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H^1 AND BMO FOR CERTAIN LOCALLY DOUBLING METRIC MEASURE SPACES OF FINITE MEASURE

BY

ANDREA CARBONARO (Genova), GIANCARLO MAUCERI (Genova) and STEFANO MEDA (Milano)

In memory of our friend Andrzej Hulanicki

Abstract. In a previous paper the authors developed an H^1-BMO theory for unbounded metric measure spaces (M,ρ,μ) of infinite measure that are locally doubling and satisfy two geometric properties, called "approximate midpoint" property and "isoperimetric" property. In this paper we develop a similar theory for spaces of finite measure. We prove that all the results that hold in the infinite measure case have their counterparts in the finite measure case. Finally, we show that the theory applies to a class of unbounded, complete Riemannian manifolds of finite measure and to a class of metric measure spaces of the form $(\mathbb{R}^d,\rho_\varphi,\mu_\varphi)$, where $\mathrm{d}\mu_\varphi = \mathrm{e}^{-\varphi}\,\mathrm{d}x$ and ρ_φ is the Riemannian metric corresponding to the length element $\mathrm{d}s^2 = (1+|\nabla\varphi|)^2(\mathrm{d}x_1^2+\cdots+\mathrm{d}x_d^2)$. This generalizes previous work of the last two authors for the Gauss space.

1. Introduction. In [3] the authors developed an H^1-BMO theory on unbounded metric measure spaces (M,ρ,μ) that are locally doubling and satisfy two additional "geometric" properties, called *approximate midpoint* (AM) property and *isoperimetric* (I) property. Roughly speaking, a space satisfies (AM) if its points do not become too sparse at infinity, and satisfies (I) if a fixed proportion of the measure of any bounded set is concentrated near the boundary.

For each scale parameter b in $\mathbb{R}^+$, we defined the spaces $H^1_b(\mu)$ and $\mathrm{BMO}_b(\mu)$ much as in the classical case of spaces of homogeneous type, in the sense of Coifman and Weiss [6], the only difference being that the balls involved have at most radius b. Then we showed that these spaces do not depend on the scale b, at least if b is sufficiently large, and that all the classical results that hold on spaces of homogeneous type, such as a John–Nirenberg inequality, the $H^1(\mu)$-$\mathrm{BMO}(\mu)$ duality, complex interpolation, hold for these spaces. Moreover, these spaces provide end-point estimates for some interesting singular integrals which arise in various settings. We also showed

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that the theory applies to noncompact complete Riemannian manifolds with Ricci curvature bounded from below and strictly positive spectrum, e.g. to noncompact Riemannian symmetric spaces.

In [3] we focused on the case where $\mu(M) = \infty$. In this paper we tackle the case where $\mu(M) < \infty$. In this case we must modify slightly the isoperimetric property, by assuming that, instead of (I), M satisfies the *complementary isoperimetric* property ($\mathrm{I}^c_{B_0}$). Roughly speaking, M satisfies ($\mathrm{I}^c_{B_0}$) if there exists a ball B_0 such that a fixed proportion of the measure of any open set contained in $M \setminus \bar{B}_0$ is concentrated near the boundary of the set.

When $\mu(M)$ is finite, the definitions of the atomic Hardy space $H^1(\mu)$ and the space $\mathrm{BMO}(\mu)$ of functions of bounded mean oscillation are quite similar to those of the corresponding spaces in the infinite measure case considered in [3].

To be specific, for each b in $\mathbb{R}^+$ denote by $\mathcal{B}_b$ the collection of balls of radius at most b. The constant b may be thought of as a "scale parameter", and the balls in $\mathcal{B}_b$ are called *admissible balls at the scale* b. An atom a is either the exceptional atom $1/\mu(M)$ or a function in $L^1(\mu)$ supported in a ball B which satisfies an appropriate "size" and cancellation condition. Fix a sufficiently large "scale parameter" b in $\mathbb{R}^+$ (how large depends on the constants that appear in the definition of the (AM) property). Then $H^1(\mu)$ is the space of all functions in $L^1(\mu)$ that admit a decomposition of the form $\sum_j \lambda_j a_j$, where the a_j's are atoms supported in balls in $\mathcal{B}_b$ or the exceptional atom, and the sequence of complex numbers $\{\lambda_j\}$ is summable.

A locally integrable function f is in $\mathrm{BMO}(\mu)$ if it is in $L^1(\mu)$ and

$$\sup_B \frac{1}{\mu(B)} \int_B |f - f_B| \, \mathrm{d}\mu < \infty,$$

where the supremum is taken over *all* balls B in $\mathcal{B}_b$, and f_B denotes the average of f over B. This definition of $\mathrm{BMO}(\mu)$ is inspired by previous work of A. Ionescu [12], who defined a similar space on rank one noncompact symmetic spaces.

We prove that these spaces indeed do not depend on the parameter b, that the topological dual of $H^1(\mu)$ is isomorphic to $\mathrm{BMO}(\mu)$, and an inequality of John–Nirenberg type holds for functions in $\mathrm{BMO}(\mu)$. Furthermore, the spaces $L^p(\mu)$ are intermediate spaces between $H^1(\mu)$ and $\mathrm{BMO}(\mu)$ for the complex interpolation methods. It is worth observing that some important operators, which are bounded on $L^p(\mu)$ for all p in $(1,\infty)$, but otherwise unbounded on $L^1(\mu)$ and on $L^\infty(\mu)$, turn out to be bounded from $H^1(\mu)$ to $L^1(\mu)$ and from $L^\infty(\mu)$ to $\mathrm{BMO}(\mu)$.

Some of the proofs of these results require only simple adaptations of the proofs of the analogous results in [3]. In these cases we shall briefly indicate the variations needed. Other proofs, like those of the duality and the interpolation results, require more substantial changes, and we give full details.

In Section 7 we show that our theory applies to unbounded complete Riemannian manifolds M of finite volume with Ricci curvature bounded from below such that Cheeger's isoperimetric constant $h(M)$ is strictly positive. It is well known that, on such manifolds, Cheeger's constant is strictly positive if and only if the Laplace–Beltrami operator $\mathcal{L}$ on M has spectral gap, i.e. if and only if 0 is an isolated eigenvalue of $\mathcal{L}$ on $L^2(\mu)$.

In [15] G. Mauceri and S. Meda defined an atomic Hardy space $H^1(\gamma)$ and a space $\mathrm{BMO}(\gamma)$ of functions of bounded mean oscillation associated to the Gauss measure $\mathrm{d}\gamma(x) = \mathrm{e}^{-|x|^2}\,\mathrm{d}x$ on $\mathbb{R}^d$. We recall briefly the definitions of these spaces. For each scale parameter b we denote by $\mathcal{B}_b^\gamma$ the set of all Euclidean balls B in $\mathbb{R}^d$ such that

$$r_B \leq b\min(1, 1/|c_B|),$$

where c_B and r_B denote the centre and the radius of B respectively. Now, $H^1(\gamma)$ is defined as $H^1(\mu)$ above, but with the family of admissible balls $\mathcal{B}_b$ replaced by $\mathcal{B}_b^\gamma$, and similarly for $\mathrm{BMO}(\gamma)$. In [15] the authors proved that $H^1(\gamma)$ and $\mathrm{BMO}(\gamma)$ possess the analogues of the properties enumerated above for $H^1(\mu)$ and $\mathrm{BMO}(\mu)$. They also showed that some important operators related to the Ornstein–Uhlenbeck operator on $\mathbb{R}^d$ that are bounded on $L^p(\gamma)$ for all p in $(1,\infty)$, but otherwise unbounded on $L^1(\gamma)$ and on $L^\infty(\gamma)$, are bounded from $H^1(\gamma)$ to $L^1(\gamma)$ and from $L^\infty(\gamma)$ to $\mathrm{BMO}(\gamma)$.

It may be worth observing that the metric measure space $(\mathbb{R}^d, \rho, \gamma)$, where ρ denotes the Euclidean distance, has finite measure and is not locally doubling.

The definition of the class $\mathcal{B}_b^\gamma$ of admissible balls in [15] suggests that on the Gauss space $(\mathbb{R}^d, \rho, \gamma)$ the Euclidean metric ρ should be replaced by the Riemannian metric associated to the length element $\mathrm{d}s^2 = (1+|x|)^2 (\mathrm{d}x_1^2 + \cdots + \mathrm{d}x_d^2)$.

In Section 8 we exploit and generalize this idea, by considering metric measure spaces of the form $(\mathbb{R}^d, \rho_\varphi, \mu_\varphi)$ where φ is a function in $C^2(\mathbb{R}^d)$, ρ_φ is the Riemannian metric on $\mathbb{R}^d$ defined by the length element $\mathrm{d}s^2 = (1+|\nabla\varphi|)^2(\mathrm{d}x_1^2 + \cdots + \mathrm{d}x_d^2)$ and $\mathrm{d}\mu_\varphi = \mathrm{e}^\varphi\,\mathrm{d}\lambda$, where λ is the Lebesgue measure on $\mathbb{R}^d$. We prove that if the function φ satisfies appropriate conditions, the space $(\mathbb{R}^d, \rho_\varphi, \mu_\varphi)$ is locally doubling and satisfies properties (AM) and $(\mathrm{I}^c_{B_0})$.

Finally, we recall that Hardy spaces and spaces of functions of bounded mean oscillation have recently been studied on various nondoubling metric measure spaces [14, 18, 19, 20]. We point out that our spaces are different and that they provide end-point estimates for singular integrals which do not satisfy the standard Calderón–Zygmund estimates at infinity, still maintaining the important property that the complex interpolation spaces between $H^1(\mu)$ and $\mathrm{BMO}(\mu)$ are the spaces $L^p(\mu)$.

2. Geometric assumptions. Suppose that (M, ρ, μ) is a metric measure space and denote by $\mathcal{B}$ the family of all balls in M. We assume that $0 < \mu(M) < \infty$. For each B in $\mathcal{B}$ we denote by c_B and r_B the centre and the radius of B respectively. Furthermore, for each $\kappa > 0$, we denote by κB the ball with centre c_B and radius κr_B. For each b in $\mathbb{R}^+$, we denote by $\mathcal{B}_b$ the family of all balls B in $\mathcal{B}$ such that $r_B \leq b$. For any subset A of M and each κ in $\mathbb{R}^+$ we denote by A_κ and A^κ the sets

$$\{x \in A : \rho(x, A^c) \leq \kappa\} \quad \text{and} \quad \{x \in A : \rho(x, A^c) > \kappa\}$$

respectively.

In this paper we assume that (M, ρ, μ) is an unbounded metric measure space of *finite measure*, which possesses the following properties:

(i) *Local doubling property* (LD): for every b in $\mathbb{R}^+$ there exists a constant D_b such that

$$\mu(2B) \leq D_b \mu(B) \qquad \forall B \in \mathcal{B}_b.$$

This property is often called *local doubling condition* in the literature, and we adhere to this terminology. Note that if (LD) holds and M is bounded, then μ is doubling.

(ii) *Property* (AM) (approximate midpoint property): there exist R_0 in $[0, \infty)$ and β in $(1/2, 1)$ such that for every pair of points x and y in M with $\rho(x, y) > R_0$ there exists a point z in M such that $\rho(x, z) < \beta\rho(x, y)$ and $\rho(y, z) < \beta\rho(x, y)$.

(iii) *Complementary isoperimetric property* ($\mathrm{I}^c_{B_0}$): there exist a ball B_0 in M, and κ_0 and C in $\mathbb{R}^+$, such that for every open set A contained in $M \setminus \bar{B}_0$,

$$\mu(A_\kappa) \geq C\kappa\mu(A) \qquad \forall \kappa \in (0, \kappa_0]. \tag{2.1}$$

Suppose that M has property ($\mathrm{I}^c_{B_0}$). For each t in $(0, \kappa_0]$ we denote by C_t the supremum over all constants C for which (2.1) holds for all κ in $(0, t]$. Then we define

$$I^c_{M,B_0} = \sup\{C_t : t \in (0, \kappa_0]\}.$$

Note that the function $t \mapsto C_t$ is decreasing on $(0, \kappa_0]$, so that

$$I^c_{M,B_0} = \lim_{t \to 0^+} C_t. \tag{2.2}$$

Remark 2.1. The first two geometric assumptions (LD) and (AM) coincide with the corresponding assumptions made in [3] for spaces of infinite measure. The isoperimetric property is sligthly different from the isoperimetric property (I) in [3], because in the infinite measure case we assumed that inequality (2.1) holds for all bounded open sets in M.

