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DISSIDENT MAPS ON THE SEVEN-DIMENSIONAL EUCLIDEAN SPACE

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Abstract. Our article contributes to the classification of dissident maps on $\mathbb{R}^7$, which in turn contributes to the classification of 8-dimensional real division algebras.

We study two large classes of dissident maps on $\mathbb{R}^7$. The first class is formed by all *composed* dissident maps, obtained from a vector product on $\mathbb{R}^7$ by composition with a definite endomorphism. The second class is formed by all *doubled* dissident maps, obtained as the purely imaginary parts of the structures of those 8-dimensional real quadratic division algebras which arise from a 4-dimensional real quadratic division algebra by doubling. For each of these two classes we exhibit a complete (but redundant) classification, given by a 49-parameter family of composed dissident maps and a 9-parameter family of doubled dissident maps respectively. The intersection of these two classes forms one isoclass of dissident maps only, namely the isoclass consisting of all vector products on $\mathbb{R}^7$.

1. Introduction. A *dissident map* on a finite-dimensional Euclidean vector space V is understood to be a linear map $\eta : V \wedge V \to V$ such that $v, w, \eta(v \wedge w)$ are linearly independent whenever $v, w \in V$ are. The notion of a dissident map provides a link between seemingly diverse aspects of real geometric algebra, thereby revealing its shifting significance. While it generalizes on the one hand the classical notion of a vector product, it specializes on the other hand the structure of a real division algebra. Moreover it yields naturally a large class of selfbijections of the projective space $\mathbb{P}(V)$, many of which are collineations, but some of which, surprisingly, are not.

Dissident maps are known to exist in dimensions 0, 1, 3 and 7 only. In dimensions 0 and 1 they are trivial. In dimension 3 they are classified completely and irredundantly. But in dimension 7 they are still far from fully understood.

Our article investigates dissident maps on a 7-dimensional Euclidean space by studying and separating two classes of them, namely the composed dissident maps and the doubled dissident maps. Once an exhaustive 49-parameter family of composed dissident maps on $\mathbb{R}^7$ and an exhaustive 9-parameter family of doubled dissident maps on $\mathbb{R}^7$ are obtained, the problem

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of restricting these two families so as to obtain a complete and irredundant classification of the two classes arises naturally. As regards characterizing when two composed dissident maps belonging to the 49-parameter family are isomorphic, we present a necessary and sufficient criterion. Regarding the analogous subproblem for the exhaustive 9-parameter family of doubled dissident maps, we present a sufficient criterion, which is conjectured, and partially proved, to be also necessary. Finally we describe those dissident maps which are both composed and doubled by proving that these coincide with the single isoclass formed by all vector products on $\mathbb{R}^7$.

For the reader's convenience we start by summarizing from the rudimentary theory of dissident maps which already appeared in print those features which the present article builds upon. For proofs and further information we refer to [12]–[18].

Throughout this article, a *Euclidean space* V is understood to be a finite-dimensional Euclidean vector space $V = (V, \langle\ \rangle)$. By a *division algebra* we mean an algebra A satisfying $0 < \dim A < \infty$ and having no zero divisors (i.e. $xy = 0$ only if $x = 0$ or $y = 0$). By a *quadratic algebra* we mean an algebra A such that $0 < \dim A < \infty$, there exists an identity element $1 \in A$ and each $x \in A$ satisfies an equation $x^2 = \alpha x + \beta 1$ with coefficients α, β in the ground field. A morphism of quadratic algebras is a linear map respecting both the multiplications and the identity elements of the quadratic algebras involved.

Now let us explain in which sense dissident maps specialize real division algebras. A *dissident triple* (V, ξ, η) consists of a Euclidean space V, a linear form $\xi : V \wedge V \to \mathbb{R}$ and a dissident map $\eta : V \wedge V \to V$. A morphism $\sigma : (V, \xi, \eta) \to (V', \xi', \eta')$ of dissident triples is an orthogonal map $\sigma : V \to V'$ satisfying both $\xi = \xi'(\sigma \wedge \sigma)$ and $\sigma\eta = \eta'(\sigma \wedge \sigma)$. Each dissident triple (V, ξ, η) determines a real quadratic division algebra $\mathcal{H}(V, \xi, \eta) = \mathbb{R} \times V$, with multiplication

$$(\alpha, v)(\beta, w) = (\alpha\beta - \langle v, w\rangle + \xi(v \wedge w), \alpha w + \beta v + \eta(v \wedge w)).$$

The assignment $(V, \xi, \eta) \mapsto \mathcal{H}(V, \xi, \eta)$ establishes a functor $\mathcal{H} : \mathcal{D} \to \mathcal{Q}$ from the category $\mathcal{D}$ of all dissident triples to the category $\mathcal{Q}$ of all real quadratic division algebras.

Proposition 1.1 [15, p. 3162]. *The functor $\mathcal{H} : \mathcal{D} \to \mathcal{Q}$ is an equivalence of categories.*

This proposition summarizes in categorical language old observations made by Frobenius [19] (cf. [24]), Dickson [11] and Osborn [29]. In order to describe an equivalence $\mathcal{I} : \mathcal{Q} \to \mathcal{D}$ which is quasi-inverse to $\mathcal{H} : \mathcal{D} \to \mathcal{Q}$, we need to recall the manner in which every real quadratic division algebra B is endowed with a natural scalar product. Frobenius's Lemma [24, p. 187] states that, for each real quadratic algebra B, the set

$$V = \{b \in B \mid b^2 \in \mathbb{R}1\} \setminus (\mathbb{R}1 \setminus \{0\})$$

of all purely imaginary elements is a linear subspace such that $B = \mathbb{R}1 \oplus V$. This decomposition of B determines a linear form $\varrho : B \to \mathbb{R}$ and a linear map $\iota : B \to V$ such that $b = \varrho(b)1 + \iota(b)$ for all $b \in B$. These in turn give rise to a quadratic form $q : B \to \mathbb{R}$, $q(b) = \varrho(b)^2 - \varrho(\iota(b)^2)$, and a linear map $\eta : V \wedge V \to V$, $\eta(v \wedge w) = \iota(vw)$. Now Osborn's Theorem [29, p. 204] asserts that B has no zero divisors if and only if q is positive definite and η is dissident. Therefore, whenever B is a real quadratic division algebra, then its purely imaginary hyperplane V is a Euclidean space $V = (V, \langle\ \rangle)$ with scalar product $\langle v, w\rangle = \frac{1}{2}(q(v+w) - q(v) - q(w)) = -\frac{1}{2}\varrho(vw + wv)$. Finally we define the linear form $\xi : V \wedge V \to \mathbb{R}$ by $\xi(v \wedge w) = \frac{1}{2}\varrho(vw - wv)$ to establish a functor $\mathcal{I} : \mathcal{Q} \to \mathcal{D}$, $\mathcal{I}(B) = (V, \xi, \eta)$.

Proposition 1.2 [15, p. 3162]. *The functor $\mathcal{I} : \mathcal{Q} \to \mathcal{D}$ is an equivalence of categories which is quasi-inverse to $\mathcal{H} : \mathcal{D} \to \mathcal{Q}$.*

Combining Proposition 1.1 with the famous theorem of Bott, Milnor [8] and Kervaire [23], asserting that each real division algebra has dimension 1, 2, 4 or 8, we obtain the following corollary.

Corollary 1.3. *If a Euclidean space V admits a dissident map $\eta : V \wedge V \to V$, then* $\dim V \in \{0, 1, 3, 7\}$.

In case $\dim V \in \{0, 1\}$, the zero map $o : V \wedge V \to V$ is the uniquely determined dissident map on V. In case $\dim V \in \{3, 7\}$, the simplest example of a dissident map on V is provided by the purely imaginary part of the structure of the real alternative division algebra $\mathbb{H}$ respectively $\mathbb{O}$ [26]. This dissident map $\pi : V \wedge V \to V$ has in fact the very special properties of a vector product (cf. Section 4, paragraph preceding Proposition 4.6). It serves as a starting point for the production of a multitude of further dissident maps, in view of the following result.

Proposition 1.4 [13, p. 19], [15, p. 3163]. *Let V be a Euclidean space endowed with a vector product $\pi : V \wedge V \to V$.*

(i) *If $\varepsilon : V \to V$ is a definite linear endomorphism*, *then $\varepsilon\pi : V \wedge V \to V$ is dissident.*

(ii) *If* $\dim V = 3$ *and $\eta : V \wedge V \to V$ is dissident*, *then there exists a unique definite linear endomorphism $\varepsilon : V \to V$ such that $\varepsilon\pi = \eta$.*

We define a *composed* dissident map to be any dissident map η on a Euclidean space V that admits a factorization $\eta = \varepsilon\pi$ into a vector product π on V and a definite linear endomorphism ε of V. By Proposition 1.4(ii), every dissident map on a 3-dimensional Euclidean space is composed. This fact leads to a complete and irredundant classification of all dissident maps on $\mathbb{R}^3$ [13, p. 21]. What is more, it even leads to a complete and irredundant classification of all 3-dimensional dissident triples and thus, in view of Proposition 1.1, also to a complete and irredundant classification of all 4-dimensional real quadratic division algebras. This assertion is made more precise in Proposition 1.5 below, whose formulation in turn requires further machinery.

First we need to recall the category $\mathcal{K}$ of configurations in $\mathbb{R}^3$ which recurs as a central theme in the series of articles [12]–[18]. Set $\mathcal{T} = \{d \in \mathbb{R}^3 \mid 0 < d_1 \leq d_2 \leq d_3\}$ and denote, for any $d \in \mathcal{T}$, by D_d the diagonal matrix in $\mathbb{R}^{3\times 3}$ with diagonal sequence d. The object set $\mathcal{K} = \mathbb{R}^3 \times \mathbb{R}^3 \times \mathcal{T}$ is endowed with the structure of a category by declaring as morphisms $S : (x, y, d) \to (x', y', d')$ those special orthogonal matrices $S \in SO_3(\mathbb{R})$ satisfying $(Sx, Sy, SD_dS^t) = (x', y', D_{d'})$. Note that the existence of a morphism $(x, y, d) \to (x', y', d')$ in $\mathcal{K}$ implies $d = d'$. The term "category of configurations" originates from the geometric interpretation of $\mathcal{K}$ obtained by identifying the objects $(x, y, d) \in \mathcal{K}$ with those configurations in $\mathbb{R}^3$ which are composed of a pair of points (x, y) and an ellipsoid $E_d = \{z \in \mathbb{R}^3 \mid z^tD_dz = 1\}$ in normal position. Then, if we identify $SO_3(\mathbb{R})$ with $SO(\mathbb{R}^3)$, the morphisms $(x, y, d) \to (x', y', d')$ in $\mathcal{K}$ are identified with those rotations of $E_d = E_{d'}$ which simultaneously send x to x' and y to y'.

Next we recall the functor $\mathcal{G} : \mathcal{K} \to \mathcal{D}_3$, where $\mathcal{D}_3$ denotes the full subcategory of $\mathcal{D}$ formed by all 3-dimensional dissident triples. To begin with, $\pi_3 : \mathbb{R}^3 \wedge \mathbb{R}^3 \xrightarrow{\sim} \mathbb{R}^3$ denotes the linear isomorphism identifying the standard basis (e_1, e_2, e_3) in $\mathbb{R}^3$ with its associated basis $(e_2 \wedge e_3, e_3 \wedge e_1, e_1 \wedge e_2)$ in $\mathbb{R}^3 \wedge \mathbb{R}^3$. Note that π_3 is a vector product on $\mathbb{R}^3$, henceforth to be referred to as the *standard* vector product on $\mathbb{R}^3$ (cf. Section 4, paragraph preceding Proposition 4.6). Every $x \in \mathbb{R}^3$ determines an antisymmetric linear endomorphism $\mu_x = \pi_3(x \wedge ?)$ of $\mathbb{R}^3$. Every $d \in \mathcal{T}$ determines a symmetric positive definite linear endomorphism $\delta_d = D_d?$ of $\mathbb{R}^3$. Every pair $(y, d) \in \mathbb{R}^3 \times \mathcal{T}$ determines a positive definite linear endomorphism $\varepsilon_{yd} = \mu_y + \delta_d$ of $\mathbb{R}^3$. Now the construction $\mathcal{G} : \mathcal{K} \to \mathcal{D}_3$, associating with any given configuration $\kappa = (x, y, d) \in \mathcal{K}$ the dissident triple $\mathcal{G}(\kappa) = (\mathbb{R}^3, \xi_x, \eta_{yd})$ defined by $\xi_x(v \wedge w) = v^t\mu_x(w)$ and $\eta_{yd} = \varepsilon_{yd}\pi_3$, is in fact functorial, acting on morphisms identically.

PROPOSITION 1.5 [18, Propositions 2.3 and 3.1]. *The functor $\mathcal{G} : \mathcal{K} \to \mathcal{D}_3$ is an equivalence of categories.*

Thus the problem of classifying $\mathcal{D}_3/\simeq$ is equivalent to the problem of describing a cross-section $\mathcal{C}$ for the set $\mathcal{K}/\simeq$ of isoclasses of configurations. Such a cross-section was first presented in [12, pp. 17–18] (see also [18, pp. 294–295]).

Let us now turn to the composed dissident maps on a 7-dimensional Euclidean space. Although here our knowledge is not as complete as in dimension 3, we do know an exhaustive 49-parameter family and we are able to characterize when two composed dissident maps belonging to this family are isomorphic. This is made precise in Proposition 1.6 below, whose formulation once more requires further notation.

