigg),\end{aligned}$$

where ξ is the column matrix of coordinates $\xi_1,\ldots,\xi_n$ and $\widehat{\varphi}$ has been computed in the proof of Lemma 7. Obviously, $|\widehat{p}|\le 1=\widehat{p}(0,\ldots,0)$ and $\widehat{p}$ is a continuous function because so is $\widehat{\varphi}$ and we may choose the eigenvalues μ_h and the matrix Ω of corresponding eigenvectors as continuous functions of Λ. The verification is analogous if n is odd, which ends the proof. ∎

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Received 5 November 2001;
revised 17 September 2003 (4128)