Remark 2.2. The local doubling property implies that for each $\tau \geq 2$ and for each b in $\mathbb{R}^+$ there exists a constant C such that

$$\mu(B') \leq C\mu(B) \tag{2.3}$$

for each pair of balls B and B' with $B \subset B'$, B in $\mathcal{B}_b$, and $r_{B'} \leq \tau r_B$. We shall denote by $D_{\tau,b}$ the smallest constant for which (2.3) holds. In particular, if (2.3) holds (with the same constant) for all balls B in $\mathcal{B}$, then μ is doubling and we shall denote by $D_{\tau,\infty}$ the smallest constant for which (2.3) holds.

Remark 2.3. Loosely speaking, the approximate midpoint property means that the points of M "do not become too sparse at infinity". The property is obviously satisfied on all length metric spaces.

Remark 2.4. In Section 7 we shall see that, on complete Riemannian manifolds, the complementary isoperimetric property is equivalent to the positivity of Cheeger's isoperimetric constant

$$h(M) = \inf \frac{\sigma(\partial A)}{\mu(A)}$$

where the infimum runs over all bounded open sets A with $\mu(A) \leq \mu(M)/2$ and with smooth boundary ∂A. Here σ denotes the induced Riemannian measure on ∂A. Moreover, if the Ricci curvature of M is bounded from below, both properties are equivalent to the existence of a spectral gap for the Laplacian.

Remark 2.5. The local doubling property is needed for all the results in this paper, but many results in Sections 2–5 depend only on some but not all of the properties (i)–(iii). In particular, all the results in Sections 3 and 4 require property (AM) but not ($\mathrm{I}^c_{B_0}$); Lemma 5.4 and Theorem 5.5, which are crucial to proving the interpolation result Theorem 5.6, require property ($\mathrm{I}^c_{B_0}$), but not (AM). Finally, all the properties (i)–(iii) above are needed for the interpolation results and the theory of singular integral operators in Section 5.

Proposition 2.6. *Suppose that M possesses property ($\mathrm{I}^c_{B_0}$). The following hold:*

(i) *for every open set A contained in $M \setminus \bar{B}_0$,*

$$\mu(A_t) \geq (1 - \mathrm{e}^{-I^c_{M,B_0} t})\mu(A) \qquad t \in \mathbb{R}^+;$$

(ii) *for every point x in M there exists a constant C, which depends on x, I^c_{M,B_0} and B_0, such that*

$$\mu(B(x,r)^c) \leq C\,\mathrm{e}^{-I^c_{M,B_0} r} \qquad \forall r > 0.$$

Proof. The proof of (i) is almost *verbatim* the same as the proof of [3, Proposition 3.1], and is omitted.

Now we prove (ii). Denote by V_r the measure of $B(x,r)^c$. Since $\mu(B(x,r)^c) \leq \mu(M)$ for every $r > 0$, it is clearly enough to prove the inequality for r sufficiently large, say $r > r_{B_0} + d(x, c_{B_0}) + 1$. Then $B(x,r-1)^c \subset M \setminus \bar{B}_0$ and $B(x,r-1)^c \setminus B(x,r)^c \supseteq (B(x,r-1)^c)_1$. Thus, by (i),

$$V_{r-1} - V_r \geq \mu((B(x,r-1)^c)_1) \geq (1 - \mathrm{e}^{-I^c_{M,B_0}}) V_{r-1}.$$

Hence

$$V_r \leq \mathrm{e}^{-I^c_{M,B_0}}\ V_{r-1}.$$

By iteration, if $r_{B_0} + d(x, c_{B_0}) + n < r \leq r_{B_0} + d(x, c_{B_0}) + n + 1$ we obtain

$$V_r \leq \mathrm{e}^{-I^c_{M,B_0} n} V_{r-n} \leq C\, \mathrm{e}^{-I^c_{M,B_0} r},$$

where $C = \exp((r_{B_0} + d(x, c_{B_0}) + 1) I^c_{M,B_0})\, \mu(M)$. ∎

3. H^1 and BMO**.** In this section we define the Hardy space $H^1(\mu)$ and the space $\mathrm{BMO}(\mu)$. The definitions are very similar to those given in [3] for metric spaces of infinite measure. The only differences are the existence of the "exceptional atom" in $H^1(\mu)$ and the fact that $\mathrm{BMO}(\mu)$ is defined as a subspace of $L^1(\mu)$.

Definition 3.1. Suppose that r is in $(1,\infty]$. A $(1,r)$-*standard atom* a is a function in $L^1(\mu)$ supported in a ball B in $\mathcal{B}$ with the following properties:

(i) $\|a\|_\infty \leq \mu(B)^{-1}$ if $r = \infty$ and

$$\left(\frac{1}{\mu(B)} \int_B |a|^r \,\mathrm{d}\mu\right)^{1/r} \leq \mu(B)^{-1} \quad \text{if } r \in (1,\infty);$$

(ii) $\int_B a \,\mathrm{d}\mu = 0$.

The constant function $1/\mu(M)$ is referred to as the *exceptional atom.*

Definition 3.2. Suppose that b is in $\mathbb{R}^+$ and that r is in $(1,\infty]$. The *Hardy space* $H^{1,r}_b(\mu)$ is the space of all functions g in $L^1(\mu)$ that admit a decomposition of the form

$$g = \sum_{k=1}^{\infty} \lambda_k a_k, \tag{3.1}$$

where a_k is either a $(1,r)$-atom *supported in a ball* B *of* $\mathcal{B}_b$ or the exceptional atom, and $\sum_{k=1}^\infty |\lambda_k| < \infty$. The norm $\|g\|_{H^{1,r}_b(\mu)}$ of g is the infimum of $\sum_{k=1}^\infty |\lambda_k|$ over all decompositions (3.1) of g.

DEFINITION 3.3. Suppose that b is in $\mathbb{R}^+$ and that q is in $[1,\infty)$. For each locally integrable function f we define $N_b^q(f)$ by

$$N_b^q(f) = \sup_{B\in\mathcal{B}_b} \left(\frac{1}{\mu(B)} \int_B |f - f_B|^q \,\mathrm{d}\mu\right)^{1/q},$$

where f_B denotes the average of f over B. We denote by $\mathrm{BMO}_b^q(\mu)$ the space of all functions f in $L^1(\mu)$ such that $N_b^q(f)$ is finite, endowed with the norm

$$\|f\|_{\mathrm{BMO}_b^q(\mu)} = \|f\|_1 + N_b^q(f).$$

Note that only balls of radius at most b enter in the definitions of $H_b^{1,r}(\mu)$ and $\mathrm{BMO}_b^q(\mu)$.

It is a nontrivial fact that $H_b^{1,r}(\mu)$ and $\mathrm{BMO}_b^q(\mu)$ are independent of the parameter b, provided b is large enough. Recall that R_0 and β are the constants which appear in the definition of the (AM) property.

PROPOSITION 3.4. *Suppose that r is in $(1,\infty]$, q is in $[1,\infty)$, and b and c are in $\mathbb{R}^+$ and satisfy $R_0/(1-\beta) < c < b$. The following hold:*

(i) *the identity is a Banach space isomorphism between $H_c^{1,r}(\mu)$ and $H_b^{1,r}(\mu)$ and between $\mathrm{BMO}_b^q(\mu)$ and $\mathrm{BMO}_c^q(\mu)$;*

(ii) *(John–Nirenberg type inequality) there exist positive constants c and C such that for all $f \in \mathrm{BMO}_b^1(\mu)$ and all B in $\mathcal{B}_b$,*

$$\mu(\{x \in B : |f(x) - f_B| > s\}) \le C\,\mathrm{e}^{-cs/N_b^1(f)}\mu(B);$$

(iii) *for each q in $(1,\infty)$ there exists a constant C such that*

$$N_b^1(f) \le N_b^q(f) \le C N_b^1(f) \qquad f \in \mathrm{BMO}_b^q(\mu).$$

Proof. The proof of (i) is *almost verbatim* the same as the proofs of [3, Prop. 4.3] and [3, Prop. 5.1] respectively, and is omitted. The proof of (ii) is the same as the proof of [3, Thm. 5.4], and the proof of (iii) follows the lines of the proof of [3, Corollary 5.5]. ∎

Suppose that b and c are in $\mathbb{R}^+$ and satisfy $R_0/(1-\beta) < c < b$. In view of Proposition 3.4(ii)–(iv), if q and r are in $[1,\infty)$, then the identity is a Banach space isomorphism between $\mathrm{BMO}_b^q(\mu)$ and $\mathrm{BMO}_c^r(\mu)$. We denote simply by $\mathrm{BMO}(\mu)$ the Banach space $\mathrm{BMO}_b^q(\mu)$ endowed with any of the equivalent norms N_b^q.

Similarly, in view of Proposition 3.4(i), if r is in $(1,\infty)$, and $R_0/(1-\beta) < c < b$, then $H_b^{1,r}(\mu)$ and $H_c^{1,r}(\mu)$ are isomorphic Banach spaces, and they will simply be denoted by $H^{1,r}(\mu)$. In Section 4 we shall prove that the topological dual of $H^{1,r}(\mu)$ may be identified with $\mathrm{BMO}^{r'}(\mu)$, where r' denotes the index conjugate to r. Suppose that $1 < r < s < \infty$. Then $(H^{1,r}(\mu))^* = (H^{1,s}(\mu))^*$, because we have proved that $\mathrm{BMO}^{r'}(\mu) = \mathrm{BMO}^{s'}(\mu)$. Observe that the identity is a continuous injection of $H^{1,s}(\mu)$ into $H^{1,r}(\mu)$, and

$H^{1,s}(\mu)$ is a dense subspace of $H^{1,r}(\mu)$. Hence we may conclude that $H^{1,s}(\mu) = H^{1,r}(\mu)$. Therefore we shall denote $H^{1,r}(\mu)$ simply by $H^1(\mu)$.

4. Duality. In this section we prove the analogue of the duality result [3, Thm. 6.1]. The proof in the finite measure case is more difficult because we must show that for every linear functional ℓ in the dual of $H^1(\mu)$ the function f^ℓ that represents the functional on $H^1(\mu) \cap L^2_c(\mu)$, constructed in [3, Thm. 6.1], is also in $L^1(\mu)$.

We need more notation and some preliminary observation. Suppose that $b > R_0/(1-\beta)$, where R_0 and β are the constants in the approximate midpoint property (AM) (see Section 2). A ball B in $\mathcal{B}_b$ is said to be *maximal* if $r_B = b$.

We shall make use of the analogues in our setting of the so-called dyadic cubes Q^k_α introduced by G. David and M. Christ [8, 5] on spaces of homogeneous type.

THEOREM 4.1. *There exists a collection of open subsets* $\{Q^k_\alpha : k \in \mathbb{Z}, \alpha \in I_k\}$ *and constants* δ *in* $(0,1)$, *and* a_0, C_1 *in* $\mathbb{R}^+$, *such that*

(i) $\bigcup_\alpha Q^k_\alpha$ *is a set of full measure in* M *for each* k *in* $\mathbb{Z}$;
(ii) *if* $l \geq k$, *then either* $Q^l_\beta \subset Q^k_\alpha$ *or* $Q^l_\beta \cap Q^k_\alpha = \emptyset$;
(iii) *for each* (k,α) *and each* $l < k$ *there is a unique* β *such that* $Q^k_\alpha \subset Q^l_\beta$;
(iv) $\operatorname{diam}(Q^k_\alpha) \leq C_1 \delta^k$;
(v) *each* Q^k_α *contains some ball* $B(z^k_\alpha, a_0\delta^k)$.

It may help to think of Q^k_α as being essentially a cube of diameter δ^k with "centre" z^k_α. Note that (iv) and (v) imply that for every integer k and each α in I_k,

$$B(z^k_\alpha, a_0\delta^k) \subset Q^k_\alpha \subset B(z^k_\alpha, C_1\delta^k/2).$$

REMARK 4.2. When we use dyadic cubes, we implicitly assume that for each k in $\mathbb{Z}$ the set $M \setminus \bigcup_{\alpha \in I_k} Q^k_\alpha$ has been permanently deleted from the space.