The object class $\mathcal{E} = \{(V, \eta) \mid \eta : V \wedge V \to V$ is a dissident map on a Euclidean space $V\}$ is endowed with the structure of a category by declaring as morphisms $\sigma : (V, \eta) \to (V', \eta')$ those orthogonal maps $\sigma : V \to V'$ satisfying $\sigma\eta = \eta'(\sigma \wedge \sigma)$. Occasionally we simply write η to denote an object $(V, \eta) \in \mathcal{E}$. By $\mathbb{R}^{7\times 7}_{\mathrm{ant}} \times \mathbb{R}^{7\times 7}_{\mathrm{syp}}$ we denote the set of all pairs (Y, D) of real 7×7-matrices such that Y is antisymmetric and D is symmetric and positive definite. The orthogonal group $O(\mathbb{R}^7)$ acts canonically from the left on the set of all vector products on $\mathbb{R}^7$, via $\sigma \cdot \pi = \sigma\pi(\sigma^{-1} \wedge \sigma^{-1})$. By $O_\pi(\mathbb{R}^7) = \{\sigma \in O(\mathbb{R}^7) \mid \sigma \cdot \pi = \pi\}$ we denote the isotropy subgroup of $O(\mathbb{R}^7)$ associated with a fixed vector product π on $\mathbb{R}^7$. In fact, $O_\pi(\mathbb{R}^7)$ is a compact, connected simple real Lie group of dimension 14 and therefore an exceptional compact Lie group of type $\mathbb{G}_2$ (cf. [30, Theorem 11.33]). We denote by π_7 the standard vector product on $\mathbb{R}^7$, as defined in Section 4, paragraph preceding Proposition 4.6.

Proposition 1.6 [13, p. 20], [15, p. 3164]. (i) *For each matrix pair* $(Y, D) \in \mathbb{R}^{7\times 7}_{\mathrm{ant}} \times \mathbb{R}^{7\times 7}_{\mathrm{syp}}$, *the linear map* $\eta_{YD} : \mathbb{R}^7 \wedge \mathbb{R}^7 \to \mathbb{R}^7$, *given by* $\eta_{YD}(v \wedge w) = (Y + D)\pi_7(v \wedge w)$ *for all* $(v, w) \in \mathbb{R}^7 \times \mathbb{R}^7$, *is a composed dissident map on* $\mathbb{R}^7$.

(ii) *Each composed dissident map* η *on a* 7*-dimensional Euclidean space is isomorphic to* η_{YD} *for some matrix pair* $(Y, D) \in \mathbb{R}^{7\times 7}_{\mathrm{ant}} \times \mathbb{R}^{7\times 7}_{\mathrm{syp}}$.

(iii) *For all matrix pairs* (Y, D) *and* (Y', D') *in* $\mathbb{R}^{7\times 7}_{\mathrm{ant}} \times \mathbb{R}^{7\times 7}_{\mathrm{syp}}$, *the composed dissident maps* η_{YD} *and* $\eta_{Y'D'}$ *are isomorphic if and only if* $(SYS^t, SDS^t) = (Y', D')$ *for some* $S \in O_{\pi_7}(\mathbb{R}^7)$.

Knowing that all dissident maps in dimensions 0, 1 and 3 are composed and observing the analogies between dissident maps in dimension 3 and composed dissident maps in dimension 7, the reader may wonder whether, even in dimension 7, every dissident map might be composed. This is not the case! The exceptional phenomenon of non-composed dissident maps, occurring in dimension 7 only, was first pointed out in [16, p. 1]. Here we shall prove it (cf. Section 4), even though not along the lines sketched in [16]. Instead our proof will emerge from the investigation of doubled dissident maps, another class of dissident maps which we proceed to introduce.

Recall that the *double* of a real quadratic algebra A is defined by $\mathcal{V}(A) = A \times A$ with multiplication $(w, x)(y, z) = (wy - \overline{z}x, x\overline{y} + zw)$, where $\overline{y}, \overline{z}$ denote the conjugates of y, z. The doubling construction provides an endofunctor $\mathcal{V}$ of the category of all real quadratic algebras, acting on morphisms by $\mathcal{V}(\varphi) = \varphi \times \varphi$ ([1]). In particular, the property of being quadratic is preserved under doubling. The additional property of having no zero divisors behaves under doubling as follows.

PROPOSITION 1.7 [14, p. 946]. *If A is a real quadratic division algebra and* $\dim A \leq 4$, *then* $\mathcal{V}(A)$ *is again a real quadratic division algebra.*

A real quadratic division algebra B will be called *doubled* if it admits an isomorphism $B \xrightarrow{\sim} \mathcal{V}(A)$ for some real quadratic division algebra A. Moreover, a dissident triple (V, ξ, η) will be called *doubled* if it admits an isomorphism $(V, \xi, \eta) \xrightarrow{\sim} \mathcal{IV}(A)$ for some real quadratic division algebra A. Finally, a dissident map η will be called *doubled* if it occurs as third component of a doubled dissident triple (V, ξ, η).

We are now in a position to describe the set-up of the present article. In Section 2 we prove that the selfmap $\eta_{\mathbb{P}} : \mathbb{P}(V) \to \mathbb{P}(V)$ induced by a dissident map $\eta : V \wedge V \to V$, introduced in [13, p. 19] and [15, p. 3163], is always bijective (Proposition 2.2). We also observe that $\eta_{\mathbb{P}}$ is collinear whenever η is composed dissident (Proposition 2.4). In Section 3 we exhibit a 9-parameter family of linear maps $\underline{\mathcal{Y}}(\kappa) : \mathbb{R}^7 \wedge \mathbb{R}^7 \to \mathbb{R}^7,\ \kappa \in \mathcal{K}$, which exhausts all isoclasses of 7-dimensional doubled dissident maps (Proposition 3.2(i),(ii)). Regarding the problem of characterizing when two doubled dissident maps $\underline{\mathcal{Y}}(\kappa)$ and $\underline{\mathcal{Y}}(\kappa')$ are isomorphic, the criterion $\kappa \xrightarrow{\sim} \kappa'$ is proved to be sufficient (Proposition 3.2(iii)) and conjectured to be necessary (Conjecture 3.3). In Section 4 we work with the exhaustive family $(\underline{\mathcal{Y}}(\kappa))_{\kappa \in \mathcal{K}}$ to prove that $\underline{\mathcal{Y}}(\kappa)_{\mathbb{P}}$ is collinear if and only if κ is formed by a double point at the origin and a sphere centred at the origin (Proposition 4.5). This implies that the dissident maps which are both composed and doubled form three isoclasses only, represented by the standard vector products on $\mathbb{R}$, $\mathbb{R}^3$ and $\mathbb{R}^7$ respectively (Corollary 4.7). In Section 5 we make inroads into a possible proof of Conjecture 3.3 by decomposing the problem into several subproblems (Proposition 5.3) and solving the simplest ones (Propositions 5.6–5.8). A complete proof of Conjecture 3.3 lies beyond the frame of the present article and is therefore postponed to a future publication. In Section 6 we summarize our results from the viewpoint of the problem of classifying all real quadratic division algebras (Theorem 6.1). The epilogue embeds our article into its historical context.

We use the following notation, conventions and terminology. We follow Bourbaki in viewing 0 as the least natural number. For each $n \in \mathbb{N}$, we set $\underline{n} = \{i \in \mathbb{N} \mid 1 \leq i \leq n\}$. We denote by $\mathbb{R}^{m \times n}$ the vector space of

([1]) The notation "$\mathcal{V}$" comes from the German term "Verdoppelung".

all real matrices of size $m \times n$. In writing down matrices, omitted entries are understood to be zero entries. We set $\mathbb{R}^m = \mathbb{R}^{m\times 1}$. The standard basis in $\mathbb{R}^m$ is denoted by $\underline{e} = (e_1, \ldots, e_m)$. The columns $y \in \mathbb{R}^m$ correspond to the diagonal matrices $D_y \in \mathbb{R}^{m\times m}$ with diagonal sequence $(y_1, \ldots, y_m)$. We denote by $1_m = \sum_{i=1}^m e_i$ the column in $\mathbb{R}^m$ all of whose entries are 1, and by $\mathbb{I}_m = D_{1_m}$ the identity matrix in $\mathbb{R}^{m\times m}$. By M^t we mean the transpose of a matrix M. If $M \in \mathbb{R}^{m\times n}$, then we denote by $M_{i\bullet}$ the ith row of M, by $M_{\bullet j}$ the jth column of M and by M_{ij} the entry of M lying in the ith row and in the jth column. Moreover, $\underline{M} : \mathbb{R}^n \to \mathbb{R}^m$ denotes the linear map given by $\underline{M}(x) = Mx$ for all $x \in \mathbb{R}^n$.

For matrices of the special size 7×21 we slightly deviate from this general convention inasmuch as we shall, for each $Y \in \mathbb{R}^{7\times 21}$, denote by $\underline{Y} : \mathbb{R}^7 \wedge \mathbb{R}^7 \to \mathbb{R}^7$ the linear map represented by Y in the standard basis of $\mathbb{R}^7$ and an associated basis of $\mathbb{R}^7 \wedge \mathbb{R}^7$, defined in the first paragraph of Section 3. Accordingly we prefer double indices to index the column set of $Y \in \mathbb{R}^{7\times 21}$.

The symbol $[v_1, \ldots, v_l]$ stands for the linear hull of vectors $v_1, \ldots, v_l$ in a vector space V. We denote by $\mathbb{I}_X$ the identity map on a set X. Given any category $\mathcal{C}$ for which a function $\dim : \mathrm{Ob}(\mathcal{C}) \to \mathbb{N}$ is defined, and any $n \in \mathbb{N}$, we denote by $\mathcal{C}_n$ the full subcategory of $\mathcal{C}$ formed by $\dim^{-1}(n)$. Non-isomorphic objects in a category will be called *heteromorphic*. Two subclasses $\mathcal{A}$ and $\mathcal{B}$ of a category $\mathcal{C}$ are called *heteromorphic* if A and B are heteromorphic for all $(A, B) \in \mathcal{A} \times \mathcal{B}$. We set $\mathbb{R}_{>0} = \{\lambda \in \mathbb{R} \mid \lambda > 0\}$.

2. The selfbijection $\eta_{\mathbb{P}}$ induced by a dissident map η. Given any dissident map $\eta : V \wedge V \to V$ and $v, w \in V$, we adopt the short notation $vw = \eta(v \wedge w)$, $vv^\perp = v(v^\perp) = \{vx \mid x \in v^\perp\}$ and $\lambda_v : V \to V$, $x \mapsto vx$. Note that $vv^\perp = v(v^\perp + [v]) = vV = \mathrm{im}\,\lambda_v$. If $v \neq 0$, then the linear endomorphism $\lambda_v : V \to V$ induces a linear isomorphism $v^\perp \xrightarrow{\sim} vv^\perp$, by dissidence of η. Because the hyperplane $vv^\perp$ only depends on the line $[v]$ spanned by v, we infer that each dissident map $\eta : V \wedge V \to V$ induces a well-defined selfmap $\eta_{\mathbb{P}} : \mathbb{P}(V) \to \mathbb{P}(V)$, $\eta_{\mathbb{P}}[v] = (vv^\perp)^\perp$, of the real projective space $\mathbb{P}(V)$. The investigation of $\eta_{\mathbb{P}}$ proves to be significant in the study of dissident maps η. Our first result in this direction is Proposition 2.2 below. To prove it we need the following lemma.

LEMMA 2.1. *Let $\eta : V \wedge V \to V$ be a dissident map on a Euclidean space V. Then for each $v \in V \setminus \{0\}$, the linear endomorphism $\lambda_v : V \to V$ induces a linear automorphism $\lambda_v : vv^\perp \xrightarrow{\sim} vv^\perp$.*

Proof. Dissidence of η implies that $v \notin vv^\perp$. Accordingly $vv^\perp + [v] = V = v^\perp + [v]$, and therefore $\lambda_v(vv^\perp) = \lambda_v(vv^\perp + [v]) = \lambda_v(v^\perp + [v]) = \lambda_v(v^\perp) = vv^\perp$. Thus the linear endomorphism $\lambda_v : V \to V$ induces a linear endomorphism $\lambda_v : vv^\perp \to vv^\perp$ which is surjective, hence bijective. ∎

PROPOSITION 2.2. *For each dissident map $\eta : V \wedge V \to V$ on a Euclidean space V, the induced selfmap $\eta_{\mathbb{P}} : \mathbb{P}(V) \to \mathbb{P}(V)$ is bijective.*

Proof. If $\dim V \in \{0,1\}$, then $\eta_{\mathbb{P}}$ is trivially bijective. Due to Corollary 1.3 we may therefore assume that $\dim V \in \{3,7\}$.

Suppose $\eta_{\mathbb{P}}$ is not injective. Then we may choose non-proportional vectors $v, w \in V$ such that $vv^{\perp} = ww^{\perp}$. Set $E = [v,w]$, $H = vv^{\perp}$ and $D = E \cap H$. Then D is non-trivial, for dimension reasons. Choose $d \in D \setminus \{0\}$ and write $d = \alpha v + \beta w$ with $\alpha, \beta \in \mathbb{R}$. Then $dd^{\perp} = (\alpha v + \beta w)V \subset vV + wV = vv^{\perp} + ww^{\perp} = H$. Equality of dimensions implies $dd^{\perp} = H$. Thus $d \in dd^{\perp}$, contradicting the dissidence of η. Hence $\eta_{\mathbb{P}}$ is injective.