We shall denote by $\mathcal{Q}^k$ the class of all dyadic cubes of "resolution" k, i.e., the family of cubes $\{Q^k_\alpha : \alpha \in I_k\}$, and by $\mathcal{Q}$ the set of all dyadic cubes. We denote by $\mathfrak{Z}^\nu$ the set $\{z^\nu_\alpha : \alpha \in I_\nu\}$, i.e. the set of "centres" of all dyadic cubes of "resolution" ν. We recall that, in Christ's construction of the family $\mathcal{Q}$ of dyadic cubes, the set $\mathfrak{Z}^\nu$ is a maximal collection of points in M such that

$$\rho(z^\nu_\alpha, z^\nu_\beta) \geq \delta^\nu$$

for all α, β in I_ν with $\alpha \neq \beta$.

We shall need the following additional properties of dyadic cubes.

LEMMA 4.3. *Choose an integer ν such that $\delta^\nu \min(1, 2a_0) > R_0$ and b in $\mathbb{R}^+$ such that $b > 4\delta^\nu \max(1/(1-\beta), a_0)$. For each z^ν_α in $\mathfrak{Z}^\nu$ denote by B_α the ball $B(z^\nu_\alpha, b)$. The following hold:*

(i) *the balls $\{B_\alpha\}$ form a locally uniformly finite covering of M, i.e. there exists an integer N_0 such that*

$$1 \leq \sum_{\alpha\in\mathbb{N}} \mathbf{1}_{B_\alpha} \leq N_0;$$

(ii) *for every pair o, z of distinct points in $\mathfrak{Z}^\nu$, there exists a chain of points $z^\nu_{\alpha_1}, \ldots, z^\nu_{\alpha_N}$ in $\mathfrak{Z}^\nu$ such that $o = z^\nu_{\alpha_1}$, $z = z^\nu_{\alpha_N}$,*

$$N \leq 4(2d/b)^{1/[1-\log_2(1+\beta)]} + 1 \quad \textit{and} \quad \rho(z^\nu_{\alpha_j}, z^\nu_{\alpha_{j+1}}) < b/2,$$

where d denotes the distance $\rho(o, z)$. Furthermore, for $1 \leq j \leq N-1$ the intersection $B_{\alpha_j} \cap B_{\alpha_{j+1}}$ contains the ball $B(z_{\alpha_{j+1}}, a_0\delta^\nu)$, and

$$\frac{\mu(B_{\alpha_{j+1}})}{\mu(B(z^\nu_{\alpha_{j+1}}, a_0\delta^\nu))} \leq D_{b/(a_0\delta^\nu), a_0\delta^\nu}. \tag{4.1}$$

Proof. First we prove (i). By the maximaliy of the collection $\mathfrak{Z}^\nu$, for each x in M there exists z^ν_α in $\mathfrak{Z}^\nu$ such that $\rho(z^\nu_\alpha, x) < \delta^\nu$. This implies the left inequality in (i).

A simple variation of the proof of [3, Prop. 3.4(iv)] shows that there exists an integer N_0, which depends on b, ν, a_0 and C_1, such that a ball of radius $2b$ intersects at most N_0 cubes in $\mathcal{Q}^\nu$. Let $A(x) = \{B_\alpha : x \in B_\alpha\}$. Since $z^\nu_\alpha \in B_\alpha$ and $\bigcup_{B_\alpha\in A(x)} B_\alpha \subset B(x, 2b)$, the cubes $Q^\nu_{z^\nu_\alpha}$, $B_\alpha \in A(x)$, intersect $B(x, 2b)$. Thus the cardinality of $A(x)$ is at most N_0. This proves the right inequality in (i).

Next we prove (ii). Recall that d denotes the distance between o and z. Denote by B^o and B^z the balls with radius b centred at o and z respectively.

First suppose that $d < b/2$. Then the chain reduces to the two points o and z. Moreover, $B^o \cap B^z$ contains the ball $B(z, a_0\delta^\nu)$. Indeed, B^z contains $B(z, a_0\delta^\nu)$ (recall that $b > 4a_0\delta^\nu$), and B^o contains $B(z, a_0\delta^\nu)$, because B^o has radius b and $b > b/2 + a_0\delta^\nu$ is equivalent to $b > 2a_0\delta^\nu$, which we assume.

Next suppose that $d \geq b/2$. Since $b/2 > R_0$, there exists a point z_1 in M such that

$$\max(\rho(z_1, o), \rho(z_1, z)) < \beta d$$

by the (AM) property. In general, z_1 need not be in $\mathfrak{Z}^\nu$. However, by the maximality of $\mathfrak{Z}^\nu$, there exists $z^\nu_{\alpha_1}$ in $\mathfrak{Z}^\nu$ such that $\rho(z^\nu_{\alpha_1}, z_1) < \delta^\nu$. We observe that

$$\max(\rho(z^\nu_{\alpha_1}, o), \rho(z^\nu_{\alpha_1}, z)) < \frac{1+\beta}{2}\, d.$$

Indeed, by the triangle inequality

$$\rho(z^\nu_{\alpha_1}, o) \le \rho(z^\nu_{\alpha_1}, z_1) + \rho(z_1, o) \le \delta^\nu + \beta d.$$

Now, note that the conditions $d \ge b/2$ and $b > 4\delta^\nu/(1-\beta)$ imply $\delta^\nu < (1-\beta)d/2$, and we may conclude that

$$\rho(z^\nu_{\alpha_1}, o) < \Big(\frac{1-\beta}{2} + \beta\Big) d = \frac{1+\beta}{2}\, d.$$

Similarly, we may show that $\rho(z^\nu_{\alpha_1}, z) < (1+\beta)d/2$.

We now have a chain consisting of three ordered points o, $z^\nu_{\alpha_1}$ and z. The distance of two subsequent points is $< (1+\beta)d/2$.

Now consider the first two points o and $z^\nu_{\alpha_1}$ of the chain. If their distance is $< b/2$, then $B^o \cap B_{\alpha_1}$ contains the ball $B(z^\nu_{\alpha_1}, a_0\delta^\nu)$. If, instead, their distance is $\ge b/2$, then we may repeat the argument above, and find $z^\nu_{\alpha_1^{(2)}}$ in $\mathfrak{Z}^\nu$ such that

$$\max(\rho(z^\nu_{\alpha_1^{(2)}}, o), \rho(z^\nu_{\alpha_1^{(2)}}, z^\nu_{\alpha_1})) < \Big(\frac{1+\beta}{2}\Big)^2 d.$$

Next we consider the two points $z^\nu_{\alpha_1}$ and z of the chain and argue similarly. Either their distance is $< b/2$, and $B^z \cap B_{\alpha_1}$ contains the ball $B(z, a_0\delta^\nu)$, or their distance is $\ge b/2$, and we may find $z^\nu_{\alpha_2^{(2)}}$ in $\mathfrak{Z}^\nu$ such that

$$\max(\rho(z^\nu_{\alpha_2^{(2)}}, z^\nu_{\alpha_1}), \rho(z^\nu_{\alpha_2^{(2)}}, z)) < \Big(\frac{1+\beta}{2}\Big)^2 d.$$

By iterating the procedure described above n times, we find a chain of points $z^\nu_{\alpha_1}, \dots, z^\nu_{\alpha_N}$ such that $o = z^\nu_{\alpha_1}$, $z = z^\nu_{\alpha_N}$, and

$$\rho(z^\nu_{\alpha_j}, z^\nu_{\alpha_{j+1}}) < \Big(\frac{1+\beta}{2}\Big)^n d \qquad j \in \{1, \dots, N-1\}.$$

If n is the least integer $\ge \log_2(2d/b)/\log_2[2/(1+\beta)]$, then

$$\Big(\frac{1+\beta}{2}\Big)^n d < b/2,$$

and for all j in $\{1, \dots, N-1\}$ the intersection $B_{\alpha_j} \cap B_{\alpha_{j-1}}$ contains the ball $B(z_{\alpha_{j+1}}, a_0\delta^\nu)$. Furthermore, the number N of points of the chain is at most

$$4(2d/b)^{1/[1-\log_2(1+\beta)]} + 1,$$

and

$$\frac{\mu(B_{\alpha_{j+1}})}{\mu(B(z^\nu_{\alpha_{j+1}}, a_0\delta^\nu))} \le D_{b/(a_0\delta^\nu), a_0\delta^\nu}$$

for all j in $\{1, \dots, N-1\}$, by the locally doubling property.

This concludes the proof of (ii). ∎

We need more notation and some preliminary observations. Let $b > 0$. For each ball B in $\mathcal{B}_b$ let $L^2_0(B)$ denote the Hilbert space of all functions f in $L^2(\mu)$ with support contained in B and $\int_B f \,\mathrm{d}\mu = 0$. We remark that a function f in $L^2_0(B)$ is a multiple of a $(1,2)$-atom, and that, for all $c \geq b$,

$$\|f\|_{H^{1,2}_c(\mu)} \leq \mu(B)^{1/2} \|f\|_{L^2(B)}. \tag{4.2}$$

Let ℓ be a bounded linear functional on $H^{1,2}(\mu)$. Then for each B in $\mathcal{B}$ the restriction of ℓ to $L^2_0(B)$ is a bounded linear functional on $L^2_0(B)$. Therefore, by the Riesz representation theorem there exists a unique function ℓ^B in $L^2_0(B)$ which represents the restriction of ℓ to $L^2_0(B)$. Note that for every constant η the function $\ell^B + \eta$ represents the same functional, though it is not in $L^2_0(B)$ unless η is equal to 0. Denote by $\|\ell\|_{H^{1,2}(\mu)^*}$ the norm of ℓ. Then, by (4.2), we have

$$\|\ell^B\|_{L^2_0(B)} \leq \mu(B)^{1/2} \|\ell\|_{H^{1,2}(\mu)^*} \tag{4.3}$$

For every f in $\mathrm{BMO}^{r'}(\mu)$ and every finite linear combination g of $(1,r)$-atoms the integral $\int_{\mathbb{R}^d} fg \,\mathrm{d}\mu$ is convergent. Let $H^{1,r}_{\mathrm{fin}}(\mu)$ denote the subspace of $H^{1,r}(\mu)$ consisting of all finite linear combinations of $(1,r)$-atoms. Then $g \mapsto \int_{\mathbb{R}^d} fg \,\mathrm{d}\mu$ defines a linear functional on $H^{1,r}_{\mathrm{fin}}(\mu)$. We observe that $H^{1,r}_{\mathrm{fin}}(\mu)$ is dense in $H^{1,r}(\mu)$.

Theorem 4.4. *Suppose that r is in $(1,\infty)$. The following hold:*

(i) *for every f in $\mathrm{BMO}^{r'}(\mu)$ the functional ℓ, initially defined on $H^{1,r}_{\mathrm{fin}}(\mu)$ by the rule*

$$\ell(g) = \int_{\mathbb{R}^d} fg \,\mathrm{d}\mu,$$

extends to a bounded functional on $H^{1,r}(\mu)$, and

$$\|\ell\|_{H^{1,r}(\mu)} \leq \|f\|_{\mathrm{BMO}^{r'}(\mu)};$$

(ii) *there exists a constant C such that for every continuous linear functional ℓ on $H^{1,r}(\mu)$ there exists a function f^ℓ in $\mathrm{BMO}^{r'}(\mu)$ such that $\|f^\ell\|_{\mathrm{BMO}^{r'}(\mu)} \leq C\|\ell\|_{H^{1,r}(\mu)^*}$ and*

$$\ell(g) = \int_{\mathbb{R}^d} f^\ell g \,\mathrm{d}\mu \quad \forall g \in H^{1,r}_{\mathrm{fin}}(\mu).$$

Proof. The proof of (i) follows the lines of the proof of [6] which is based on the classical result of C. Fefferman [10, 11]. We omit the details.

Now we prove (ii) in the case where r is equal to 2. The proof for r in $(1,\infty) \setminus \{2\}$ is similar and is omitted.

Let ℓ be a bounded linear functional on $H^{1,2}(\mu)$. Fix $\nu \in \mathbb{Z}$ and $b \in \mathbb{R}^+$ as in Lemma 4.3, such that b is also greater than $R_0/(1-\beta)$, where R_0 and

β are the constants of assumption (AM). Recall that for all $b' \geq b$ the space $H^{1,2}(\mu)$ is isomorphic to $H^{1,2}_{b'}(\mu)$ with norm $\|\cdot\|_{H^{1,2}_{b'}(\mu)}$, by Proposition 3.4. Thus, we may interpret ℓ as a continuous linear functional on $H^{1,2}_{b'}(\mu)$ for all $b' \geq b$. Fix a point o in $\mathfrak{Z}^\nu$. For each $b' \geq b$ there exists a function $\ell^{B(o,b')}$ in $L^2_0(B(o,b'))$ that represents ℓ as a functional on $L^2_0(B(o,b'))$. Since both $\ell^{B(o,b)}$ and the restriction of $\ell^{B(o,b')}$ to $B(0,b)$ represent the same functional on $L^2_0(B(o,b))$, there exists a constant $\eta^{B(0,b')}$ such that

$$\ell^{B(o,b)} - \ell^{B(o,b')} = \eta^{B(0,b')}$$

on $B(o,b)$. By integrating both sides of this equality on $B(o,b)$ we see that

$$\eta^{B(0,b')} = -\frac{1}{\mu(B(o,b))} \int_{B(o,b)} \ell^{B(o,b)} \, \mathrm{d}\mu.$$

Note that, since $\ell^{B(o,b)} \in L^2_0(B(o,b))$,

$$\eta^{B(o,b)} = 0. \tag{4.4}$$

Define

$$f^\ell(x) = \ell^{B(0,b')}(x) + \eta^{B(0,b')} \qquad \forall x \in B(o,b') \ \forall b' \geq b.$$

It is straightforward to check that this is a good definition.

We claim that the function f^ℓ is in $\mathrm{BMO}(\mu)$ and there exists a constant C such that

$$\|f^\ell\|_{\mathrm{BMO}(\mu)} \leq C\|\ell\|_{H^{1,2}(\mu)^*} \qquad \forall \ell \in H^{1,2}(\mu)^*.$$

First we show that $N^2_b(f^\ell) \leq \|\ell\|_{H^{1,2}(\mu)^*}$. Indeed, choose a ball B in $\mathcal{B}_b$. Then there exists a function ℓ^B in $L^2_0(B)$ that represents the restriction of ℓ to $L^2_0(B)$ and a constant η^B such that

$$f^\ell|_B = \ell^B + \eta^B. \tag{4.5}$$

By integrating both sides on B, we see that $\eta^B = (f^\ell)_B$. Thus, by (4.5) and (4.3),

$$\left(\frac{1}{\mu(B)} \int_B |f^\ell - (f^\ell)_B|^2 \, \mathrm{d}\mu\right)^{1/2} = \left(\frac{1}{\mu(B)} \int_B |\ell^B|^2 \, \mathrm{d}\mu\right)^{1/2} \leq \|\ell\|_{H^{1,2}(\mu)^*},$$

so that $N^2_b(f^\ell) \leq \|\ell\|_{H^{1,2}(\mu)^*}$, as required.

Next we show that f^ℓ is in $L^1(\mu)$ and that $\|f^\ell\|_1 \leq C\|\ell\|_{H^{1,2}(\mu)^*}$. Let $\{B_\alpha\}$ be the covering described in Lemma 4.3. For each integer $h \geq 2$ let A_h denote the annulus $B(o,hb) \setminus B(o,(h-1)b)$. For the sake of brevity denote $B(o,b)$ by B^o. Observe that $M = B^o \cup \bigcup_{h=2}^\infty A_h$. The left inequality in Lemma 4.3(i) implies that

$$\begin{aligned} \|f^\ell\|_1 &= \|f^\ell\|_{L^1(B^o)} + \sum_{h=2}^{\infty} \|f^\ell\|_{L^1(A_h)} \\ &\leq \|\ell^{B^o}\|_{L^1(B^o)} + \sum_{h=2}^{\infty} \sum_{\{B_\alpha : B_\alpha \cap A_h \neq \emptyset\}} \|f^\ell\|_{L^1(B_\alpha)}. \end{aligned} \tag{4.6}$$

By (4.5), the triangle inequality, the Schwarz inequality and (4.3),

$$\begin{aligned} \|f^\ell\|_{L^1(B_\alpha)} &\leq \mu(B_\alpha)^{1/2} \|\ell^{B_\alpha}\|_{L^2_0(B_\alpha)} + \mu(B_\alpha)|\eta^{B_\alpha}| \\ &\leq \mu(B_\alpha)\|\ell\|_{H^{1,2}(\mu)^*} + \mu(B_\alpha)|\eta^{B_\alpha}|. \end{aligned} \tag{4.7}$$

Now, we claim that if $B_\alpha \cap A_h \neq \emptyset$, then

$$|\eta^{B_\alpha}| \leq 8(2d/b)^{1/[1-\log_2(1+\beta)]}\sqrt{D}\,\|\ell\|_{H^{1,2}(\mu)^*}, \tag{4.8}$$

where $D = D_{b/(a_0\delta^\nu), a_0\delta^\nu}$ is the doubling constant corresponding to the parameters $b/(a_0\delta^\nu)$ and $a_0\delta^\nu$ (see Remark 2.2), and d denotes the distance of o from the centre z^ν_α of B_α.

By Lemma 4.3(ii) there exists a chain of points $z^\nu_{\alpha_1}, \ldots, z^\nu_{\alpha_N}$ such that $o = z^\nu_{\alpha_1}$, $z^\nu_\alpha = z^\nu_{\alpha_N}$, with

$$N \leq 4(2d/b)^{1/[1-\log_2(1+\beta)]} + 1,$$

and such that for all j in $\{1, \ldots, N-1\}$ the intersection $B_{\alpha_j} \cap B_{\alpha_{j-1}}$ contains the ball $B(z^\nu_{\alpha_{j+1}}, a_0\delta^\nu)$. Denote by B'_{α_j} the ball $B(z^\nu_{\alpha_j}, a_0\delta^\nu)$. Since, by (4.5), $\ell^{B_{\alpha_{j-1}}} + \eta^{B_{\alpha_{j-1}}} = \ell^{B_{\alpha_j}} + \eta^{B_{\alpha_j}}$ on $B_{\alpha_{j-1}} \cap B_{\alpha_j}$, it follows that on B'_{α_j},

$$\begin{aligned} |\eta^{B_{\alpha_j}}| &\leq |(\ell^{B_{\alpha_{j-1}}} + \eta^{B_{\alpha_{j-1}}})_{B'_{\alpha_j}}| + |(\ell^{B_{\alpha_j}})_{B'_{\alpha_j}}| \\ &\leq \left(\frac{1}{\mu(B'_{\alpha_j})} \int_{B'_{\alpha_j}} |\ell^{B_{\alpha_{j-1}}}|^2 \,\mathrm{d}\mu\right)^{1/2} + |\eta^{B_{\alpha_{j-1}}}| \\ &\quad + \left(\frac{1}{\mu(B'_{\alpha_j})} \int_{B'_{\alpha_j}} |\ell^{B_{\alpha_j}}|^2 \,\mathrm{d}\mu\right)^{1/2} \end{aligned}$$

by the triangle inequality and Schwarz's inequality. Now we use (4.3) to estimate the first and the third summands and obtain

$$\begin{aligned} |\eta^{B_{\alpha_j}}| &\leq \sqrt{\frac{\mu(B_{\alpha_{j-1}})}{\mu(B'_{\alpha_j})}}\,\|\ell\|_{H^{1,2}(\mu)^*} + |\eta^{B_{\alpha_{j-1}}}| + \sqrt{\frac{\mu(B_{\alpha_j})}{\mu(B'_{\alpha_j})}}\,\|\ell\|_{H^{1,2}(\mu)^*} \\ &\leq 2\sqrt{D}\,\|\ell\|_{H^{1,2}(\mu)^*} + |\eta^{B_{\alpha_{j-1}}}|. \end{aligned} \tag{4.9}$$

Note that we have used (4.1) of Lemma 4.3(ii) in the last inequality. Hence, iterating this inequality, we obtain

$$\begin{aligned}|\eta^{B_\alpha}| = |\eta^{B_{\alpha_N}}| &\leq 2(N-1)\sqrt{D}\,\|\ell\|_{H^{1,2}(\mu)^*} + |\eta^{B_0}| \\ &\leq 8(2d/b)^{1/[1-\log_2(1+\beta)]}\sqrt{D}\,\|\ell\|_{H^{1,2}(\mu)^*},\end{aligned}$$

because $\eta^{B_0} = 0$. This proves the claim (4.8).

Now (4.7) and (4.8) imply that for all the balls of the covering $\{B_\alpha\}$,

$$\|f^\ell\|_{L^1(B_\alpha)} \leq [1 + 8(2d/b)^{1/[1-\log_2(1+\beta)]}\sqrt{D}\,]\mu(B_\alpha)\|\ell\|_{H^{1,2}(\mu)^*}, \tag{4.10}$$

where d denotes the distance $\rho(z_\alpha^\nu, o)$. Note that if $B_\alpha \cap A_h \neq \emptyset$ then $d \leq (h+1)b$.

We estimate the first summand in (4.6) by Schwarz's inequality and (4.3), while we use (4.10) to estimate the other summands, and obtain

$$\|f^\ell\|_1 \leq \|\ell\|_{H^{1,2}(\mu)^*}\Big(\mu(B^o) + C\sum_{h=2}^{\infty}(h+1)^{1/[1-\log_2(1+\beta)]}\sum_{\{\alpha : B_\alpha \cap A_h \neq \emptyset\}}\mu(B_\alpha)\Big).$$

Since by Lemma 4.3(i) the balls $\{B_{\alpha_j}\}$ have the finite intersection property, and each intersects at most three annuli A_h, we have

$$\|f^\ell\|_1 \leq \mu(B^o)\|\ell\|_{H^{1,2}(\mu)^*} + C\|\ell\|_{H^{1,2}(\mu)^*}\sum_{h=2}^{\infty}(h+1)^{1/[1-\log_2(1+\beta)]}\sum_{j=h-2}^{h+2}\mu(A_j).$$

By Proposition 2.6(ii) there exist constants η in $(0,1)$ and $C > 0$ such that $\mu(A_j) \leq C\eta^j$. Thus

$$\sum_{h=2}^{\infty}(h+1)^{1/[1-\log_2(1+\beta)]}\sum_{j=h-2}^{h+2}\mu(A_j) < \infty,$$

and we may conclude that

$$\|f^\ell\|_1 \leq C\|\ell\|_{H^{1,2}(\mu)^*},$$

thereby proving that f^ℓ is in $L^1(\mu)$. ∎

Remark 4.5. Note that the proof of Theorem 4.4 does not apply, strictly speaking, to the case where r is equal to ∞. However, a straightforward, though tedious, adaptation to the case where μ is only locally doubling of a classical result [6] shows that $H^{1,\infty}(\mu)$ and $H^{1,2}(\mu)$ coincide, with equivalence of norms. Consequently, the dual space of $H^{1,\infty}(\mu)$ is $\mathrm{BMO}(\mu)$.

5. Interpolation. In this section we prove, for the finite measure case, the analogues of the interpolation theorems proved in [3] when $\mu(M) = \infty$. Because of the close similarity with the infinite measure case, we shall be rather sketchy in our exposition and we shall only indicate the necessary modifications to the statements and proofs.

The first technical ingredient in the proof of the interpolation theorems in [3] is a covering lemma (see [3, Prop. 5.3]). To prove the analogous result for spaces that satisfy the complementary isoperimetric property we need a lemma. We recall that B_0 is the ball in the complementary isoperimetric property $(\mathrm{I}^c_{B_0})$ (see Section 2).

LEMMA 5.1. *Suppose that A is an open subset of M such that $A\cap \bar B_0$ is contained in A_t for some t in $\mathbb{R}^+$. Then*

$$\mu(A_t) \geq (1-\mathrm{e}^{-I^c_{M,B_0}t/2})\mu(A)/2.$$

Proof. First we prove that $(A\cap \bar B_0^c)_t$ is contained in A_{2t}.