To prove that $\eta_{\mathbb{P}}$ is surjective, let $L \in \mathbb{P}(V)$ be given. Set $H = L^{\perp}$ and consider the short exact sequence

$$0 \to H \xrightarrow{\iota} V \xrightarrow{\psi} L \to 0$$

formed by the inclusion map ι and the orthogonal projection ψ. Then the map $\alpha : V \to \mathrm{Hom}_{\mathbb{R}}(H, L)$, $v \mapsto \psi\lambda_v\iota$, is linear and has non-trivial kernel, for dimension reasons. Thus we may choose $v \in \ker\alpha \setminus \{0\}$. Now it suffices to prove that $vv^{\perp} = H$. To do that, consider $I = vv^{\perp} \cap H$. The linear endomorphism $\lambda_v : V \to V$ induces both a linear automorphism $\lambda_v : vv^{\perp} \xrightarrow{\sim} vv^{\perp}$ (Lemma 2.1) and a linear endomorphism $\lambda_v : H \to H$ (since $v \in \ker\alpha$), hence a linear automorphism $\lambda_v : I \xrightarrow{\sim} I$. If now $vv^{\perp} \neq H$, then $\dim I \in \{1,5\}$ and therefore $\lambda_v : I \xrightarrow{\sim} I$ has a non-zero eigenvalue, contradicting the dissidence of η. Accordingly $vv^{\perp} = H$, i.e. $\eta_{\mathbb{P}}[v] = L$. ∎

Recall that a selfbijection $\psi : \mathbb{P}(V) \to \mathbb{P}(V)$ is called *collinear* (or synonymously a *collineation*) if $\dim(L_1 + L_2 + L_3) = 2$ implies $\dim(\psi(L_1) + \psi(L_2) + \psi(L_3)) = 2$ for all $L_1, L_2, L_3 \in \mathbb{P}(V)$. Each $\varphi \in GL(V)$ induces a collineation $\mathbb{P}(\varphi) : \mathbb{P}(V) \to \mathbb{P}(V)$, $\mathbb{P}(\varphi)(L) = \varphi(L)$. Following Proposition 2.2, the natural question arises whether the selfbijection $\eta_{\mathbb{P}}$ induced by a dissident map η is collinear. The answer turns out to depend on the isoclass of η only (Lemma 2.3). Moreover, the answer is positive for all composed dissident maps (Proposition 2.4), while for doubled dissident maps it is in general negative (Proposition 4.5).

LEMMA 2.3. *If $\sigma : (V, \eta) \xrightarrow{\sim} (V', \eta')$ is an isomorphism of dissident maps, then*

(i) $\mathbb{P}(\sigma) \circ \eta_{\mathbb{P}} = \eta'_{\mathbb{P}} \circ \mathbb{P}(\sigma)$,

(ii) *$\eta_{\mathbb{P}}$ is collinear if and only if $\eta'_{\mathbb{P}}$ is collinear.*

Proof. (i) For each $v \in V \setminus \{0\}$ we have $(\mathbb{P}(\sigma) \circ \eta_{\mathbb{P}})[v] = \sigma((\eta(v \wedge v^{\perp}))^{\perp}) = (\sigma\eta(v \wedge v^{\perp}))^{\perp} = (\eta'(\sigma(v) \wedge \sigma(v^{\perp})))^{\perp} = \eta'_{\mathbb{P}}[\sigma(v)] = (\eta'_{\mathbb{P}} \circ \mathbb{P}(\sigma))[v]$.

(ii) Since both $\mathbb{P}(\sigma)$ and $\mathbb{P}(\sigma)^{-1} = \mathbb{P}(\sigma^{-1})$ are collinear and the composition of collinear maps is collinear, (ii) follows directly from (i). ∎

PROPOSITION 2.4 [13, p. 19], [15, p. 3163]. *For each composed dissident map η on a Euclidean space V, the induced selfbijection $\eta_{\mathbb{P}} : \mathbb{P}(V) \to \mathbb{P}(V)$ is collinear. More precisely, the identity $\eta_{\mathbb{P}} = \mathbb{P}(\varepsilon^{-*})$ holds for any factorization $\eta = \varepsilon\pi$ of η into a vector product π on V and a definite linear endomorphism ε of V.*

3. Doubled dissident maps. The standard basis $\underline{e} = (e_1, e_2, e_3 \mid e_4 \mid e_5, e_6, e_7)$ in $\mathbb{R}^7$ gives rise to the subset $\{\pm e_i \wedge e_j \mid 1 \le i < j \le 7\}$ of $\mathbb{R}^7 \wedge \mathbb{R}^7$ which after any choice of signs and total order becomes a basis in $\mathbb{R}^7 \wedge \mathbb{R}^7$, denoted by $\underline{e} \wedge \underline{e}$. We choose signs and total order such that $\underline{e} \wedge \underline{e} = (e_{23}, e_{31}, e_{12} \mid e_{72}, e_{17}, e_{61} \mid e_{14}, e_{24}, e_{34} \mid e_{15}, e_{26}, e_{37} \mid e_{45}, e_{46}, e_{47} \mid e_{36}, e_{53}, e_{25} \mid e_{76}, e_{57}, e_{65})$, using the shorthand $e_{ij} = e_i \wedge e_j$. For each matrix $Y \in \mathbb{R}^{7\times 21}$ we denote by $\underline{Y} : \mathbb{R}^7 \wedge \mathbb{R}^7 \to \mathbb{R}^7$ the linear map represented by Y in the bases $\underline{e}$ and $\underline{e} \wedge \underline{e}$.

To build up an exhaustive 9-parameter family of doubled dissident maps on $\mathbb{R}^7$ we start from the category $\mathcal{K}$ of configurations in $\mathbb{R}^3$, described in the introduction. For each configuration $\kappa = (x, y, d) \in \mathcal{K}$ we set

$$E_{yd} = \begin{pmatrix} d_1 & -y_3 & y_2 \\ y_3 & d_2 & -y_1 \\ -y_2 & y_1 & d_3 \end{pmatrix},$$

$$\mathcal{Y}(\kappa) = \left(\begin{array}{c|c|c|c|c|c|c} E_{yd} & 0 & 0 & 0 & \mathbb{I}_3 & 0 & E_{yd} \\ \hline 0 & -x^t & 0 & -1_3^t & 0 & -x^t & 0 \\ \hline 0 & E_{yd} & \mathbb{I}_3 & 0 & 0 & E_{yd} & 0 \end{array}\right),$$

thus defining the map $\mathcal{Y} : \mathcal{K} \to \mathbb{R}^{7\times 21}$. Recall that $x^t = (x_1\ x_2\ x_3)$ and $1_3^t = (1\ 1\ 1)$. Note that the block partition of $\mathcal{Y}(\kappa)$ corresponds to the partitions of $\underline{e}$ and $\underline{e} \wedge \underline{e}$ respectively, indicated by "|" above. Also note that E_{yd} represents the positive definite linear endomorphism $\varepsilon_{yd} = \mu_y + \delta_d$ of $\mathbb{R}^3$ in the standard basis. Composing $\mathcal{Y}$ with the linear isomorphism

$$\mathbb{R}^{7\times 21} \xrightarrow{\sim} \mathrm{Hom}_{\mathbb{R}}(\mathbb{R}^7 \wedge \mathbb{R}^7, \mathbb{R}^7), \qquad Y \mapsto \underline{Y},$$

we obtain the map

$$\underline{\mathcal{Y}} : \mathcal{K} \to \mathrm{Hom}_{\mathbb{R}}(\mathbb{R}^7 \wedge \mathbb{R}^7, \mathbb{R}^7), \qquad \underline{\mathcal{Y}}(\kappa) = \underline{\mathcal{Y}(\kappa)}.$$

Some properties of $\underline{\mathcal{Y}}$ are collected in Proposition 3.2 below. To prove it we need an explicit description of the functor $\mathcal{J} : \mathcal{D}_3 \to \mathcal{D}_7$ defined as composition $\mathcal{J} = \mathcal{IVH}$ of the functors $\mathcal{H}, \mathcal{V}$ and $\mathcal{I}$ described in the introduction. Motivated by the commutative square

$$\begin{array}{ccc} \mathcal{D}_7 & \xleftarrow{\mathcal{I}} & \mathcal{Q}_8 \\ {\scriptstyle \mathcal{J}}\uparrow & & \uparrow{\scriptstyle \mathcal{V}} \\ \mathcal{D}_3 & \xrightarrow[\mathcal{H}]{} & \mathcal{Q}_4 \end{array}$$

we view $\mathcal{J}$ as doubling functor for dissident triples, corresponding to the doubling functor $\mathcal{V}$ for real quadratic division algebras by means of the equivalences of categories $\mathcal{H}$ and $\mathcal{I}$. Indeed, a dissident triple $(V, \xi, \eta) \in \mathcal{D}_7$ is doubled (in the sense defined in the introduction) if and only if $(V, \xi, \eta) \xrightarrow{\sim} \mathcal{J}(V_0, \xi_0, \eta_0)$ for some dissident triple $(V_0, \xi_0, \eta_0) \in \mathcal{D}_3$.

LEMMA 3.1. *The functor $\mathcal{J} : \mathcal{D}_3 \to \mathcal{D}_7$ admits the following explicit description.*

(i) *If (V, ξ, η) is an object in $\mathcal{D}_3$ and $\mathcal{J}(V, \xi, \eta) = (V^{\mathrm{d}}, \xi^{\mathrm{d}}, \eta^{\mathrm{d}})$, then*

$$V^{\mathrm{d}} = V \times \mathbb{R} \times V,$$

$$\xi^{\mathrm{d}}\left(\begin{pmatrix} v \\ \alpha \\ w \end{pmatrix} \wedge \begin{pmatrix} v' \\ \alpha' \\ w' \end{pmatrix}\right) = \xi(v \wedge v') - \xi(w \wedge w'),$$

$$\eta^{\mathrm{d}}\left(\begin{pmatrix} v \\ \alpha \\ w \end{pmatrix} \wedge \begin{pmatrix} v' \\ \alpha' \\ w' \end{pmatrix}\right) = \begin{pmatrix} \alpha w' - \alpha' w + \eta(v \wedge v') - \eta(w \wedge w') \\ -\langle v, w' \rangle + \langle w, v' \rangle - \xi(v \wedge w') - \xi(w \wedge v') \\ -\alpha v' + \alpha' v - \eta(v \wedge w') - \eta(w \wedge v') \end{pmatrix}$$

for all $\begin{pmatrix} v \\ \alpha \\ w \end{pmatrix}, \begin{pmatrix} v' \\ \alpha' \\ w' \end{pmatrix} \in V^{\mathrm{d}}$.

(ii) *If $\varphi : (V, \xi, \eta) \to (V', \xi', \eta')$ is a morphism in $\mathcal{D}_3$, then $\mathcal{J}(\varphi) = \varphi \times \mathbb{I}_{\mathbb{R}} \times \varphi$.*

Proof. Apply the functors $\mathcal{H}, \mathcal{V}$ and $\mathcal{I}$ successively (and identify the purely imaginary hyperplane $\{0\} \times V \times \mathbb{R} \times V$ in $\mathcal{VH}(V, \xi, \eta)$ with $V \times \mathbb{R} \times V$). ∎

PROPOSITION 3.2. (i) *For each configuration $\kappa \in \mathcal{K}$, the linear map $\underline{\mathcal{Y}}(\kappa)$ is a doubled dissident map on $\mathbb{R}^7$.*

(ii) *Each doubled dissident map η on a 7-dimensional Euclidean space is isomorphic to $\underline{\mathcal{Y}}(\kappa)$ for some configuration $\kappa \in \mathcal{K}$.*

(iii) *If κ and κ' are isomorphic configurations in $\mathcal{K}$, then $\underline{\mathcal{Y}}(\kappa)$ and $\underline{\mathcal{Y}}(\kappa')$ are isomorphic doubled dissident maps.*

Proof. (i) Given $\kappa = (x, y, d) \in \mathcal{K}$, consider $\mathcal{G}(\kappa) = (\mathbb{R}^3, \xi_x, \eta_{yd})$ (cf. introduction) and $\mathcal{J}(\mathbb{R}^3, \xi_x, \eta_{yd}) = ((\mathbb{R}^3)^{\mathrm{d}}, \xi_x^{\mathrm{d}}, \eta_{yd}^{\mathrm{d}})$. Then η_{yd}^{d} is a doubled dissident map on $(\mathbb{R}^3)^{\mathrm{d}}$. Lemma 3.1(i) shows that $(\mathbb{R}^3)^{\mathrm{d}} = \mathbb{R}^7$ and $\eta_{yd}^{\mathrm{d}} = \underline{\mathcal{Y}}(\kappa)$.