Indeed, suppose that x is in $(A\cap \bar B_0^c)_t$. Then either x is in A_t, hence in A_{2t}, or x is in $(A\cap\bar B_0^c)_t\setminus A_t$. In the latter case x is in $A\cap\bar B_0^c$, and $\rho(x,A^c)>t$. Furthermore $\rho(x,B_0)\leq t$, for otherwise the ball $B(x,t)$ would be contained in $A\cap\bar B_0^c$, i.e., $\rho(x,A^c\cup B_0)>t$, contradicting the fact that x is in $(A\cap\bar B_0^c)_t$.

Therefore the ball $B(x,t)$ is contained in A and there exists a point y in $A\cap\bar B_0$ such that $\rho(x,y)<t$. By assumption y is in A_t, whence

$$\rho(x,A^c)<\rho(x,y)+\rho(y,A^c)<2t,$$

as required.

Now,

$$\begin{aligned}\mu(A) &= \mu(A\cap\bar B_0)+\mu(A\cap\bar B_0^c) \leq \mu(A_t)+(1-\mathrm{e}^{-I^c_{M,B_0}t})^{-1}\mu((A\cap\bar B_0^c)_t)\\ &\leq \mu(A_{2t})+(1-\mathrm{e}^{-I^c_{M,B_0}t})^{-1}\mu(A_{2t}) = \frac{2-\mathrm{e}^{-I^c_{M,B_0}t}}{1-\mathrm{e}^{-I^c_{M,B_0}t}}\,\mu(A_{2t}),\end{aligned}$$

from which the desired estimate follows directly. ∎

LEMMA 5.2. *Suppose that ν is an integer. For every κ in $\mathbb{R}^+$, every open subset A of M such that $A\cap\bar B_0\subseteq A_\kappa$ and every collection $\mathcal{C}$ of dyadic cubes of resolution at least ν such that $\bigcup_{Q\in\mathcal{C}}Q=A$, there exist mutually disjoint cubes $Q_1,\ldots,Q_k$ in $\mathcal{C}$ such that*

(i) $\sum_{j=1}^k\mu(Q_j)\geq(1-\mathrm{e}^{-I^c_{M,B_0}\kappa/2})\mu(A)/4$;

(ii) $\rho(Q_j,A^c)\leq\kappa$ *for every j in* $\{1,\ldots,k\}$.

Proof. The proof is almost *verbatim* the same as the proof of [3, Proposition 3.5]. The only difference is that we use Lemma 5.1 in the proof of (i). ∎

REMARK 5.3. Observe that in Remark 2.4 we may replace B_0 with any ball containing B_0. Therefore we may assume that $r_{B_0}\geq C_1\delta^2$.

The second technical ingredient is a relative distributional inequality for the *noncentred dyadic maximal function*

$$\mathcal{M}_2f(x)=\sup_Q\frac{1}{\mu(Q)}\int_Q|f|\,\mathrm{d}\mu\qquad\forall x\in M, \tag{5.1}$$

where the supremum is taken over all dyadic cubes of resolution ≥ 2 that contain x, and the local sharp function

$$f^{\sharp,b}(x) = \sup_{B\in\mathcal{B}_b(x)} \frac{1}{\mu(B)} \int_B |f - f_B| \,\mathrm{d}\mu \qquad \forall x \in M.$$

Observe that f is in $\mathrm{BMO}(\mu)$ if and only if $f \in L^1(\mu)$ and $\|f^{\sharp,b}\|_\infty$ is finite for some (hence for all) $b > R_0/(1-\beta)$.

Note that the maximal operator $\mathcal{M}_2$ is of weak type 1. We denote by $|||\mathcal{M}_2|||_{1;1,\infty}$ its weak type 1 quasi norm.

For every $\alpha > 0$ denote by $A(\alpha)$ and $S(\alpha)$ the level sets $\{\mathcal{M}_2 f > \alpha\}$ and $\{f^{\sharp,b'} > \alpha\}$ respectively. Thus, for $\alpha, \epsilon > 0$,

$$\{\mathcal{M}_2 f > \alpha,\ f^{\sharp,b'} \leq \epsilon\alpha\} = A(\alpha) \cap S(\epsilon\alpha)^c.$$

The following lemma is the analogue of [3, Lemma 7.2] for spaces of finite measure that satisfy the complementary isoperimetric property.

LEMMA 5.4. *Let B_0 be as in Remark 2.4, with $r_{B_0} \geq C_1\delta^2$. Define constants b', σ and D by*

$$b' = 2C_1 + C_0, \qquad \sigma = (1 - \mathrm{e}^{-I^c_{M,B_0} C_1\delta^2/2})/4 \quad \text{and} \quad D = D_{b'/a_0,a_0},$$

where a_0, C_1 and δ are as in Theorem 4.1, and $D_{b'/a_0,a_0}$ is defined in Remark 2.2. Denote by ω the number

$$\inf\{\mu(Q) : Q \in \mathcal{Q}^2,\ Q \cap \bar{B}_0 \neq \emptyset\},$$

and by $\mathfrak{M}$ a constant $> |||\mathcal{M}_2|||_{1;1,\infty}/\omega$. Then for every η' in $(0,1)$, for all positive $\varepsilon < (1-\eta')/(2D)$, and for every f in $L^1(\mu)$,

$$\mu(A(\alpha) \cap S(\varepsilon\alpha)^c) \leq \eta\mu(A(\eta'\alpha)) \qquad \forall \alpha \geq \frac{\mathfrak{M}}{\eta'} \|f\|_{L^1(\mu)}$$

where

$$\eta = 1 - \sigma + \frac{2\varepsilon D}{\sigma(1-\eta')}. \tag{5.2}$$

Proof. First we prove that ω is strictly positive. Indeed, suppose that Q^2_α is a dyadic cube of resolution 2 with nonempty intersection with B_0; the cube Q^2_α contains the ball $B(z^2_\alpha, a_0\delta^2)$ by Theorem 4.1(v) and is contained in the ball $2B_0$ by the triangle inequality.

Denote by D the doubling constant $D_{a_0\delta^2/(2r_{B_0}),a_0\delta^2}$. By the local doubling property,

$$\mu(2B_0) \leq D\mu(B(z^2_\alpha, a_0\delta^2)) \leq D\mu(Q^2_\alpha).$$

Therefore $\omega \geq D^{-1}\mu(2B_0) > 0$, as required. For the rest of this proof we shall write κ instead of $C_1\delta^2$. Suppose that $\alpha \geq \mathfrak{M}\|f\|_{L^1(\mu)}/\eta'$. Since f is in

$L^1(\mu)$, we have

$$\mu(A(\eta'\alpha)) \le \frac{|\!|\!|\mathcal{M}_2|\!|\!|_{1;1,\infty}}{\eta'\alpha}\|f\|_{L^1(\mu)} \le \frac{|\!|\!|\mathcal{M}_2|\!|\!|_{1;1,\infty}}{\mathfrak{M}} < \omega. \tag{5.3}$$

We claim that $(A(\eta'\alpha))^\kappa = \{x \in A(\eta'\alpha) : \rho(x, A(\eta'\alpha)^c) > \kappa\}$ is contained in $\bar{B}_0^c$. Indeed, if x is in $(A(\eta'\alpha))^\kappa$, and Q is the dyadic cube of resolution 2 that contains x, then Q is contained in $A(\eta'\alpha)$ by the triangle inequality. Therefore $\mu(Q) \le \mu(A(\eta'\alpha)) < \omega$ by (5.3). Hence x is not in $\bar{B}_0$ by the definition of ω. The claim proved above implies that $A(\eta'\alpha) \cap \bar{B}_0 \subseteq (A(\eta'\alpha))_\kappa$.

The rest of the proof is the same as that of [3, Lemma 7.2]. The only difference is that we use Lemma 5.2 instead of [3, Prop. 5.3]. ∎

Next, we prove the analogue of [3, Thm. 7.3].

THEOREM 5.5. *For each p in $(1,\infty)$ there exists a positive constant C such that*

$$\|f\|_{L^1(\mu)} + \|f^{\sharp,b'}\|_{L^p(\mu)} \ge C\|f\|_{L^p(\mu)} \quad \forall f \in L^p(\mu).$$

Proof. Observe that it suffices to show that

$$\|f\|_{L^1(\mu)} + \|f^{\sharp,b'}\|_{L^p(\mu)} \ge C\|\mathcal{M}_2 f\|_{L^p(\mu)}, \tag{5.4}$$

because $\mathcal{M}_2 f \ge |f|$ by the differentiation theorem for integrals.

Let σ and $\mathfrak{M}$ be as in the statement of Lemma 5.4. Fix $\eta' = (1-\sigma/4)^{1/p}$ and let η be as in (5.2). Denote by ξ the number $\mathfrak{M}\|f\|_{L^1(\mu)}/\eta'$. Then

$$\begin{aligned}\|\mathcal{M}_2 f\|_p^p &= p\int_0^\infty \alpha^{p-1}\mu(A(\alpha))\,\mathrm{d}\alpha \\ &= p\int_\xi^\infty \alpha^{p-1}[\mu(A(\alpha)\cap S(\varepsilon\alpha)^c) + \mu(A(\alpha)\cap S(\varepsilon\alpha))]\,\mathrm{d}\alpha \\ &\quad + p\int_0^\xi \alpha^{p-1}\mu(A(\alpha))\,\mathrm{d}\alpha,\end{aligned}$$

so that, by Lemma 5.4,

$$\begin{aligned}\|\mathcal{M}_2 f\|_p^p &\le p\eta\int_0^\infty \alpha^{p-1}\mu(A(\eta'\alpha))\,\mathrm{d}\alpha + p\int_0^\infty \alpha^{p-1}\mu(S(\varepsilon\alpha))\,\mathrm{d}\alpha \\ &\quad + p\mu(M)\int_0^\xi \alpha^{p-1}\,\mathrm{d}\alpha \\ &= p\eta\eta'^{-p}\int_0^\infty \gamma^{p-1}\mu(A(\gamma))\,\mathrm{d}\gamma + p\varepsilon^{-p}\int_0^\infty \gamma^{p-1}\mu(S(\gamma))\,\mathrm{d}\gamma + \mu(M)\xi^p \\ &\le \eta\eta'^{-p}\|\mathcal{M}_2 f\|_p^p + \varepsilon^{-p}\|f^{\sharp,b'}\|_p^p + \mu(M)\,\frac{\mathfrak{M}^p}{(\eta')^p}\,\|\mathcal{M}_2 f\|_1^p.\end{aligned}$$

Now we choose ε small enough so that $\eta \leq 1 - \sigma/2$. Therefore $\eta\eta'^{-p} < 1$ and (5.4) follows. ∎

If X and Y are Banach spaces and θ is in $(0,1)$, we denote by $(X,Y)_{[\theta]}$ the complex interpolation space between X and Y with parameter θ.

Now that all the groundwork has been laid, we may proceed to state the interpolation theorems without further ado. The proofs are adaptations of classical results. We refer the reader to [3, Ths. 7.4 and 7.5] for more details.

THEOREM 5.6. *Suppose that θ is in $(0,1)$. The following hold:*

(i) *if p_θ is $2/(1-\theta)$, then $(L^2(\mu), \mathrm{BMO}(\mu))_{[\theta]} = L^{p_\theta}(\mu)$;*
(ii) *if p_θ is $2/(2-\theta)$, then $(H^1(\mu), L^2(\mu))_{[\theta]} = L^{p_\theta}(\mu)$.*

THEOREM 5.7. *Let S denote the strip $\{z \in \mathbb{C} : \Re z \in (0,1)\}$. Suppose that $\{\mathcal{T}_z\}_{z\in\bar{S}}$ is a family of uniformly bounded operators on $L^2(\mu)$ such that $z \mapsto \int_{\mathbb{R}^d} T_z f g \,\mathrm{d}\mu$ is holomorphic in S and continuous in $\bar{S}$ for all f and g in $L^2(\mu)$. Further, assume that there exists a constant A such that*

$$\|T_{is}\|_{L^2(\mu)} \leq A \quad \textit{and} \quad \|T_{1+is}\|_{L^\infty(\mu);\mathrm{BMO}(\mu)} \leq A.$$

Then for every θ in $(0,1)$ the operator T_θ is bounded on $L^{p_\theta}(\mu)$, where $p_\theta = 2/(1-\theta)$, and

$$\|T_\theta\|_{L^{p_\theta}(\mu)} \leq A_\theta,$$

where A_θ depends only on A and on θ.

6. Singular integrals. In this section we state the analogue of [3, Thm. 8.2]. Assume that $\mathcal{T}$ is a bounded linear operator on $L^2(\mu)$ with kernel k, i.e. k is a function on $M \times M$ which is locally integrable off the diagonal in $M \times M$ and such that for every function f with support of finite measure,

$$\mathcal{T}f(x) = \int_M k(x,y) f(y)\,\mathrm{d}\mu(y) \qquad \forall x \notin \operatorname{supp} f.$$

THEOREM 6.1. *Suppose that b is in $\mathbb{R}^+$ and $b > R_0/(1-\beta)$, where R_0 and β appear in the definition of property* (AM). *Suppose that $\mathcal{T}$ is a bounded operator on $L^2(\mu)$ and that its kernel k is locally integrable off the diagonal of $M \times M$. Let υ_k and ν_k be defined by*

$$\upsilon_k = \sup_{B\in\mathcal{B}_b} \sup_{x,x'\in B} \int_{(2B)^c} |k(x,y) - k(x',y)|\,\mathrm{d}\mu(y),$$

$$\nu_k = \sup_{B\in\mathcal{B}_b} \sup_{y,y'\in B} \int_{(2B)^c} |k(x,y) - k(x,y')|\,\mathrm{d}\mu(x).$$

The following hold:

(i) *if ν_k is finite, then $\mathcal{T}$ extends to a bounded operator on $L^p(\mu)$ for all p in $(1,2]$ and from $H^1(\mu)$ to $L^1(\mu)$; furthermore, there exists a constant C such that*

$$|\!|\!|\mathcal{T}|\!|\!|_{H^1(\mu);L^1(\mu)} \leq C(\nu_k + |\!|\!|\mathcal{T}|\!|\!|_{L^2(\mu)});$$

(ii) *if υ_k is finite, then $\mathcal{T}$ extends to a bounded operator on $L^p(\mu)$ for all p in $[2,\infty)$ and from $L^\infty(\mu)$ to* BMO(μ)*; furthermore, there exists a constant C such that*

$$|\!|\!|\mathcal{T}|\!|\!|_{L^\infty(\mu);\mathrm{BMO}(\mu)} \leq C(\upsilon_k + |\!|\!|\mathcal{T}|\!|\!|_{L^2(\mu)});$$

(iii) *if $\mathcal{T}$ is self-adjoint on $L^2(\mu)$ and ν_k is finite, then $\mathcal{T}$ extends to a bounded operator on $L^p(\mu)$ for all p in $(1,\infty)$, from $H^1(\mu)$ to $L^1(\mu)$ and from $L^\infty(\mu)$ to* BMO(μ).

Proof. The proof is *almost verbatim* the same as the proof of [3, Thm. 8.2], and is omitted. ∎

REMARK 6.2. It is worth observing that in the case where M is a Riemannian manifold and the kernel k is "regular", the condition $\upsilon_k < \infty$ of Theorem 6.1(i) may be replaced by the condition $\upsilon_k' < \infty$, where

$$\upsilon_k' = \sup_{B\in\mathcal{B}_b} r_B \sup_{x\in B} \int_{(2B)^c} |\nabla_x k(x,y)| \,\mathrm{d}\mu(y). \tag{6.1}$$

Similarly, the condition $\nu_k < \infty$ of Theorem 6.1(ii) may be replaced by the condition $\nu_k' < \infty$, where

$$\nu_m' = \sup_{B\in\mathcal{B}_b} r_B \sup_{y\in B} \int_{(2B)^c} |\nabla_y k(x,y)| \,\mathrm{d}\mu(x). \tag{6.2}$$

7. Riemannian manifolds. Let (M,ρ,μ) be a complete Riemannian manifold of dimension d, endowed with the Riemannian metric ρ and the corresponding Riemannian measure μ. Let $h(M)$ be Cheeger's isoperimetric costant, defined by

$$h(M) = \inf \frac{\sigma(\partial A)}{\mu(A)}$$

where the infimum is taken over all bounded open sets A with smooth boundary ∂A such that $\mu(A) \leq \mu(M)/2$. Here σ denotes the induced $(d-1)$-dimensional Riemannian measure on ∂A. Note that the condition $\mu(A) \leq \mu(M)/2$ is automatically satisfied if $\mu(M) = \infty$.

In [3, Section 9] we proved that, on Riemannian manifolds of infinite measure, the isoperimetric property (I) is equivalent to the positivity of $h(M)$. Moreover, if the Ricci curvature is bounded from below, both properties are equivalent to the positivity of the bottom of the spectrum of M,

$$b(M) = \inf\Big\{ \int_M |\nabla f|^2 \,\mathrm{d}\mu : f \in C^1_c(M), \|f\|_2 = 1\Big\}.$$

Here we shall prove that when M has finite measure, an analogous characterization holds for the complementary isoperimetric property $(\mathrm{I}^c_{B_0})$, provided that we replace $b(M)$ by the spectral gap of the Laplacian,

$$\lambda_1(M) = \inf\Big\{ \int_M |\nabla f|^2 \,\mathrm{d}\mu : f \in C^1_c(M), \|f\|_2 = 1 \text{ and } \int_M f \,\mathrm{d}\mu = 0\Big\}.$$

Again, since the arguments coincide to a large extent with those used to prove [3, Thm. 9.5], we only point out the differences, referring the reader to [3] for details and unexplained terminology.

Given a measurable set E in M, we shall denote by $P(E)$ its perimeter, i.e. the total variation $\mathrm{Var}(\mathbf{1}_E, M)$ in M of the indicator function $\mathbf{1}_E$ of E. The following lemma is the counterpart of [3, Prop. 9.2] in the finite measure case.

Lemma 7.1. *Suppose that M is a complete unbounded Riemannian manifold of finite volume. If $h(M) > 0$, then for every measurable set E with $\mu(E) \le \mu(M)/2$,*

$$P(E) \ge h(M)\mu(E).$$

Proof. Let f be a real-valued function in $C^1_c(M)$ whose support has measure less than $\mu(M)/2$. By the coarea formula [4],

$$\int_M |\nabla f| \,\mathrm{d}\mu \ge h(M) \int_M |f| \,\mathrm{d}\mu.$$

By [17, Prop. 1.4], there exists a sequence (f_n) of functions in $C^1_c(M)$ with supports of measure less than $\mu(M)/2$ and such that $f_n \to \mathbf{1}_E$ in $L^1(M)$ and $\int_M |\nabla f_n| \,\mathrm{d}\mu \to \mathrm{Var}(\mathbf{1}_E, M) = P(E)$. Hence, passing to the limit, we get $P(E) \ge h(M)\mu(E)$. ∎

Now we are ready to state the main result of this section. We recall that the constant I^c_{M,B_0} is defined in (2.2).

Theorem 7.2. *Suppose that M is a complete unbounded Riemannian manifold of finite volume and Ricci curvature bounded from below. Then the following are equivalent:*

(i) $h(M) > 0$;
(ii) *M possesses property* $(\mathrm{I}^c_{B_0})$;
(iii) $\lambda_1(M) > 0$.

Proof. To prove that (i) implies (ii), we fix a ball B_0 such that $\mu(B_0) > \mu(M)/2$ and we consider an open set A in $M \setminus \bar{B}_0$. Fix $t > 0$ and let f be the function defined by

$$f(x) = \begin{cases} t^{-1}\rho(x, A^c) & \text{if } x \in A_t, \\ 1 & \text{if } x \in A \setminus A_t, \\ 0 & \text{if } x \in A^c. \end{cases}$$

Then f is Lipschitz and $|\nabla f(x)| = t^{-1}$ for almost every x in A_t, $|\nabla f(x)| = 0$ elsewhere. Thus, by the coarea formula for functions of bounded variation [9, 16] and Lemma 7.1,

$$\begin{aligned} t^{-1}\mu(A_t) &= \int |\nabla f| \, \mathrm{d}\mu = \int_0^1 P(\{f > s\}) \, \mathrm{d}s \\ &\geq h(M) \int_0^1 \mu(\{f > s\}) \, \mathrm{d}s = h(M) \int f \, \mathrm{d}\mu \\ &\geq h(M)(\mu(A) - \mu(A_t)). \end{aligned}$$

Thus

$$\mu(A_t) \geq \frac{h(M)}{1 + h(M)t} \, t\mu(A) \qquad \forall t > 0.$$

Hence M satisfies property $(\mathrm{I}^c_{B_0})$ and by (2.2) the constant I^c_{M,B_0} is at least $h(M)$.

Next, we prove that (ii) implies (iii). Let A be a bounded open set with regular boundary, contained in $M \setminus \bar{B}_0$. Then $\mu(A_t) \geq (1 - \mathrm{e}^{-I_{M,\bar{B}_0} t})\mu(A)$ for all $t > 0$, by Proposition 2.6. Since the boundary of A is regular,

$$\sigma(\partial A) = \lim_{t \to 0+} \frac{\mu(A_t)}{t} \geq I_{M,B_0} \mu(A).$$

Hence, by the coarea formula, for every real-valued function f in $C^\infty_c(M \setminus \bar{B}_0)$,

$$I_{M,B_0} \int_M |f| \, \mathrm{d}\mu \leq \int_M |\nabla f| \, \mathrm{d}\mu. \tag{7.1}$$

By replacing f with f^2 in (7.1), we obtain

$$\inf \frac{\int_M |\nabla f|^2 \, \mathrm{d}\mu}{\int_M |f|^2 \, \mathrm{d}\mu} \geq \frac{I^2_{M,B_0}}{4}, \tag{7.2}$$

where the infimum is taken over all real f in $C^\infty_c(M \setminus \bar{B}_0)$ such that $\|f\|_2 \neq 0$. Hence the bottom of the essential spectrum $b_{\mathrm{ess}}(M)$ of the Laplace–Beltrami operator on M is positive, by the variational characterization of $b_{\mathrm{ess}}(M)$ [1]. Thus 0 is an isolated point in the spectrum and $\lambda_1(M) > 0$.

Finally, to prove that (iii) implies (i), we use the fact that if the Ricci curvature is bounded below by $-K$ for some $K \geq 0$, then

$$\lambda_1(M) \leq C(\sqrt{K}\, h(M) + h(M)^2),$$

where C is a constant which depends only on the dimension of M [2, 13]. ∎

8. Another family of metric spaces. In this section we shall construct another family of metric measure spaces which are locally doubling and satisfy the approximate midpoint property and the isoperimetric property. They may have either infinite or finite measure. In the first case they satisfy property (I), in the latter case, property ($\mathrm{I}^c_{B_0}$) (see Remark 2.1 or [3] for the definition of property (I)). The spaces we consider are of the form $(\mathbb{R}^d, \rho_\varphi, \mu_\varphi)$ or $(\mathbb{R}^d, \rho_\varphi, \mu_{-\varphi})$, where φ is a function in $C^2(\mathbb{R}^d)$ which satisfies certain additional conditions specified later, ρ_φ is the Riemannian measure on $\mathbb{R}^d$ defined by the length element $\mathrm{d}s^2 = (1 + |\nabla\varphi|)^2(\mathrm{d}x_1^2 + \cdots + \mathrm{d}x_d^2)$ and $\mathrm{d}\mu_{\pm\varphi} = \mathrm{e}^{\pm\varphi}\,\mathrm{d}\lambda$. Note that $\mu_{\pm\varphi}$ is not the Riemannian metric on $(\mathbb{R}^d, \rho_\varphi)$. First we need some preliminaries on Riemannian metrics of the form $\mathrm{d}s^2 = m^2(\mathrm{d}x_1^2 + \cdots + \mathrm{d}x_d^2)$, where m is a continuous positive function on $\mathbb{R}^d$ which tends to infinity at infinity.

We say that a positive function $m \in C^0(\mathbb{R}^d)$ is *tame* if for every $R > 0$ there exists a constant $C(R) \geq 1$ such that

$$C(R)^{-1} \leq \frac{m(x)}{m(y)} \leq C(R) \quad \forall x, y \in \mathbb{R}^d \text{ such that } |x - y| < R.$$

The following lemma provides a simple criterion for establishing tameness.