(ii) Given a doubled dissident map $\eta : V \wedge V \to V$ with $\dim V = 7$, there exist a linear form $\xi : V \wedge V \to \mathbb{R}$ and a dissident triple $(V_0, \xi_0, \eta_0) \in \mathcal{D}_3$ such that $(V, \xi, \eta) \xrightarrow{\sim} \mathcal{J}(V_0, \xi_0, \eta_0)$. By Proposition 1.5 there exists a configuration $\kappa = (x, y, d) \in \mathcal{K}$ such that $(V_0, \xi_0, \eta_0) \xrightarrow{\sim} \mathcal{G}(\kappa) = (\mathbb{R}^3, \xi_x, \eta_{yd})$. Hence $(V, \xi, \eta) \xrightarrow{\sim} \mathcal{J}(\mathbb{R}^3, \xi_x, \eta_{yd}) = (\mathbb{R}^7, \xi_x^{\mathrm{d}}, \eta_{yd}^{\mathrm{d}})$, where $\eta_{yd}^{\mathrm{d}} = \underline{\mathcal{Y}}(\kappa)$. So $\eta \xrightarrow{\sim} \underline{\mathcal{Y}}(\kappa)$.

(iii) If $\kappa = (x, y, d)$ and $\kappa' = (x', y', d')$ are isomorphic configurations, then $(\mathbb{R}^7, \xi_x^{\mathrm{d}}, \eta_{yd}^{\mathrm{d}}) = \mathcal{J}\mathcal{G}(\kappa) \xrightarrow{\sim} \mathcal{J}\mathcal{G}(\kappa') = (\mathbb{R}^7, \xi_{x'}^{\mathrm{d}}, \eta_{y'd'}^{\mathrm{d}})$ and hence $\underline{\mathcal{Y}}(\kappa) = \eta_{yd}^{\mathrm{d}} \xrightarrow{\sim} \eta_{y'd'}^{\mathrm{d}} = \underline{\mathcal{Y}}(\kappa')$. ■

We conjecture that the converse of Proposition 3.2(iii) also holds true.

CONJECTURE 3.3. *If κ and κ' are configurations in $\mathcal{K}$ such that $\underline{\mathcal{Y}}(\kappa)$ and $\underline{\mathcal{Y}}(\kappa')$ are isomorphic, then κ and κ' are isomorphic.*

If we denote by $\mathcal{E}^{\mathrm{d}}$ the full subcategory of $\mathcal{E}$ formed by all doubled dissident maps, Proposition 3.2 can be rephrased by stating that the map $\underline{\mathcal{Y}} : \mathcal{K} \to \mathrm{Hom}_{\mathbb{R}}(\mathbb{R}^7 \wedge \mathbb{R}^7, \mathbb{R}^7)$ induces a map $\underline{\mathcal{Y}} : \mathcal{K} \to \mathcal{E}_7^{\mathrm{d}}$ which in turn induces a surjection $\overline{\underline{\mathcal{Y}}} : \mathcal{K}/\simeq \to \mathcal{E}_7^{\mathrm{d}}/\simeq$. The validity of Conjecture 3.3 would imply that $\overline{\underline{\mathcal{Y}}}$ is in fact a bijection. Thus it would solve the problem of classifying all doubled dissident maps, because, starting from the known cross-section $\mathcal{C}$ for $\mathcal{K}/\simeq$ (cf. [12, pp. 17–18], [18, pp. 294–295]), one would obtain the cross-section $\underline{\mathcal{Y}}(\mathcal{C})$ for $\mathcal{E}_7^{\mathrm{d}}/\simeq$.

The obstacle to proving Conjecture 3.3 comes from the fact that the doubling functor $\mathcal{V} : \mathcal{Q}_4 \to \mathcal{Q}_8$ is indeed faithful, but not full. Nevertheless there is evidence for the truth of Conjecture 3.3 (cf. Section 5).

4. Doubled dissident maps η with collinear $\eta_{\mathbb{P}}$. While we already know that the object class $\mathcal{E}_7^{\mathrm{d}}$ is exhausted by a 9-parameter family (Proposition 3.2), the main result of the present section asserts that the subclass $\{(V, \eta) \in \mathcal{E}_7^{\mathrm{d}} \mid \eta_{\mathbb{P}} \text{ is collinear}\}$ is exhausted by a single 1-parameter family, and that $\eta_{\mathbb{P}} = \mathbb{I}_{\mathbb{P}(V)}$ for each (V, η) in this subclass (Proposition 4.5). The proof rests on a series of lemmas investigating the selfbijection $\underline{\mathcal{Y}}(\kappa)_{\mathbb{P}} : \mathbb{P}(\mathbb{R}^7) \to \mathbb{P}(\mathbb{R}^7)$ induced by the doubled dissident map $\underline{\mathcal{Y}}(\kappa) : \mathbb{R}^7 \wedge \mathbb{R}^7 \to \mathbb{R}^7$, for any $\kappa \in \mathcal{K}$. The entire present section forms a streamlined version of [27, pp. 8–12].

We introduce the shorthand $\mathcal{Y}(\kappa)_{ij} = \underline{\mathcal{Y}}(\kappa)(e_i \wedge e_j)$ for all $ij \in \underline{7}^2$. If $e_i \wedge e_j$ belongs to the basis $\underline{e} \wedge \underline{e}$ in $\mathbb{R}^7 \wedge \mathbb{R}^7$ chosen in the first paragraph of Section 3, then $\mathcal{Y}(\kappa)_{ij}$ is just the column of $\mathcal{Y}(\kappa)$ with column index ij. We denote by $(v_1 : \ldots : v_7)$ the line $[v]$ spanned by $v = (v_1 \ldots v_7)^t \in \mathbb{R}^7 \setminus \{0\}$.

LEMMA 4.1. *For each configuration $\kappa = (x, y, d) \in \mathcal{K}$, the selfbijection $\underline{\mathcal{Y}}(\kappa)_{\mathbb{P}} : \mathbb{P}(\mathbb{R}^7) \to \mathbb{P}(\mathbb{R}^7)$ acts on the coordinate axes $[e_1], \ldots, [e_7]$ as follows.*

$$\underline{\mathcal{Y}}(\kappa)_{\mathbb{P}}[e_1] = (y_1^2 + d_2 d_3 : y_1 y_2 + y_3 d_3 : y_1 y_3 - y_2 d_2 : 0 : 0 : 0 : 0),$$
$$\underline{\mathcal{Y}}(\kappa)_{\mathbb{P}}[e_2] = (y_1 y_2 - y_3 d_3 : y_2^2 + d_1 d_3 : y_2 y_3 + y_1 d_1 : 0 : 0 : 0 : 0),$$
$$\underline{\mathcal{Y}}(\kappa)_{\mathbb{P}}[e_3] = (y_1 y_3 + y_2 d_2 : y_2 y_3 - y_1 d_1 : y_3^2 + d_1 d_2 : 0 : 0 : 0 : 0),$$
$$\underline{\mathcal{Y}}(\kappa)_{\mathbb{P}}[e_4] = [e_4],$$

$$\underline{\mathcal{Y}}(\kappa)_{\mathbb{P}}[e_5] = (0:0:0:0:y_1^2+d_2d_3:y_1y_2+y_3d_3:y_1y_3-y_2d_2),$$
$$\underline{\mathcal{Y}}(\kappa)_{\mathbb{P}}[e_6] = (0:0:0:0:y_1y_2-y_3d_3:y_2^2+d_1d_3:y_2y_3+y_1d_1),$$
$$\underline{\mathcal{Y}}(\kappa)_{\mathbb{P}}[e_7] = (0:0:0:0:y_1y_3+y_2d_2:y_2y_3-y_1d_1:y_3^2+d_1d_2).$$

Proof. If (V,ξ,η) is any object in $\mathcal{D}_3$ and $\mathcal{J}(V,\xi,\eta)=(V^{\mathrm{d}},\xi^{\mathrm{d}},\eta^{\mathrm{d}})$, then Lemma 3.1(i) shows that

$$\eta^{\mathrm{d}}\left(\begin{pmatrix} v \\ 0 \\ 0 \end{pmatrix} \wedge V^{\mathrm{d}}\right) = \eta(v\wedge V)\times\mathbb{R}\times V,$$

$$\eta^{\mathrm{d}}\left(\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \wedge V^{\mathrm{d}}\right) = V\times\{0\}\times V,$$

$$\eta^{\mathrm{d}}\left(\begin{pmatrix} 0 \\ 0 \\ w \end{pmatrix} \wedge V^{\mathrm{d}}\right) = V\times\mathbb{R}\times\eta(w\wedge V)$$

for all $v,w\in V\setminus\{0\}$. If in particular $(V,\xi,\eta)=(\mathbb{R}^3,\xi_x,\eta_{yd})=\mathcal{G}(\kappa)$, for any given configuration $\kappa=(x,y,d)$, then $\eta^{\mathrm{d}}=\eta^{\mathrm{d}}_{yd}=\underline{\mathcal{Y}}(\kappa)$ and thus the asserted formulae for $\underline{\mathcal{Y}}(\kappa)_{\mathbb{P}}[e_i]$ can be read off at once from the above identities. ∎

With any configuration $\kappa=(x,y,d)\in\mathcal{K}$ we associate, along with E_{yd}, the real 3×3-matrix

$$\widehat{E}_{yd} = \begin{pmatrix} y_1^2+d_2d_3 & y_1y_2-y_3d_3 & y_1y_3+y_2d_2 \\ y_1y_2+y_3d_3 & y_2^2+d_1d_3 & y_2y_3-y_1d_1 \\ y_1y_3-y_2d_2 & y_2y_3+y_1d_1 & y_3^2+d_1d_2 \end{pmatrix}.$$

LEMMA 4.2. *If $\kappa=(x,y,d)\in\mathcal{K}$ and $K\in GL_7(\mathbb{R})$ are related by the identity $\mathbb{P}(\underline{K})=\underline{\mathcal{Y}}(\kappa)_{\mathbb{P}}$, then there exist scalars $\alpha,\beta\in\mathbb{R}\setminus\{0\}$ such that*

$$K=\beta\begin{pmatrix} \widehat{E}_{yd} & & \\ & \alpha & \\ & & \widehat{E}_{yd} \end{pmatrix}.$$

Proof. Evaluating $\mathbb{P}(\underline{K})=\underline{\mathcal{Y}}(\kappa)_{\mathbb{P}}$ at $[e_i]$ for any $i\in\underline{7}$, we obtain $[K_{\bullet i}]=\mathbb{P}(\underline{K})[e_i]=\underline{\mathcal{Y}}(\kappa)_{\mathbb{P}}[e_i]$, which, together with Lemma 4.1, implies the existence of scalars $c_1,\dots,c_7\in\mathbb{R}\setminus\{0\}$ such that

$$(*)\qquad K=\begin{pmatrix} \widehat{E}_{yd} & & \\ & 1 & \\ & & \widehat{E}_{yd} \end{pmatrix}\begin{pmatrix} c_1 & & \\ & \ddots & \\ & & c_7 \end{pmatrix}.$$

Evaluating $\mathbb{P}(\underline{K}) = \underline{\mathcal{Y}}(\kappa)_{\mathbb{P}}$ at $[e_i + e_j]$ for any $ij \in \underline{7}^2$ such that $i < j$, we obtain

$$\begin{aligned}[K_{\bullet i} + K_{\bullet j}] &= \mathbb{P}(\underline{K})[e_i + e_j] = \underline{\mathcal{Y}}(\kappa)_{\mathbb{P}}[e_i + e_j] \\ &= (\underline{\mathcal{Y}}(\kappa)((e_i + e_j) \wedge (e_i + e_j)^{\perp}))^{\perp} \\ &= [\underline{\mathcal{Y}}(\kappa)((e_i + e_j) \wedge (e_i - e_j)),\ \underline{\mathcal{Y}}(\kappa)((e_i + e_j) \wedge e_k)]^{\perp}_{k \in \underline{7} \setminus \{i,j\}} \\ &= [\mathcal{Y}(\kappa)_{ij},\ \mathcal{Y}(\kappa)_{ik} + \mathcal{Y}(\kappa)_{jk}]^{\perp}_{k \in \underline{7} \setminus \{i,j\}},\end{aligned}$$

or equivalently

$$(*)_{ij} \qquad (K_{\bullet i} + K_{\bullet j})^t (\mathcal{Y}(\kappa)_{ij} \mid \mathcal{Y}(\kappa)_{ik} + \mathcal{Y}(\kappa)_{jk})_{k \in \underline{7} \setminus \{i,j\}} = 0.$$

If we substitute $K_{\bullet i} + K_{\bullet j}$ by means of $(*)$, the complicated looking system of polynomial equations $(*)_{ij}$ gets a very simple interpretation. Namely, straightforward verifications show that $(*)_{ij}$ is equivalent to $c_i = c_j$ for all $ij \in \{12, 23, 56, 67\}$, while $(*)_{35}$ is equivalent to $c_3 = c_5 \ \wedge \ y_2 = 0$. Summarizing, we obtain $c_1 = c_2 = c_3 = c_5 = c_6 = c_7$, which, together with $(*)$, completes the proof on setting $\alpha = c_1^{-1} c_4$ and $\beta = c_1$. ■

LEMMA 4.3. *If* $\kappa = (x, y, d) \in \mathcal{K}$ *and*

$$K = \begin{pmatrix} \widehat{E}_{yd} & & \\ & \alpha & \\ & & \widehat{E}_{yd} \end{pmatrix} \in GL_7(\mathbb{R})$$

are related by $\mathbb{P}(\underline{K}) = \underline{\mathcal{Y}}(\kappa)_{\mathbb{P}}$, *then* $(x, y, d) = (0, 0, d_1 1_3)$ *and* $K = d_1^2 \mathbb{I}_7$.