LEMMA 8.1. *Let m be a function in $C^1(\mathbb{R}^d)$ such that $m \geq 1$ and $|\nabla m| \leq Cm^\alpha$ for some α in $[0,1]$ and some $C > 0$. Then m is tame.*

Proof. By the mean value theorem, for all x, y in $\mathbb{R}^d$ such that $|x - y| < R$,

$$\left|\log\frac{m(x)}{m(y)}\right| \leq |x - y| \max_{z\in\mathbb{R}^d} \frac{|\nabla m(z)|}{m(z)} \leq CR. \ \blacksquare$$

It is easy to see that the functions $m(x) = 1 + |x|^\alpha$ with $\alpha \geq 0$ are tame. The function $\mathrm{e}^{|x|^\alpha}$ is tame if and only if $0 \leq \alpha \leq 1$.

PROPOSITION 8.2. *Let m be a tame function such that $\lim_{x\to\infty} m(x) = \infty$. Denote by ρ the Riemannian metric on $\mathbb{R}^d$ defined by the length element $\mathrm{d}s^2 = m(x)^2(\mathrm{d}x_1^2 + \cdots + \mathrm{d}x_d^2)$. Then the manifold $(\mathbb{R}^d, \rho)$ is complete. Moreover, for every $R > 0$, there exists a constant $C(R) \geq 1$ such that for all x, y in $\mathbb{R}^d$ with $\rho(x, y) < R$,*

$$C(R)^{-1} m(x)|x - y| \leq \rho(x, y) \leq C(R) m(x)|x - y|. \tag{8.1}$$

Proof. The function m has a positive minimum on $\mathbb{R}^d$, which we may assume to be greater than or equal to one, by multiplying m by a positive constant if necessary. If γ is a path in $\mathbb{R}^d$ we shall denote by $\ell(\gamma)$ its length with respect to the Riemannian metric ρ and by $\ell_e(\gamma)$ its Euclidean length. Since the minimum of m on $\mathbb{R}^d$ is at least 1 we have $\ell(\gamma) \geq \ell_e(\gamma)$ for all paths γ. Hence

$$\rho(x,y) \geq |x-y| \quad \forall x,y \in \mathbb{R}^d. \tag{8.2}$$

Let x and y be two points in $\mathbb{R}^d$ such that $\rho(x,y) < R$ and denote by γ the line segment joining them. Since $|x-y| \leq \rho(x,y) < R$ and m is tame,

$$\rho(x,y) \leq \ell(\gamma) = \int_0^1 m(\gamma(t))|\gamma'(t)|\,\mathrm{d}t \leq C(R)m(x)|x-y|. \tag{8.3}$$

This proves the second inequality in (8.1).

Together the two inequalities (8.2) and (8.3) imply that the manifold $(\mathbb{R}^d, \rho)$ is complete. In particular, any two points in $(\mathbb{R}^d, \rho)$ may be joined by a minimizing geodesic by the Hopf–Rinow theorem.

It remains to prove the first inequality in (8.1). We observe that there exists a constant A such that for all $S > 0$ there exists a compact set $K(S)$ in $\mathbb{R}^d$ such that

$$A^{-1} \leq \frac{m(x)}{m(y)} \leq A \quad \forall x,y \in \mathbb{R}^d \text{ with } x \notin K(S),\ |x-y| < S/m(x). \tag{8.4}$$

Indeed, by the definition of tame function it suffices to choose $A = C(1)$ and $K(S) = \{x \in \mathbb{R}^d : m(x) \leq S\}$.

Fix $R > 0$ and let x, y in $\mathbb{R}^d$ be such that $\rho(x,y) < R$. Assume first that $x \notin K(AR)$ and let $\gamma : [0, \rho(x,y)] \to \mathbb{R}^d$ be a minimizing geodesic joining x and y. We claim that $|\gamma(t) - x| < AR/m(x)$ for all t in $[0, \rho(x,y)]$. Indeed, suppose by contradiction that there exists t_0 in $[0, \rho(x,y)]$ such that $|\gamma(t_0) - x| = AR/m(x)$ and $|\gamma(t) - x| < AR/m(x)$ for all t in $[0, t_0)$. Then, by (8.4),

$$\rho(x,y) \geq \int_0^{t_0} m(\gamma(t))|\gamma'(t)|\,\mathrm{d}t \geq A^{-1}m(x)|\gamma(t_0) - x| = R,$$

which contradicts the assumption $\rho(x,y) < R$. Thus the claim is proved. Hence by (8.4),

$$\rho(x,y) = \int_0^{\rho(x,y)} m(\gamma(t))|\gamma'(t)|\,\mathrm{d}t \geq A^{-1}m(x)|y-x|.$$

Finally, if $x \in K(AR)$ then by (8.2),

$$m(x)|x-y| \leq m(x)\rho(x,y) \leq \max_{x \in K(AR)} m(x)\rho(x,y).$$

This concludes the proof of the proposition. ∎

Proposition 8.3. *Let φ be a function in $C^1(\mathbb{R}^d)$ such that $1 + |\nabla\varphi|$ is tame and $\lim_{x\to\infty} |\nabla\varphi(x)| = \infty$. Then the metric measure spaces $(\mathbb{R}^d, \rho_\varphi, \mu_\varphi)$ and $(\mathbb{R}^d, \rho_\varphi, \mu_{-\varphi})$ are locally doubling.*

Proof. Write $m(x) = 1 + |\nabla\varphi(x)|$ for the sake of brevity. Let $B_e(x,r)$ denote the Euclidean ball of centre x and radius r in $\mathbb{R}^d$. We claim that for every $R > 0$ there exists a constant $D(R)$ such that

$$D(R)^{-1}\,\mathrm{e}^{\varphi(x)} \le \mathrm{e}^{\varphi(y)} \le D(R)\,\mathrm{e}^{\varphi(x)} \qquad \forall y \in B_e(x, R/m(x)). \tag{8.5}$$

Indeed, by the mean value theorem and the fact that m is tame,

$$\begin{aligned}|\varphi(x) - \varphi(y)| &\le \max\{|\nabla\varphi(z)|\,|x-y| : z \in B_e(x, R/m(x))\} \\ &\le C(R)m(x)\,|x-y| \le C(R)R,\end{aligned}$$

whence (8.5) follows with $D(R) = \mathrm{e}^{C(R)R}$. Thus for every $R > 0$,

$$\begin{aligned}D(R)^{-1}\,\mathrm{e}^{\varphi(x)} &\le \frac{\mu_\varphi(B_e(x, r/m(x)))}{\lambda(B_e(x, r/m(x)))} \\ &\le D(R)\,\mathrm{e}^{\varphi(x)} \qquad \forall x \in \mathbb{R}^d,\ 0 < r \le R.\end{aligned} \tag{8.6}$$

Thus $(\mathbb{R}^d, \rho_\varphi, \mu_\varphi)$ is locally doubling, because by Proposition 8.2 there exists a constant C (which depends on R but not on r) such that

$$B_e(x, C^{-1}r/m(x)) \subset B(x,r) \subset B(x,2r) \subset B_e(x, 2Cr/m(x)) \quad \forall r \in [0,R]$$

and the Lebesgue measure is doubling. The proof for $(\mathbb{R}^d, \rho_\varphi, \mu_{-\varphi})$ is similar. ∎

Next, we look for sufficient conditions that guarantee that the spaces $(\mathbb{R}^d, \rho_\varphi, \mu_\varphi)$ and $(\mathbb{R}^d, \rho_\varphi, \mu_{-\varphi})$ satisfy the isoperimetric property.

DEFINITION 8.4. Let φ be function in $C^1(\mathbb{R}^d)$. We say that φ is *admissible* if

(i) there exists $\tau_0 > 0$ such that φ is C^2 for $|x| \ge \tau_0$;
(ii) $1 + |\nabla\varphi|$ is tame and

$$\lim_{x\to\infty} |\nabla\varphi(x)| = \infty, \qquad \lim_{x\to\infty} \frac{|\operatorname{Hess}\varphi(x)|}{|\nabla\varphi(x)|^2} = 0;$$

(iii) the radial derivative $\partial_r\varphi = \frac{x}{|x|}\cdot\nabla\varphi$ satisfies

$$\liminf_{x\to\infty} \frac{\partial_r\varphi(x)}{|\nabla\varphi(x)|} > 0.$$

It is easy to see that the functions $|x|^\alpha$ with $\alpha > 1$ are admissible. The function $\mathrm{e}^{|x|^\alpha}$ is not admissible if $\alpha > 1$.

LEMMA 8.5. *Let $\psi : [0,\infty) \to \mathbb{R}$ be a continuous function such that $\psi \in C^2([\tau_0,\infty))$ for some $\tau_0 > 0$. Assume that*

$$\liminf_{r\to\infty} \psi'(r) > 0, \qquad \lim_{r\to\infty} \frac{\psi''(r)}{(\psi'(r))^2} = 0.$$

Let h be a positive function in $C^0([0,\infty))$ such that

$$\liminf_{r\to\infty} h(r)\psi'(r) > 0.$$

Then for every $d \geq 1$ there exists a positive constant C such that

$$\int_\tau^{\tau+ah(\tau)} \mathrm{e}^{\psi(r)} r^{d-1}\,\mathrm{d}r \geq Ca \int_0^{\tau+ah(\tau)} \mathrm{e}^{\psi(r)} r^{d-1}\,\mathrm{d}r \qquad \forall \tau \in \mathbb{R}_+\ \forall a \in [0,1].$$

Proof. It is clearly enough to prove that

$$\int_\tau^{\tau+ah(\tau)} \mathrm{e}^{\psi(r)} r^{d-1}\,\mathrm{d}r \geq Ca \int_0^{\tau} \mathrm{e}^{\psi(r)} r^{d-1}\,\mathrm{d}r \qquad \forall \tau \in \mathbb{R}_+.$$

The integral on the right hand side is asymptotic to $\mathrm{e}^{\psi(\tau)}\tau^{d-1}/\psi'(\tau)$ as τ tends to infinity, by l'Hôpital's rule and the assumptions on ψ. Let $\tau_1 > \tau_0$ be such that

$$\int_0^{\tau} \mathrm{e}^{\psi(r)} r^{d-1}\,\mathrm{d}r \leq 2\,\mathrm{e}^{\psi(\tau)}\,\frac{\tau^{d-1}}{\psi'(\tau)} \qquad \forall \tau \geq \tau_1. \tag{8.7}$$

The assumptions on ψ and h imply that if we choose τ_1 sufficiently large there exists $\eta > 0$ such that

$$\psi'(\tau) \geq \eta, \qquad h(\tau)\psi'(\tau) \geq \eta \qquad \forall \tau \geq \tau_1.$$

Thus, if $\tau > \tau_1$ the function ψ is increasing. Hence for $\tau > \tau_1$,

$$\begin{aligned}\int_\tau^{\tau+ah(\tau)} \mathrm{e}^{\psi(r)} r^{d-1}\,\mathrm{d}r &\geq \mathrm{e}^{\psi(\tau)}\tau^{d-1}ah(\tau) \geq \eta a\,\mathrm{e}^{\psi(\tau)}\,\frac{\tau^{d-1}}{\psi'(\tau)} \\ &\geq \frac{\eta}{2}\,a\int_0^{\tau} \mathrm{e}^{\psi(r)} r^{d-1}\,\mathrm{d}r,\end{aligned}$$

where in the last inequality we have used (8.7). It remains to prove the desired inequality for τ in $[0,\tau_1]$. Set $m_0 = \min_{[0,\infty]}\psi$, $M_0 = \max_{[0,\tau_1]}\psi$ and $h_0 = \min_{[0,\tau_1]} h$. Then for $\tau \in [0,\tau_1]$,

$$\int_0^{\tau} \mathrm{e}^{\psi(r)} r^{d-1}\,\mathrm{d}r \leq \mathrm{e}^{M_0}\,\frac{\tau^d}{d}$$

and

$$\int_\tau^{\tau+ah(\tau)} \mathrm{e}^{\psi(r)} r^{d-1}\,\mathrm{d}r \geq \mathrm{e}^{m_0}\tau^{d-1}ah(\tau) \geq \mathrm{e}^{m_0}\tau^d ah_0/\tau_1.$$