Proof. The system of polynomial equations $(*)_{ij}$ derived in the previous proof is still valid for each $ij \in \underline{7}^2$ such that $i < j$. Elimination of α from $(*)_{ij}$ for selected values of ij reveals the following conditions imposed on κ.

$$(*)_{14} \quad \text{implies} \quad \begin{cases} x_2(y_1^2 + d_2 d_3) = y_1 y_3 - y_2 d_2, & (1) \\ x_3(y_1^2 + d_2 d_3) = -y_1 y_2 - y_3 d_3, & (2) \end{cases}$$

$$(*)_{24} \quad \text{implies} \quad \begin{cases} x_1(y_2^2 + d_1 d_3) = -y_2 y_3 - y_1 d_1, & (3) \\ x_3(y_2^2 + d_1 d_3) = y_1 y_2 - y_3 d_3, & (4) \end{cases}$$

$$(*)_{34} \quad \text{implies} \quad \begin{cases} x_1(y_3^2 + d_1 d_2) = y_2 y_3 - y_1 d_1, & (5) \\ x_2(y_3^2 + d_1 d_2) = -y_1 y_3 - y_2 d_2, & (6) \end{cases}$$

$$(*)_{45} \quad \text{implies} \quad \begin{cases} x_2(y_1^2 + d_2 d_3) = -y_1 y_3 + y_2 d_2, & (7) \\ x_3(y_1^2 + d_2 d_3) = y_1 y_2 + y_3 d_3, & (8) \end{cases}$$

$$(*)_{46} \quad \text{implies} \quad \begin{cases} x_1(y_2^2 + d_1 d_3) = y_2 y_3 + y_1 d_1, & (9) \\ x_3(y_2^2 + d_1 d_3) = -y_1 y_2 + y_3 d_3. & (10) \end{cases}$$

Now (3) + (9) implies $x_1 = 0$, which in turn, combined with (3) + (5), implies $y_1 = 0$. Similarly (1) $\wedge$ (7) $\wedge$ (6) implies $x_2 = y_2 = 0$, and (2) $\wedge$ (8) $\wedge$ (4) implies $x_3 = y_3 = 0$. So $x = y = 0$.

Evaluating $\mathbb{P}(\underline{K}) = \underline{\mathcal{Y}}(\kappa)_{\mathbb{P}}$ at $[\sum_{i=1}^4 e_i]$ and working with $(\sum_{i=1}^4 e_i)^\perp = [e_1 - e_j, e_k]_{\substack{j=2,3,4\\k=5,6,7}}$ we obtain, arguing as in the previous proof, the system

$$(*)_{\underline{4}} \qquad \Big(\sum_{i=1}^4 K_{\bullet i}\Big)^t \Big(\sum_{i=1}^4 (\mathcal{Y}(\kappa)_{i1} - \mathcal{Y}(\kappa)_{ij}) \;\Big|\; \sum_{i=1}^4 \mathcal{Y}(\kappa)_{ik}\Big)_{\substack{j=2,3,4\\k=5,6,7}} = 0.$$

Reading off the relevant columns from the matrices K and $\mathcal{Y}(\kappa)$, and taking into account that $x = y = 0$, we find that $(*)_{\underline{4}}$ is equivalent to $d_2d_3 = d_1d_3 = d_1d_2 = \alpha$. This proves both $d_1 = d_2 = d_3$ and $K = d_1^2\mathbb{I}_7$. ∎

With any given configuration $\kappa = (x, y, d) \in \mathcal{K}$ we associate, along with E_{yd} and $\widehat{E}_{yd}$, the real 3×3-matrices

$$M_x = \begin{pmatrix} 0 & -x_3 & x_2 \\ x_3 & 0 & -x_1 \\ -x_2 & x_1 & 0 \end{pmatrix}, \quad F_{yd} = \begin{pmatrix} d_3 - d_2 & y_3 & y_2 \\ -y_3 & d_3 - d_1 & -y_1 \\ y_2 & y_1 & d_2 - d_1 \end{pmatrix}.$$

In fact, M_x represents the antisymmetric linear endomorphism $\mu_x = \pi_3(x \wedge ?)$ of $\mathbb{R}^3$ in the standard basis. Recall that D_y denotes the diagonal matrix with diagonal sequence (y_1, y_2, y_3). Moreover, with any $v \in \mathbb{R}^7$ we associate $v_{<4} = (v_1\ v_2\ v_3)^t$ and $v_{>4} = (v_5\ v_6\ v_7)^t$ in $\mathbb{R}^3$.

LEMMA 4.4. *For each configuration $\kappa = (x, y, d) \in \mathcal{K}$ and each $v \in \mathbb{R}^7$, the following assertions are equivalent.*

(i) $\langle \underline{\mathcal{Y}}(\kappa)(u \wedge v), w\rangle = \langle u, \underline{\mathcal{Y}}(\kappa)(v \wedge w)\rangle$ *for all* $(u, w) \in \mathbb{R}^7 \times \mathbb{R}^7$.

(ii) $\begin{cases} M_x v_{<4} = M_x v_{>4} = 0, \\ D_y v_{<4} = D_y v_{>4} = 0, \\ F_{yd} v_{<4} = F_{yd} v_{>4} = 0. \end{cases}$

Proof. The given data κ and v determine a linear endomorphism $\underline{\mathcal{Y}}(\kappa)(v \wedge ?)$ on $\mathbb{R}^7$ which is represented in $\underline{e}$ by a matrix $L_{\kappa v} \in \mathbb{R}^{7\times 7}$. Assertion (i) holds if and only if $L_{\kappa v}$ is antisymmetric. If we write down $L_{\kappa v}$ explicitly, a closer look reveals (by elementary but lengthy arguments) that $L_{\kappa v}$ is antisymmetric if and only if the system (ii) is valid. ∎

PROPOSITION 4.5. *For each doubled dissident map η on a 7-dimensional Euclidean space V and for each configuration $\kappa \in \mathcal{K}$ such that $\eta \xrightarrow{\sim} \underline{\mathcal{Y}}(\kappa)$, the following statements are equivalent.*

(i) $\eta_{\mathbb{P}}$ *is collinear.*
(ii) $\kappa = (0, 0, \lambda 1_3)$ *for some* $\lambda > 0$.
(iii) $\langle \eta(u \wedge v), w\rangle = \langle u, \eta(v \wedge w)\rangle$ *for all* $(u, v, w) \in V^3$.
(iv) $\eta_{\mathbb{P}} = \mathbb{I}_{\mathbb{P}(V)}$.

Proof. (i)⇒(ii). If $\eta_{\mathbb{P}}$ is collinear then $\underline{\mathcal{Y}}(\kappa)_{\mathbb{P}}$ is collinear, by Lemma 2.3(ii). Hence we may apply the fundamental theorem of projective geometry (cf. [2, p. 88]) which asserts the existence of an invertible matrix $K \in GL_7(\mathbb{R})$ such that $\mathbb{P}(\underline{K}) = \underline{\mathcal{Y}}(\kappa)_{\mathbb{P}}$. According to Lemma 4.2 we may assume that

$$K = \begin{pmatrix} \widehat{E}_{yd} & & \\ & \alpha & \\ & & \widehat{E}_{yd} \end{pmatrix}$$

for some $\alpha \in \mathbb{R} \setminus \{0\}$. From Lemma 4.3 we conclude that $\kappa = (0, 0, d_1 1_3)$.

(ii)⇒(iii). If κ is of the special form $(x, y, d) = (0, 0, \lambda 1_3)$, then $M_x = D_y = F_{yd} = 0$. Thus (iii) holds for $\eta = \underline{\mathcal{Y}}(0, 0, \lambda 1_3)$, by Lemma 4.4. Consequently, (iii) also holds for each $(V, \eta) \in \mathcal{E}_7^{\mathrm{d}}$ admitting an isomorphism $\eta \xrightarrow{\sim} \underline{\mathcal{Y}}(0, 0, \lambda 1_3)$.

(iii)⇒(iv). If $(V, \eta) \in \mathcal{E}_7^{\mathrm{d}}$ satisfies (iii), then we find in particular for all $v \in V \setminus \{0\}$ and $w \in v^\perp$ that $\langle v, \eta(v \wedge w)\rangle = \langle \eta(v \wedge v), w\rangle = 0$. This means $\eta(v \wedge v^\perp) = v^\perp$, or equivalently $\eta_{\mathbb{P}}[v] = [v]$. So $\eta_{\mathbb{P}} = \mathbb{I}_{\mathbb{P}(V)}$.

(iv)⇒(i) is trivially true. ■

Recall that a *vector product* on a Euclidean space V is, by definition, a linear map $\pi : V \wedge V \to V$ satisfying the following two conditions:

(a) $\langle \pi(u \wedge v), w\rangle = \langle u, \pi(v \wedge w)\rangle$ for all $(u, v, w) \in V^3$.
(b) $|\pi(u \wedge v)| = 1$ for all orthonormal pairs $(u, v) \in V^2$.

Every vector product is a dissident map. More precisely, the equivalence of categories $\mathcal{H} : \mathcal{D} \to \mathcal{Q}$ (Proposition 1.1) induces an equivalence between the full subcategories $\{(V, \xi, \eta) \in \mathcal{D} \mid \xi = o$ and η is a vector product$\}$ and $\mathcal{A} = \{A \in \mathcal{Q} \mid A$ is alternative$\}$ (cf. [26]). Moreover, the famous theorems of Frobenius [19] and Zorn [32] assert that $\mathcal{A}$ is classified by $\{\mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}\}$ (cf. [24], [25]). Accordingly there exist four isoclasses of vector products only, one in each of the dimensions 0, 1, 3 and 7. The *standard* vector products are the chosen representatives $\pi_m : \mathbb{R}^m \wedge \mathbb{R}^m \to \mathbb{R}^m$, $m \in \{0, 1, 3, 7\}$, defined by $\pi_0 = o$, $\pi_1 = o$, $(\pi_3(e_2 \wedge e_3), \pi_3(e_3 \wedge e_1), \pi_3(e_1 \wedge e_2)) = (e_1, e_2, e_3)$ and $\pi_7 = \underline{\mathcal{Y}}(0, 0, 1_3)$.

PROPOSITION 4.6. *For each doubled dissident map η on a 7-dimensional Euclidean space V and for each configuration $\kappa \in \mathcal{K}$ such that $\eta \xrightarrow{\sim} \underline{\mathcal{Y}}(\kappa)$, the following statements are equivalent.*

(i) *η is composed.*
(ii) $\kappa = (0, 0, 1_3)$.
(iii) *η is a vector product.*

Proof. (i)⇒(ii). If η admits a factorization $\eta = \varepsilon\pi$ into a vector product π on V and a definite linear endomorphism ε of V, then $\eta_{\mathbb{P}} = \mathbb{P}(\varepsilon^{-*})$ is collinear, by Proposition 2.4. Applying Proposition 4.5 we conclude that

$\kappa = (0,0,\lambda 1_3)$ for some $\lambda > 0$ and $\eta_{\mathbb{P}} = \mathbb{I}_{\mathbb{P}(V)}$. Hence $\varepsilon = \mu \mathbb{I}_V$ for some $\mu \in \mathbb{R} \setminus \{0\}$, and therefore $\eta = \mu\pi \xrightarrow{\sim} \underline{\mathcal{Y}}(0,0,\lambda 1_3)$. Accordingly we see for all $1 \leq i < j \leq 7$ that $|\underline{\mathcal{Y}}(0,0,\lambda 1_3)(e_i \wedge e_j)| = |\mu|$. Special choices of (i,j) yield $\lambda = |\underline{\mathcal{Y}}(0,0,\lambda 1_3)(e_1 \wedge e_2)| = |\mu| = |\underline{\mathcal{Y}}(0,0,\lambda 1_3)(e_3 \wedge e_4)| = 1$, proving that $\kappa = (0,0,1_3)$.

(ii)⇒(iii). If $\kappa = (0,0,1_3)$, then $\eta \xrightarrow{\sim} \underline{\mathcal{Y}}(0,0,1_3) = \pi_7$ is a vector product.

(iii)⇒(i) is trivially true. ∎

COROLLARY 4.7. *The class of all dissident maps on a Euclidean space which are both composed and doubled coincides with the class of all vector products on a non-zero Euclidean space. This object class constitutes three isoclasses, represented by the standard vector products* π_1, π_3 *and* π_7.

Proof. Let η be a dissident map on V which is both composed and doubled. Being doubled dissident means, by definition, that $(V,\xi,\eta) \xrightarrow{\sim} \mathcal{IV}(A)$ for some linear form $\xi : V \wedge V \to \mathbb{R}$ and some real quadratic division algebra A. Since $\dim V \in \{0,1,3,7\}$ and $\dim A \in \{1,2,4,8\}$ are related by $\dim V = 2\dim A - 1$, we infer that $\dim V \in \{1,3,7\}$ and $\dim A \in \{1,2,4\}$. If $\dim V = 1$, then $(V,\xi,\eta) \xrightarrow{\sim} (\mathbb{R}^1, o, \pi_1)$ holds trivially. From Proposition 1.1 we conclude that $\{\mathbb{C}\}$ classifies $\mathcal{Q}_2$. Hence if $\dim V = 3$, then

$$(V,\xi,\eta) \xrightarrow{\sim} \mathcal{IV}(\mathbb{C}) \xrightarrow{\sim} \mathcal{I}(\mathbb{H}) \xrightarrow{\sim} (\mathbb{R}^3, o, \pi_3).$$

Finally if $\dim V = 7$, then we conclude from Proposition 4.6 directly that η is a vector product.

Conversely, let π be a vector product on a non-zero Euclidean space V. Then $\pi = \mathbb{I}_V \pi$ is trivially composed dissident. Moreover $B = \mathcal{H}(V,o,\pi)$ is a real alternative division algebra such that $\dim B \geq 2$. Hence B is isomorphic to one of the chosen representatives $\mathbb{C} = \mathcal{V}(\mathbb{R})$, $\mathbb{H} = \mathcal{V}(\mathbb{C})$ or $\mathbb{O} = \mathcal{V}(\mathbb{H})$. Accordingly $(V,o,\pi) \xrightarrow{\sim} \mathcal{IH}(V,o,\pi) \xrightarrow{\sim} \mathcal{IV}(A)$ for some $A \in \{\mathbb{R},\mathbb{C},\mathbb{H}\}$, proving that π is also doubled dissident. ∎

Let us record two interesting features that are implicit in the preceding results. Whereas η composed dissident always implies $\eta_{\mathbb{P}}$ collinear (Proposition 2.4), the converse is in general not true. Namely each doubled dissident map $\underline{\mathcal{Y}}(0,0,\lambda 1_3)$, $\lambda > 0$, induces the collinear selfbijection $\underline{\mathcal{Y}}(0,0,\lambda 1_3)_{\mathbb{P}} = \mathbb{I}_{\mathbb{P}(\mathbb{R}^7)}$ (Proposition 4.5), while $\underline{\mathcal{Y}}(0,0,\lambda 1_3)$ is composed dissident if and only if $\lambda = 1$ (Proposition 4.6).

Moreover we have already obtained two sufficient criteria for the heteromorphism of doubled dissident maps, in terms of their underlying configurations.

(1) If $\kappa \in \mathcal{K} \setminus \{(0,0,\lambda 1_3) \mid \lambda > 0\}$, then $\underline{\mathcal{Y}}(\kappa) \not\xrightarrow{\sim} \underline{\mathcal{Y}}(0,0,\lambda 1_3)$ for all $\lambda > 0$.

(2) If $\lambda \in \mathbb{R}_{>0} \setminus \{1\}$, then $\underline{\mathcal{Y}}(0,0,\lambda 1_3) \not\xrightarrow{\sim} \underline{\mathcal{Y}}(0,0,1_3)$.

Indeed, (1) follows from Proposition 4.5 and Lemma 2.3(ii), while (2) follows from Proposition 4.6. The next section is devoted to refinements of the sufficient criteria (1) and (2).

5. On the isomorphism problem for doubled dissident maps. With any dissident map η on a Euclidean space V we associate the subspace $V_\eta = \{v \in V \mid \langle \eta(u\wedge v), w\rangle = \langle u, \eta(v\wedge w)\rangle \text{ for all } (u,w) \in V^2\}$ of V. Dissident maps (V,η) with $V_\eta = V$ are called *weak* vector products [16]. In general, the subspace $V_\eta \subset V$ measures how close η comes to being a weak vector product. The investigation of V_η proves to be useful in our search for refined sufficient criteria for the heteromorphism of doubled dissident maps.

LEMMA 5.1. *Each isomorphism of dissident maps* $\sigma : (V,\eta) \xrightarrow{\sim} (V',\eta')$ *induces an isomorphism of Euclidean spaces* $\sigma : V_\eta \xrightarrow{\sim} V'_{\eta'}$.

Proof. Let $\sigma : (V,\eta) \xrightarrow{\sim} (V',\eta')$ be an isomorphism of dissident maps. If $v \in V_\eta$, then we obtain for all $u, w \in V$ the chain of identities

$$\begin{array}{ccccc} \langle \eta'(\sigma(u)\wedge\sigma(v)), \sigma(w)\rangle' & = & \langle \sigma\eta(u\wedge v), \sigma(w)\rangle' & = & \langle \eta(u\wedge v), w\rangle \\ & & & & \| \\ \langle \sigma(u), \eta'(\sigma(v)\wedge\sigma(w))\rangle' & = & \langle \sigma(u), \sigma\eta(v\wedge w)\rangle' & = & \langle u, \eta(v\wedge w)\rangle \end{array}$$

which proves that $\sigma(v) \in V'_{\eta'}$. So σ induces a morphism of Euclidean spaces $\sigma : V_\eta \to V'_{\eta'}$. Applying the same argument to $\sigma^{-1} : (V',\eta') \xrightarrow{\sim} (V,\eta)$, we find that the induced morphism $\sigma : V_\eta \to V'_{\eta'}$ is an isomorphism. ∎

Let $\kappa = (x,y,d) \in \mathcal{K}$ be any configuration. Application of Lemma 3.1(i) to $\mathcal{G}(\kappa) = (\mathbb{R}^3, \xi_x, \eta_{yd})$, together with $\eta_{yd}^{\mathrm{d}} = \underline{\mathcal{Y}}(\kappa)$, shows that

$$\underline{\mathcal{Y}}(\kappa)\left(\begin{pmatrix}0\\1\\0\end{pmatrix} \wedge \begin{pmatrix}v\\ \alpha\\ w\end{pmatrix}\right) = \begin{pmatrix}w\\0\\-v\end{pmatrix} \quad \text{for all } \begin{pmatrix}v\\ \alpha\\ w\end{pmatrix} \in \mathbb{R}^7.$$

Accordingly the linear endomorphism $\underline{\mathcal{Y}}(\kappa)(e_4 \wedge ?) : \mathbb{R}^7 \to \mathbb{R}^7$ is antisymmetric and does not depend on κ. So $e_4 \in \mathbb{R}^7_{\underline{\mathcal{Y}}(\kappa)}$ for all $\kappa \in \mathcal{K}$.

We proceed by refining this preliminary observation to a complete and explicit description of the subspaces $\mathbb{R}^7_{\underline{\mathcal{Y}}(\kappa)} \subset \mathbb{R}^7$ for all configurations $\kappa \in \mathcal{K}$. This description is prepared by partitioning $\mathcal{K}$ into the pairwise disjoint subsets

$$\begin{aligned} \mathcal{K}_7 &= \{(x,y,d) \in \mathcal{K} \mid x = y = 0 \ \wedge\ d_1 = d_2 = d_3\}, \\ \mathcal{K}_{31} &= \{(x,y,d) \in \mathcal{K} \mid x \neq 0 \ \wedge\ y = 0 \ \wedge\ d_1 = d_2 = d_3\}, \\ \mathcal{K}_{32} &= \{(x,y,d) \in \mathcal{K} \mid x_1 = x_2 = 0 \ \wedge\ y = 0 \ \wedge\ d_1 = d_2 < d_3\}, \\ \mathcal{K}_{33} &= \{(x,y,d) \in \mathcal{K} \mid x_2 = x_3 = 0 \ \wedge\ y = 0 \ \wedge\ d_1 < d_2 = d_3\}, \end{aligned}$$

$$\mathcal{K}_{34} = \{(x, y, d) \in \mathcal{K} \mid x \in [x_{yd}] \ \wedge \ y = \pm\varrho_d e_2 \ \wedge \ d_1 < d_2 < d_3\},$$
$$\mathcal{K}_1 = \mathcal{K} \setminus (\mathcal{K}_7 \cup \mathcal{K}_{31} \cup \mathcal{K}_{32} \cup \mathcal{K}_{33} \cup \mathcal{K}_{34}),$$

where in the definition of $\mathcal{K}_{34}$ we used the notation $x_{yd} = (-y_2 \ \ 0 \ \ d_3 - d_2)^t$ and $\varrho_d = \sqrt{(d_3 - d_2)(d_2 - d_1)}$. We introduce moreover the linear injections $\iota_{<4} : \mathbb{R}^3 \to \mathbb{R}^7$, $\iota_{<4}(x) = \sum_{i=1}^3 x_i e_i$, and $\iota_{>4} : \mathbb{R}^3 \to \mathbb{R}^7$, $\iota_{>4}(x) = \sum_{i=1}^3 x_i e_{4+i}$, identifying $\mathbb{R}^3$ with the first, respectively last factor of the product $\mathbb{R}^3 \times \mathbb{R} \times \mathbb{R}^3 = \mathbb{R}^7$.

LEMMA 5.2. *The subspace* $\mathbb{R}^7_{\underline{\mathcal{Y}}(\kappa)} \subset \mathbb{R}^7$ *determined by any configuration* $\kappa = (x, y, d) \in \mathcal{K}$ *admits the following description.*

(i) *If* $\kappa \in \mathcal{K}_7$ *then* $\mathbb{R}^7_{\underline{\mathcal{Y}}(\kappa)} = \mathbb{R}^7$.
(ii) *If* $\kappa \in \mathcal{K}_{31}$ *then* $\mathbb{R}^7_{\underline{\mathcal{Y}}(\kappa)} = [\iota_{<4}(x), e_4, \iota_{>4}(x)]$.
(iii) *If* $\kappa \in \mathcal{K}_{32}$ *then* $\mathbb{R}^7_{\underline{\mathcal{Y}}(\kappa)} = [e_3, e_4, e_7]$.
(iv) *If* $\kappa \in \mathcal{K}_{33}$ *then* $\mathbb{R}^7_{\underline{\mathcal{Y}}(\kappa)} = [e_1, e_4, e_5]$.
(v) *If* $\kappa \in \mathcal{K}_{34}$ *then* $\mathbb{R}^7_{\underline{\mathcal{Y}}(\kappa)} = [\iota_{<4}(x_{yd}), e_4, \iota_{>4}(x_{yd})]$.
(vi) *If* $\kappa \in \mathcal{K}_1$ *then* $\mathbb{R}^7_{\underline{\mathcal{Y}}(\kappa)} = [e_4]$.

Proof. The statements (i)–(vi) are easy consequences of Lemma 4.4, by straightforward linear algebraic arguments. ■

We introduce the map $\delta : \mathcal{K} \to \{0, 1, \ldots, 7\}$, $\delta(\kappa) = \dim \mathbb{R}^7_{\underline{\mathcal{Y}}(\kappa)}$. Moreover we set $\mathcal{K}_3 = \bigcup_{i=1}^4 \mathcal{K}_{3i}$.

PROPOSITION 5.3. (i) *The image and the non-empty fibres of* δ *are given by* $\operatorname{im}\delta = \{1, 3, 7\}$ *and* $\delta^{-1}(m) = \mathcal{K}_m$ *for all* $m \in \{1, 3, 7\}$.

(ii) *If* κ *and* κ' *are configurations in* $\mathcal{K}$ *such that* $\underline{\mathcal{Y}}(\kappa)$ *and* $\underline{\mathcal{Y}}(\kappa')$ *are isomorphic*, *then* $\delta(\kappa) = \delta(\kappa')$.

Proof. (i) can be read off directly from Lemma 5.2. (ii) follows immediately from Lemma 5.1. ■

Proposition 5.3 decomposes the problem of proving Conjecture 3.3 into three separate subproblems which one obtains by restricting $\mathcal{K}$ to the subsets $\mathcal{K}_1$, $\mathcal{K}_3$ and $\mathcal{K}_7$ respectively. In the present article we content ourselves with solving the subproblem given by $\mathcal{K}_7$ (Proposition 5.6), along with a slightly weakened version of the subproblem given by $\mathcal{K}_3$ (Propositions 5.7 and 5.8). The proofs of Propositions 5.6–5.8 make use of Lemmas 5.4 and 5.5, which in turn rest upon the following elementary observation.

Given any configuration $\kappa \in \mathcal{K}$ and any vector $v \in \mathbb{R}^7_{\underline{\mathcal{Y}}(\kappa)} \setminus \{0\}$, the linear endomorphism $\underline{\mathcal{Y}}(\kappa)(v \wedge ?)$ of $\mathbb{R}^7$ has kernel $[v]$ and induces an antisymmetric linear automorphism of $v^\perp$. Accordingly there exist an orthonormal basis $\underline{b}$

in $\mathbb{R}^7$ and an ascending triple t of positive real numbers $0 < t_1 \leq t_2 \leq t_3$ such that $\underline{\mathcal{Y}}(\kappa)(v \wedge ?)$ is represented in $\underline{b}$ by the matrix

$$N_t = \begin{pmatrix} 0 & & & & & & \\ & 0 & -t_1 & & & & \\ & t_1 & 0 & & & & \\ & & & 0 & -t_2 & & \\ & & & t_2 & 0 & & \\ & & & & & 0 & -t_3 \\ & & & & & t_3 & 0 \end{pmatrix}.$$

Here $t \in \mathcal{T}$ (see introduction) is uniquely determined by the given data $\kappa \in \mathcal{K}$ and $v \in \mathbb{R}^7_{\underline{\mathcal{Y}}(\kappa)} \setminus \{0\}$. We express this by introducing for any $\kappa \in \mathcal{K}$ the map $\tau_\kappa : \mathbb{R}^7_{\underline{\mathcal{Y}}(\kappa)} \setminus \{0\} \to \mathcal{T},\ \tau_\kappa(v) = t$.