This implies that the desired inequality holds also for τ in $[0,\tau_1]$. ∎

Lemma 8.6. *Let ψ and h be two functions which satisfy the assumptions of Lemma* 8.5. *Assume further that*

$$\lim_{r\to\infty} (r - h(r)) = \infty.$$

Then for every $d \geq 1$ there exist positive constants C and T such that

$$\int_{\tau-ah(\tau)}^{\tau} \mathrm{e}^{-\psi(r)} r^{d-1}\,\mathrm{d}r \geq Ca \int_{\tau-ah(\tau)}^{\infty} \mathrm{e}^{-\psi(r)} r^{d-1}\,\mathrm{d}r \quad \forall \tau \geq T\ \forall a \in [0,1].$$

Proof. It is clearly enough to prove that

$$\int_{\tau-ah(\tau)}^{\tau} \mathrm{e}^{-\psi(r)} r^{d-1}\,\mathrm{d}r \geq Ca \int_{\tau}^{\infty} \mathrm{e}^{-\psi(r)} r^{d-1}\,\mathrm{d}r \quad \forall \tau \geq T\ \forall a \in [0,1].$$

The integral on the right hand side is asymptotic to $\mathrm{e}^{-\psi(\tau)}\tau^{d-1}/\psi'(\tau)$ as τ tends to infinity, by l'Hôpital's rule and the assumptions on ψ. Thus there exists $\tau_1 > \tau_0$ such that

$$\int_{\tau}^{\infty} \mathrm{e}^{-\psi(r)} r^{d-1}\,\mathrm{d}r \leq 2\,\mathrm{e}^{-\psi(\tau)}\,\frac{\tau^{d-1}}{\psi'(\tau)} \quad \forall \tau \geq \tau_1. \tag{8.8}$$

The assumptions on ψ and h imply that if we choose τ_1 sufficiently large there exists $\eta > 0$ such that

$$\psi'(\tau) \geq \eta, \quad h(\tau)\psi'(\tau) \geq \eta, \quad r\psi'(r) > d-1 \quad \forall \tau \geq \tau_1.$$

Note that the last inequality implies that the function $r \mapsto \mathrm{e}^{-\psi(r)} r^{d-1}$ is decreasing for $r > \tau_1$. Choose $T > \tau_1$ such that $\tau - h(\tau) > \tau_1$ for $\tau \geq T$. Then for $\tau \geq T$,

$$\begin{aligned}\int_{\tau-ah(\tau)}^{\tau} \mathrm{e}^{-\psi(r)} r^{d-1}\,\mathrm{d}r &\geq \mathrm{e}^{-\psi(\tau)}\tau^{d-1} a h(\tau) \geq \eta a \mathrm{e}^{-\psi(\tau)}\,\frac{\tau^{d-1}}{\psi'(\tau)} \\ &\geq \frac{\eta}{2}\, a \int_{\tau}^{\infty} \mathrm{e}^{-\psi(r)} r^{d-1}\,\mathrm{d}r\end{aligned}$$

where in the last inequality we have used (8.8). This concludes the proof of the lemma. ∎

THEOREM 8.7. *Suppose that the function φ is admissible. Then*

(i) *the metric measure space $(\mathbb{R}^d, \rho_\varphi, \mu_\varphi)$ is locally doubling, $\mu_\varphi(\mathbb{R}^d) = \infty$, and satisfies property* (I)*;*

(ii) *the space $(\mathbb{R}^d, \rho_\varphi, \mu_{-\varphi})$ is locally doubling, $\mu_{-\varphi}(\mathbb{R}^d) < \infty$, and satisfies property $(\mathrm{I}^c_{B_0})$ for some ball $B_0 \subset \mathbb{R}^d$.*

Proof. Both spaces are locally doubling by Proposition 8.3. It easily follows from the assumptions on φ that $\mu_\varphi(\mathbb{R}^d) = \infty$ and $\mu_{-\varphi}(\mathbb{R}^d) < \infty$. To prove that $(\mathbb{R}^d, \rho_\varphi, \mu_\varphi)$ also has property (I) we must prove that there exists a constant C such that for every bounded open set A and every κ in $[0,1)$,

$$\mu_\varphi(A_\kappa) \geq C\mu_\varphi(A),$$

where we recall that $A_\kappa = \{x \in A : \rho(x, A^c) < \kappa\}$.

Henceforth we shall write $m = 1 + |\nabla\varphi|$, for the sake of brevity. Since m is tame there exists a constant $C_1 \geq 1$ such that

$$C_1^{-1} \leq \frac{m(x)}{m(y)} \leq C_1 \qquad \forall x, y \text{ such that } |x - y| < 1.$$

Let d_e denote the Euclidean distance in $\mathbb{R}^d$ and set

$$A'_\kappa = \left\{ x \in A : d_e(x, A^c) < \frac{\kappa}{C_1 m(x)} \right\}.$$

We observe that if $x \in A'_\kappa$, then there exists y in A^c such that

$$|x - y| < \frac{\kappa}{C_1 m(x)} \leq 1.$$

Thus, by (8.1), we get $\rho(x, y) < C_1 m(x)|x - y| < \kappa$. Hence $A'_\kappa \subset A_\kappa$ and it suffices to prove that there exists a constant C such that

$$\mu_\varphi(A'_\kappa) \geq C\mu_\varphi(A).$$

For every ω in the unit sphere S^{d-1} let μ_φ^ω denote the measure on $\mathbb{R}_+$ defined by

$$\mu_\varphi^\omega(E) = \int_E \mathrm{e}^{\varphi(r\omega)} r^{d-1} \, \mathrm{d}r$$

for every measurable subset E of $\mathbb{R}_+$.

The functions $\psi_\omega(r) = \varphi(r\omega)$ and $h_\omega(r) = 1/m(r\omega)$ satisfy the assumptions of Lemma 8.5 uniformly with respect to ω in S^{d-1}. Thus for all $a \in [0, 1]$ there exists a constant $C > 0$ such that

$$\mu_\varphi^\omega((\tau, \tau + ah_\omega(\tau))) \geq Ca\mu_\varphi^\omega([0, \tau + ah_\omega(\tau))) \qquad \forall \tau \in \mathbb{R}_+ \ \forall \omega \in S^{d-1}. \tag{8.9}$$

If F is a measurable subset of $\mathbb{R}^d$ let $F(\omega)$ denote the set $\{r \in \mathbb{R}_+ : r\omega \in F\}$.

If the set $(A \setminus A'_\kappa)(\omega)$ is empty then obviously

$$\mu_\varphi^\omega(A'_\kappa(\omega)) = \mu_\varphi^\omega(A(\omega)).$$

Otherwise, set $\tau_\omega = \sup(A \setminus A'_\kappa)(\omega)$. Observe that $\tau_\omega\omega \in A \setminus A'_\kappa$. Indeed, by the definition of τ_ω, there exists a sequence $s_n \to \tau_\omega$ such that $s_n\omega \in A \setminus A'_\kappa$. By the continuity of m,

$$d(\tau_\omega\omega, A^c) = \lim_n d(s_n\omega, A^c) \geq \lim_n \frac{\kappa}{C_1 m(s_n\omega)} = \frac{\kappa}{C_1 m(\tau_\omega\omega)} > 0.$$

This implies that $\tau_\omega\omega \in A \setminus A'_\kappa$.

The set $(A \setminus A'_\kappa)(\omega)$ is obviously contained in the interval $[0, \tau_\omega)$. We claim that the set $A'_\kappa(\omega)$ contains the interval $(\tau_\omega, \tau_\omega + C_1^{-1}\kappa h_\omega(\tau_\omega))$. Indeed, if $s \in (\tau_\omega, \tau_\omega + C_1^{-1}\kappa h_\omega(\tau_\omega))$, then $d(\tau_\omega\omega, s\omega) < \kappa/(C_1 m(\tau_\omega\omega))$. Hence $s\omega \in A$, because otherwise $\tau_\omega\omega$ would be in A'_κ. Since $s\omega \notin A \setminus A'_\kappa$ by the definition of τ_ω, the claim is proved.

Then, writing $a = C_1^{-1}\kappa$ for the sake of brevity, using the fact that for every positive number δ the function $x \mapsto x/(\delta + x)$ is increasing and (8.9), we see that

$$\begin{aligned}\frac{\mu_\varphi^\omega(A'_\kappa(\omega))}{\mu_\varphi^\omega(A(\omega))} &= \frac{\mu_\varphi^\omega(A'_\kappa(\omega))}{\mu_\varphi^\omega((A\setminus A'_\kappa)(\omega)) + \mu_\varphi^\omega(A'_\kappa(\omega))}\\ &\geq \frac{\mu_\varphi^\omega((\tau_\omega, \tau_\omega + a h_\omega(\tau_\omega)))}{\mu_\varphi^\omega([0,\tau_\omega)) + \mu_\varphi^\omega((\tau_\omega, \tau_\omega + a h_\omega(\tau_\omega)))}\\ &= \frac{\mu_\varphi^\omega((\tau_\omega, \tau_\omega + a h_\omega(\tau_\omega)))}{\mu_\varphi^\omega([0,\tau_\omega + a h_\omega(\tau_\omega)))} \geq Ca = CC_1^{-1}\kappa.\end{aligned}$$

Thus, integrating in polar coordinates, one has

$$\begin{aligned}\mu_\varphi(A'_\kappa) &= \int_{S^{d-1}} \mu_\varphi^\omega(A'_\kappa(\omega))\,\mathrm{d}\sigma(\omega)\\ &\geq C\kappa \int_{S^{d-1}} \mu_\varphi^\omega(A(\omega))\,\mathrm{d}\sigma(\omega) = C\kappa\mu_\varphi(A).\end{aligned}$$

This concludes the proof of property (I) for $(\mathbb{R}^d, \rho_\varphi, \mu_\varphi)$.

The proof of property $(\mathrm{I}^c_{B_0})$ for $(\mathbb{R}^d, \rho_{-\varphi}, \mu_{-\varphi})$ is similar. The main differences are the following:

(i) the set A is an open set contained in $\{x \in \mathbb{R}^d : |x| \geq T\}$ for some $T > 0$ which depends only on φ;
(ii) τ_ω is now $\inf\,(A\setminus A'_\kappa)(\omega)$;
(iii) $(A\setminus A'_\kappa)(\omega) \subseteq (\tau_\omega, \infty)$ and $A'_\kappa(\omega) \supseteq (\tau_\omega - C_1^{-1}\kappa h_\omega(\tau_\omega), \tau_\omega)$;
(iv) Lemma 8.6 is used instead of Lemma 8.5.

We omit the details. ∎

Remark 8.8. We point out that the H^1-BMO theory for the Gaussian space $(\mathbb{R}^d, \gamma)$ developed in [15] is a particular case of the theory developed in the present paper. Indeed, $\gamma = \mu_{-\varphi}$ with $\varphi(x) = |x|^2$. Moreover, in [15] the family of admissible balls is the set $\mathcal{B}_1^\gamma$ of all Euclidean balls B in $\mathbb{R}^d$ such that $r_B \leq \min(1, 1/|c_B|)$, where c_B and r_B denote the centre and the radius of B respectively, while the family $\mathcal{B}_1$ of admissible balls in $(\mathbb{R}^d, \rho_\varphi, \gamma)$ is the set of all balls of radius at most one with respect to the metric $\mathrm{d}s^2 = (1+|x|)^2(\mathrm{d}x_1^2 + \cdots + \mathrm{d}x_d^2)$. By Proposition 8.2, every ball in $\mathcal{B}_1^\gamma$ is contained in a ball in $\mathcal{B}_1$ of comparable measure and vice versa. Thus the spaces $H^1(\gamma)$ and $\mathrm{BMO}(\gamma)$ defined in [15] coincide with those defined in the present paper.

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Andrea Carbonaro, Giancarlo Mauceri
Dipartimento di Matematica
Università di Genova
via Dodecaneso 35, 16146 Genova, Italia
E-mail: carbonaro.a@gmail.com
mauceri@dima.unige.it

Stefano Meda
Dipartimento di Matematica e Applicazioni
Università di Milano-Bicocca
via R. Cozzi 53
20125 Milano, Italy
E-mail: stefano.meda@unimib.it

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