LEMMA 5.4. *Let κ and κ' be configurations in $\mathcal{K}$. If $\sigma : \underline{\mathcal{Y}}(\kappa) \xrightarrow{\sim} \underline{\mathcal{Y}}(\kappa')$ is an isomorphism of dissident maps, then $\tau_{\kappa'}\sigma(v) = \tau_\kappa(v)$ for all $v \in \mathbb{R}^7_{\underline{\mathcal{Y}}(\kappa)} \setminus \{0\}$.*

Proof. By Lemma 5.1, the isomorphism σ induces a bijection $\sigma : \mathbb{R}^7_{\underline{\mathcal{Y}}(\kappa)} \setminus \{0\} \xrightarrow{\sim} \mathbb{R}^7_{\underline{\mathcal{Y}}(\kappa')} \setminus \{0\}$. Given $v \in \mathbb{R}^7_{\underline{\mathcal{Y}}(\kappa)} \setminus \{0\}$, set $\tau_\kappa(v) = t$. This means that the linear endomorphism $\underline{\mathcal{Y}}(\kappa)(v \wedge ?)$ of $\mathbb{R}^7$ is represented by N_t in some orthonormal basis $\underline{b}$. Accordingly the linear endomorphism $\underline{\mathcal{Y}}(\kappa')(\sigma(v) \wedge ?)$ of $\mathbb{R}^7$ is represented by N_t in the orthonormal basis $\sigma(\underline{b})$. Hence $\tau_{\kappa'}\sigma(v) = t = \tau_\kappa(v)$. ■

In order to exploit Lemma 5.4 we need explicit descriptions of the maps τ_κ. These we attain as follows. Given $\kappa \in \mathcal{K}$ and $v \in \mathbb{R}^7_{\underline{\mathcal{Y}}(\kappa)} \setminus \{0\}$, we denote by $L_{\kappa v}$ the antisymmetric matrix representing $\underline{\mathcal{Y}}(\kappa)(v \wedge ?)$ in the standard basis of $\mathbb{R}^7$. Subtle calculations with $L^2_{\kappa v}$ will reveal the eigenspace decomposition of $v^\perp$ with respect to the symmetric linear automorphism induced by $\underline{\mathcal{Y}}(\kappa)(v \wedge ?)^2$ on $v^\perp$. This insight being gained, the desired explicit formula for $\tau_\kappa(v)$ follows trivially. The cases $\mathcal{K}_7$ and $\mathcal{K}_{31}$ are covered by the following lemma.

LEMMA 5.5. *If $\kappa = (x_1 e_1, 0, \lambda 1_3) \in \mathcal{K}$, with $x_1 \geq 0$ and $v \in \mathbb{R}^7_{\underline{\mathcal{Y}}(\kappa)} \setminus \{0\}$, then*

$$\tau_\kappa(v) = \begin{cases} (\varepsilon, \varepsilon, |v|) & \text{if } 0 < \lambda \leq 1, \\ (|v|, \varepsilon, \varepsilon) & \text{if } 1 \leq \lambda < \infty, \end{cases}$$

where $\varepsilon = \sqrt{\lambda^2(|v_{<4}|^2 + |v_{>4}|^2) + v_4^2}$.

Proof. If κ and v are as in the statement, then

$$L_{\kappa v} = \left(\begin{array}{c|c|c} \lambda M_{v_{<4}} & -v_{>4} & v_{<4}\mathbb{I}_3 - \lambda M_{v_{>4}} \\ \hline -(v_{>4})^t & 0 & -(v_{<4})^t \\ \hline -v_4\mathbb{I}_3 - \lambda M_{v_{>4}} & v_{<4} & -\lambda M_{v_{<4}} \end{array}\right).$$

Observing that $M_a b = \pi_3(a \wedge b)$ for all $a, b \in \mathbb{R}^3$ and using both the Graßmann identity and Jacobi identity for π_3, one derives the identity

$$(*) \qquad L_{\kappa v}^2 w = -\varepsilon^2 w + (1-\lambda^2)\left(w_4(v_4 v - |v|^2 e_4) + \begin{vmatrix} v_{<4} & w_{<4} \\ v_{>4} & w_{>4} \end{vmatrix} \begin{pmatrix} v_{>4} \\ 0 \\ -v_{<4} \end{pmatrix}\right)$$

for all $w \in v^\perp$, where

$$\begin{vmatrix} v_{<4} & w_{<4} \\ v_{>4} & w_{>4} \end{vmatrix} = \langle v_{<4}, w_{>4}\rangle - \langle v_{>4}, w_{<4}\rangle.$$

Denote by E_α the eigenspace in $v^\perp$ corresponding to a non-zero eigenvalue α of $L_{\kappa v}^2$. The eigenspace decomposition of $v^\perp$ with respect to $L_{\kappa v}^2$ is now easily read off from $(*)$. If $\lambda = 1$ or $v \in [e_4] \setminus \{0\}$, then $v^\perp = E_{-|v|^2}$. If $\lambda \neq 1$ and $v \notin [e_4]$, then $v^\perp = E_{-|v|^2} \oplus E_{-\varepsilon^2}$, where $E_{-|v|^2} = [v_4 v - |v|^2 e_4, \iota_{<4}(v_{>4}) - \iota_{>4}(v_{<4})]$ is 2-dimensional. This results in the claimed description of $\tau_\kappa(v)$. ∎

Proposition 5.6. *If $\kappa = (0, 0, \lambda 1_3)$ and $\kappa' = (0, 0, \lambda' 1_3)$ are configurations in $\mathcal{K}_7$ such that the dissident maps $\underline{\mathcal{Y}}(\kappa)$ and $\underline{\mathcal{Y}}(\kappa')$ are isomorphic, then $\lambda = \lambda'$.*

Proof. Let $\sigma : \underline{\mathcal{Y}}(\kappa) \xrightarrow{\sim} \underline{\mathcal{Y}}(\kappa')$ be an isomorphism of dissident maps. We may assume that $\lambda' \neq 1$. Applying Lemmas 5.4 and 5.5 to $v = e_4$ we obtain $\tau_{\kappa'}\sigma(e_4) = \tau_\kappa(e_4) = (1,1,1)$. This implies $\sigma(e_4) = \pm e_4$ and hence $\sigma(e_4^\perp) = e_4^\perp$. Applying the same two lemmas now to any $v \in e_4^\perp$ with $|v| = 1$, we deduce that $\{1, \lambda\} = \{1, \lambda'\}$, hence $\lambda = \lambda'$. ∎

Proposition 5.7. *If $\kappa = (x, 0, \lambda 1_3)$ and $\kappa' = (x', 0, \lambda' 1_3)$ are configurations in $\mathcal{K}_{31}$ such that the dissident triples $(\mathbb{R}^7, \xi_x^{\mathrm{d}}, \underline{\mathcal{Y}}(\kappa))$ and $(\mathbb{R}^7, \xi_{x'}^{\mathrm{d}}, \underline{\mathcal{Y}}(\kappa'))$ are isomorphic, then κ and κ' are isomorphic.*

Proof. Let $\mathcal{F} : \mathcal{K} \to \mathcal{D}_7$ be the composed functor $\mathcal{F} = \mathcal{J}\mathcal{G}$. Recall that $\mathcal{F}(\kappa) = (\mathbb{R}^7, \xi_x^{\mathrm{d}}, \eta_{yd}^{\mathrm{d}}) = (\mathbb{R}^7, \xi_x^{\mathrm{d}}, \underline{\mathcal{Y}}(\kappa))$ for all $\kappa = (x, y, d) \in \mathcal{K}$. If in particular $\kappa = (x, 0, \lambda 1_3) \in \mathcal{K}_{31}$, then we choose $\kappa_n = (|x|e_1, 0, \lambda 1_3) \in \mathcal{K}_{31}$ as its normal form. Any $R \in SO_3(\mathbb{R})$ with $Rx = |x|e_1$ is an isomorphism $R : \kappa \xrightarrow{\sim} \kappa_n$ in $\mathcal{K}$, determining an isomorphism $\mathcal{F}(R) : \mathcal{F}(\kappa) \xrightarrow{\sim} \mathcal{F}(\kappa_n)$ in $\mathcal{D}_7$. This reduces the proof to the special case where both κ and κ' are in normal form.

So let $\kappa = (x, 0, \lambda 1_3)$ and $\kappa' = (x', 0, \lambda' 1_3)$ be configurations in $\mathcal{K}_{31}$ satisfying $x = x_1 e_1$ with $x_1 > 0$ and $x' = x_1' e_1$ with $x_1' > 0$. Moreover, let $\sigma : (\mathbb{R}^7, \xi^{\rm d}_x, \underline{\mathcal{Y}}(\kappa)) \xrightarrow{\sim} (\mathbb{R}^7, \xi^{\rm d}_{x'}, \underline{\mathcal{Y}}(\kappa'))$ be an isomorphism of dissident triples, i.e. an isomorphism of dissident maps $\sigma : \underline{\mathcal{Y}}(\kappa) \xrightarrow{\sim} \underline{\mathcal{Y}}(\kappa')$ which in addition satisfies $\xi^{\rm d}_x = \xi^{\rm d}_{x'}(\sigma \wedge \sigma)$. Just as in the previous proof, Lemmas 5.4 and 5.5 show that the first property implies $\lambda = \lambda'$. The second property is equivalent to $S\mathcal{X}(x)S^t = \mathcal{X}(x')$, where $S \in O_7(\mathbb{R})$ is the matrix representing σ in $\underline{e}$ and $\mathcal{X}(x), \mathcal{X}(x') \in \mathbb{R}^{7\times 7}_{\rm ant}$ are given by $\mathcal{X}(x)_{ij} = \xi^{\rm d}_x(e_i \wedge e_j)$ and $\mathcal{X}(x')_{ij} = \xi^{\rm d}_{x'}(e_i \wedge e_j)$ respectively. Accordingly the eigenvalues of $\mathcal{X}(x)^2$ and of $\mathcal{X}(x')^2$ coincide. In view of Lemma 3.1(i) this means that $\{0, -x_1^2\} = \{0, -(x_1')^2\}$, hence $x_1 = x_1'$. ■

Arguing along similar lines we are able to generalize Proposition 5.7 from $\mathcal{K}_{31}$ to $\mathcal{K}_3$. Because the technical effort needed for a proof is abominable we content ourselves here with the mere statement and refer for details to the forthcoming article [28].

PROPOSITION 5.8. *If $\kappa = (x, y, d)$ and $\kappa' = (x', y', d')$ are configurations in $\mathcal{K}_3$ such that the dissident triples $(\mathbb{R}^7, \xi^{\rm d}_x, \underline{\mathcal{Y}}(\kappa))$ and $(\mathbb{R}^7, \xi^{\rm d}_{x'}, \underline{\mathcal{Y}}(\kappa'))$ are isomorphic, then κ and κ' are isomorphic.*

6. On the classification of real quadratic division algebras. So far we strongly emphasized the viewpoint of dissident maps. However, in view of Proposition 1.1, any insight gained into dissident maps entails insight into real quadratic division algebras. Let us now bring in the harvest and summarize what the results of the previous sections mean for the problem of classifying all real quadratic division algebras.

To this end we need to introduce more terminology and notation. Let B be a real quadratic division algebra, with corresponding dissident triple $\mathcal{I}(B) = (V, \xi, \eta)$ (cf. introduction). We call B *disguised doubled* in case η is doubled, and *composed* in case η is composed. Furthermore we denote by $\mathcal{Q}_8^{\rm d}$, $\mathcal{Q}_8^{\rm dd}$ and $\mathcal{Q}_8^{\rm c}$ the full subcategories of $\mathcal{Q}_8$ formed by all objects $B \in \mathcal{Q}_8$ which are doubled, disguised doubled and composed respectively. These full subcategories are partially ordered by inclusion, with inclusion diagram

$$\begin{array}{ccccc} & & \mathcal{Q}_8 & & \\ & \nearrow & & \nwarrow & \\ \mathcal{Q}_8^{\rm dd} & & & & \mathcal{Q}_8^{\rm c} \\ \uparrow & & & & \\ \mathcal{Q}_8^{\rm d} & & & & \end{array}$$

Moreover, they occur as codomains of dense functors $\mathcal{F}^{\rm d} : \mathcal{K} \to \mathcal{Q}_8^{\rm d}$, $\mathcal{F}^{\rm dd} : \mathbb{R}^{7\times 7}_{\rm ant} \times \mathcal{K} \to \mathcal{Q}_8^{\rm dd}$ and $\mathcal{F}^{\rm c} : \mathcal{L} \to \mathcal{Q}_8^{\rm c}$ which we proceed to describe.

The composed functor $\mathcal{VHG} : \mathcal{K} \to \mathcal{Q}_8$ (cf. introduction) induces a dense and faithful (but not full) functor $\mathcal{F}^{\mathrm{d}} : \mathcal{K} \to \mathcal{Q}_8^{\mathrm{d}}$, which in turn induces an equivalence relation $\sim$ on $\mathcal{K}$, defined by setting $\kappa \sim \kappa'$ if and only if $\mathcal{F}^{\mathrm{d}}(\kappa) \xrightarrow{\sim} \mathcal{F}^{\mathrm{d}}(\kappa')$.

Recall that $\mathbb{R}^{7\times 7}_{\mathrm{ant}} = \{X \in \mathbb{R}^{7\times 7} \mid X^t = -X\}$. The object set $\mathbb{R}^{7\times 7}_{\mathrm{ant}} \times \mathcal{K}$ is endowed with the structure of a category by declaring as morphisms $S : (X, \kappa) \to (X', \kappa')$ the $\mathcal{K}$-morphisms $S : \kappa \to \kappa'$ which satisfy $\widetilde{S} X \widetilde{S}^t = X'$, where

$$\widetilde{S} = \begin{pmatrix} S & & \\ & 1 & \\ & & S \end{pmatrix}.$$

The functor $\mathcal{F}^{\mathrm{dd}} : \mathbb{R}^{7\times 7}_{\mathrm{ant}} \times \mathcal{K} \to \mathcal{Q}_8^{\mathrm{dd}}$, given on objects by $\mathcal{F}^{\mathrm{dd}}(X, \kappa) = \mathcal{H}(\mathbb{R}^7, \xi_X, \underline{\mathcal{Y}}(\kappa))$, where $\xi_X(v \wedge w) = v^t X w$, and on morphisms by $\mathcal{F}^{\mathrm{dd}}(S) = \mathcal{H}(\widetilde{S})$, is dense and faithful, but not full. The functor $\mathcal{F}^{\mathrm{dd}}$ induces an equivalence relation $\sim$ on $\mathbb{R}^{7\times 7}_{\mathrm{ant}} \times \mathcal{K}$, defined by setting $(X, \kappa) \sim (X', \kappa')$ if and only if $\mathcal{F}^{\mathrm{dd}}(X, \kappa) \xrightarrow{\sim} \mathcal{F}^{\mathrm{dd}}(X', \kappa')$.

The object set $\mathcal{L} = \mathbb{R}^{7\times 7}_{\mathrm{ant}} \times \mathbb{R}^{7\times 7}_{\mathrm{ant}} \times \mathbb{R}^{7\times 7}_{\mathrm{syp}}$ (cf. notation preceding Proposition 1.6) is endowed with the structure of a category by declaring as morphisms $S : (X, Y, D) \to (X', Y', D')$ those orthogonal matrices $S \in O_{\pi_7}(\mathbb{R}^7)$ satisfying $(SXS^t, SYS^t, SDS^t) = (X', Y', D')$. Denote by $\mathcal{D}_7^{\mathrm{c}}$ the full subcategory of $\mathcal{D}_7$ formed by all objects $(V, \xi, \eta) \in \mathcal{D}_7$ such that η is composed. The functor $\mathcal{G}_7 : \mathcal{L} \to \mathcal{D}_7^{\mathrm{c}}$, given on objects by $\mathcal{G}_7(X, Y, D) = (\mathbb{R}^7, \xi_X, \eta_{YD})$, where $\xi_X(v \wedge w) = v^t X w$ and $\eta_{YD}(v \wedge w) = (Y + D)\pi_7(v \wedge w)$, and acting on morphisms identically, is an equivalence of categories. (This is the categorical version of [13, Theorem 10], [15, Theorem 8], emphasizing the analogy to Proposition 1.5.) Moreover, the equivalence of categories $\mathcal{H} : \mathcal{D} \to \mathcal{Q}$ (Proposition 1.1) induces an equivalence of full subcategories $\mathcal{H}_7^{\mathrm{c}} : \mathcal{D}_7^{\mathrm{c}} \to \mathcal{Q}_8^{\mathrm{c}}$. Hence the composition $\mathcal{F}^{\mathrm{c}} = \mathcal{H}_7^{\mathrm{c}} \mathcal{G}_7$ is an equivalence of categories $\mathcal{F}^{\mathrm{c}} : \mathcal{L} \to \mathcal{Q}_8^{\mathrm{c}}$.

The functors $\mathcal{F}^{\mathrm{d}}, \mathcal{F}^{\mathrm{dd}}$ and $\mathcal{F}^{\mathrm{c}}$ enable "in principle" the classification of $\mathcal{Q}_8^{\mathrm{d}}, \mathcal{Q}_8^{\mathrm{dd}}$ and $\mathcal{Q}_8^{\mathrm{c}}$ to be attained by restricting these functors to cross-sections for the equivalence relations induced on their respective domains. It is however still a very hard problem to present such cross-sections explicitly. To date, our knowledge in this respect is expressed in Theorem 6.1(a)–(d) below. In statement (a), the symbol $[\mathbb{O}]$ denotes the isoclass of the octonion algebra.

THEOREM 6.1. (i) *The object class $\mathcal{Q}$ of all real quadratic division algebras decomposes into the pairwise heteromorphic subclasses $\mathcal{Q}_1$, $\mathcal{Q}_2$, $\mathcal{Q}_4$ and $\mathcal{Q}_8$.*

(ii) *The subclasses $\mathcal{Q}_1$ and $\mathcal{Q}_2$ are classified by $\{\mathbb{R}\}$ and $\{\mathbb{C}\}$ respectively.*

(iii) *The subclass* $\mathcal{Q}_4$ *is classified by* $\mathcal{HG}(\mathcal{C})$ *whenever* $\mathcal{C}$ *is a cross-section for* $\mathcal{K}/\simeq$. *Such a cross-section* $\mathcal{C}$ *is presented explicitly in* [12], [18], [27].

(iv) *The subclass* $\mathcal{Q}_8$ *contains the object classes* $\mathcal{Q}_8^{\mathrm{d}}$, $\mathcal{Q}_8^{\mathrm{dd}}$ *and* $\mathcal{Q}_8^{\mathrm{c}}$ *which admit the following description.*

(a) $\mathcal{Q}_8^{\mathrm{d}} \cap \mathcal{Q}_8^{\mathrm{c}} = [\mathbb{O}]$.

(b) *The object class* $\mathcal{Q}_8^{\mathrm{d}}$ *is classified by* $\mathcal{F}^{\mathrm{d}}(\mathcal{C}^{\mathrm{d}})$ *whenever* $\mathcal{C}^{\mathrm{d}}$ *is a cross-section for* $\mathcal{K}/\sim$. *There exists a cross-section* $\mathcal{C}^{\mathrm{d}}$ *which is contained in the cross-section* $\mathcal{C}$ *presented in* [12], [18], [27]. *A subset of such a cross-section* $\mathcal{C}^{\mathrm{d}}$ *is given by the* 2*-parameter family* $\{(x_1e_1, 0, \lambda 1_3) \in \mathcal{K} \mid x_1 \geq 0 \ \wedge \ \lambda > 0\}$ *of pairwise non-equivalent configurations. A complete cross-section* $\mathcal{C}^{\mathrm{d}}$ *is not known as yet.*

(c) *The object class* $\mathcal{Q}_8^{\mathrm{dd}}$ *is classified by* $\mathcal{F}^{\mathrm{dd}}(\mathcal{C}^{\mathrm{dd}})$ *whenever* $\mathcal{C}^{\mathrm{dd}}$ *is a cross-section for* $(\mathbb{R}^{7\times 7}_{\mathrm{ant}} \times \mathcal{K})/\sim$. *Such a cross-section* $\mathcal{C}^{\mathrm{dd}}$ *is not known as yet.*

(d) *The object class* $\mathcal{Q}_8^{\mathrm{c}}$ *is classified by* $\mathcal{F}^{\mathrm{c}}(\mathcal{C}^{\mathrm{c}})$ *whenever* $\mathcal{C}^{\mathrm{c}}$ *is a cross-section for* $\mathcal{L}/\simeq$. *A subset of such a cross-section* $\mathcal{C}^{\mathrm{c}}$, *forming a* 49*-parameter family of pairwise heteromorphic objects in* $\mathcal{L}$, *is presented explicitly in* [13], [15]. *A complete cross-section* $\mathcal{C}^{\mathrm{c}}$ *is not known as yet.*

Proof. (i) is the $(1,2,4,8)$-Theorem of Bott, Milnor [8] and Kervaire [23], specialized to real quadratic division algebras.

(ii) follows by Proposition 1.1 from the trivial fact that $\mathcal{D}_0$ is classified by $\{(\{0\}, o, o)\}$ and $\mathcal{D}_1$ is classified by $\{(\mathbb{R}, o, o)\}$.

(iii) The composed functor $\mathcal{HG} : \mathcal{K} \to \mathcal{Q}_4$ is an equivalence of categories, by Propositions 1.5 and 1.1.

(iv) (a) Let $B \in [\mathbb{O}]$. Then $B \in \mathcal{Q}_8^{\mathrm{d}}$ because $B \xrightarrow{\sim} \mathcal{V}(\mathbb{H})$, and $B \in \mathcal{Q}_8^{\mathrm{c}}$ because $\mathcal{I}(B) = (V, o, \pi)$, where π is a vector product on V (cf. [26]).

Conversely, let $B \in \mathcal{Q}_8^{\mathrm{d}} \cap \mathcal{Q}_8^{\mathrm{c}}$, with corresponding dissident triple $\mathcal{I}(B) = (V, \xi, \eta)$. Since B is doubled, there exists a configuration $\kappa = (x, y, d) \in \mathcal{K}$ such that $B \xrightarrow{\sim} \mathcal{F}^{\mathrm{d}}(\kappa) = \mathcal{VHG}(\kappa)$. Applying Lemma 3.1(i) we conclude that $(V, \xi, \eta) \xrightarrow{\sim} (\mathbb{R}^7, \xi_x^{\mathrm{d}}, \underline{\mathcal{Y}}(\kappa))$. Since B is both doubled and composed, $\underline{\mathcal{Y}}(\kappa)$ is both doubled and composed, which by Proposition 4.6 implies that $\kappa = (0, 0, 1_3)$. Hence $(\mathbb{R}^7, \xi_x^{\mathrm{d}}, \underline{\mathcal{Y}}(\kappa)) = (\mathbb{R}^7, o, \pi_7)$, and therefore

$$B \xrightarrow{\sim} \mathcal{HI}(B) \xrightarrow{\sim} \mathcal{H}(\mathbb{R}^7, o, \pi_7) \xrightarrow{\sim} \mathbb{O}.$$

(b) The first statement is due to the density of the functor $\mathcal{F}^{\mathrm{d}} : \mathcal{K} \to \mathcal{Q}_8^{\mathrm{d}}$. The second is due to the trivial fact that $\kappa \xrightarrow{\sim} \kappa'$ only if $\kappa \sim \kappa'$. The third follows from the Propositions 1.1, 5.3, 5.6 and 5.7.

(c) The functor $\mathcal{F}^{\mathrm{dd}} : \mathbb{R}^{7\times 7}_{\mathrm{ant}} \times \mathcal{K} \to \mathcal{Q}_8^{\mathrm{dd}}$ is dense.

(d) The functor $\mathcal{F}^{\mathrm{c}} : \mathcal{L} \to \mathcal{Q}_8^{\mathrm{c}}$ is an equivalence of categories. ∎

7. Epilogue. The problem of constructing and, ultimately, classifying all real division algebras originated in the discovery of the quaternion algebra $\mathbb{H}$ (Hamilton 1843) and the octonion algebra $\mathbb{O}$ (Graves 1843, Cayley 1845). The once vivid interest in this problem was severely inhibited by the theorems of Frobenius [19] and Zorn [32], asserting that the associative real division algebras are classified by $\{\mathbb{R}, \mathbb{C}, \mathbb{H}\}$ and the alternative real division algebras are classified by $\{\mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}\}$. Hopf's contribution [22] awoke the interest of topologists and launched a new phase in this subject, culminating in Bott, Milnor and Kervaire's $(1, 2, 4, 8)$-Theorem [8], [23] and Adams's Formula [1] for the span of $\mathbb{S}^{n-1}$. Real division algebras seemed to have been wrested from algebraists for good. Many a mathematician interpreted the (1,2,4,8)-Theorem as the final word on the subject, overlooking that the triumphant progress of topology had not produced a single new example of a real division algebra. The erroneous view that $\{\mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}\}$ classifies all real division algebras spread and became "folklore knowledge", documented even in print in a widely used and otherwise highly reputed textbook (cf. [17]).

Attempting to recover the algebraic view of real division algebras by generalizing the results of Frobenius and Zorn, it is natural to aim at a classification of all power-associative real division algebras. These coincide with the quadratic real division algebras (cf. [12, Lemma 5.3]). An approach to the latter was opened by Osborn's Theorem [29, p. 204], which, however, took effect only hesitantly. Its true impact was obscured for decades by applications of Osborn [29] and Hefendehl-Hebeker [20], [21] which partly contain a misleading flaw (cf. [15]) and partly conceal the conceptual core of the matter in technical complications (cf. [18]). Osborn's Theorem was rediscovered by Dieterich [13] and it reappears, in categorical formulation, as our Proposition 1.1. In this shape it forms the foundation for most of the present article.

Another natural class of real division algebras generalizing the alternative ones is formed by the real flexible division algebras. These were studied in [4], where their classification essentially is reduced to the classification of all real quadratic flexible division algebras. The latter problem was solved in [9] by use of "vectorial isotopy", a concept which independently was introduced in [16]. It is not difficult to see that the real quadratic flexible division algebras which in addition are doubled or composed are classified by the two one-parameter families $\{\mathcal{VH}(\mathbb{R}^3, o, \lambda\pi_3) \mid \lambda > 0\}$ and $\{\mathcal{H}(\mathbb{R}^7, o, \mu\pi_7) \mid \mu > 0\}$, intersecting (according to Proposition 4.6) in the single isoclass $[\mathbb{O}]$ corresponding to the parameter values $\lambda = \mu = 1$ (cf. [10]).

A more general approach to the classification problem of real division algebras by investigating their Lie algebras of derivations was pursued by Benkart and Osborn in a series of articles [3]–[6].

Shafarevich suggested the structure of a real division algebra as a test problem for a possible future understanding of various types of algebras from a unified point of view (cf. [31, p. 201]). The present article intends to contribute to a solution of this challenging test problem.

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