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NON-LEBESGUE MULTIRESOLUTION ANALYSES

BY

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Abstract. Classical notions of wavelets and multiresolution analyses deal with the Hilbert space $L^2(\mathbb{R})$ and the standard translation and dilation operators. Key in the study of these subjects is the low-pass filter, which is a periodic function $h \in L^2([0,1))$ that satisfies the classical quadrature mirror filter equation $|h(x)|^2 + |h(x+1/2)|^2 = 2$. This equation is satisfied almost everywhere with respect to Lebesgue measure on the torus. Generalized multiresolution analyses and wavelets exist in abstract Hilbert spaces with more general translation and dilation operators. Moreover, the concept of the low-pass filter has been generalized in various ways. It may be a matrix-valued function, it may not satisfy any obvious analog of a filter equation, and it may be an element of a non-Lebesgue L^2 space. In this article we discuss the last of these generalizations, i.e., filters that are elements of non-Lebesgue L^2 spaces. We give examples of such filters, and we derive ageneralization of the filter equation.

1. Introductory remarks. Some time in the early 1960's, Andrzej Hulanicki and I both sat in on a course in Lie algebras that was being given by Ramesh Gangolli at the University of Washington. Though I didn't meet Andrzej again for fifteen years, I remembered the stimulating times we had enjoyed in that course, as well as many Friday afternoons in a local tavern, and I was thrilled to be invited by him to a conference in Poland in 1978. At that time our friendship was rekindled, and afterward it flourished from then on with several visits by him to us in Boulder and visits by me and my wife Christy to him and his family in Wrocław. Andrzej spent the month of January in 1988 at the Mathematical Sciences Research Institute in Berkeley, and I was privileged to spend the following month there. The day Christy and I arrived in Berkeley was the day before Andrzej's departure, and we spent the night and next morning with him. Among other typical Andrzej conversations, e.g., stories, rumors, and speculations about our common mathematical friends, he told me "Larry, the hot item here at MSRI is wavelets." Indeed, the mathematicians leading the research program at MSRI that spring were emphasizing the "new" subject of wavelets. During those few hours, Andrzej tried to explain to me what these peculiarly-named objects were, but I honestly gave it very little attention at that time, preferring to

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stick with my traditional mathematical inquiry on unitary representation theory. Not until 1996 did my research interests develop enough to make me sufficiently interested to find out what Andrzej had been recommending. By that time, however, he said he thought wavelets were "dead, finished." Andrzej never wanted to be seen as being behind the curve, and as always I took his opinion very seriously. But, I have spent the years since then working on wavelet and multiresolution theory, and I truly wish I could show my old friend some of my discoveries. I miss him, I miss his insight, his humor, and perhaps most of all I miss his gossips. I feel privileged and honored to be able to dedicate this article to my dear friend Andrzej Hulanicki.

2. Preliminaries. Let $\mathcal{H}$ be a Hilbert space, and let T and D be unitary operators on $\mathcal{H}$ satisfying $DT = T^2 D$. Traditionally, the Hilbert space $\mathcal{H}$ is $L^2(\mathbb{R})$, T is the translation operator $[T(f)](x) = f(x+1)$, and the operator D is the dilation operator $[D(f)](x) = \sqrt{2}^{-1} f(x/2)$. However, in this paper we will be studying other examples of $\mathcal{H}$, T, and D.

Definition 2.1. Let $\{V_j\}_{j=-\infty}^{\infty}$ be a collection of closed subspaces of $\mathcal{H}$ satisfying:

(1) $V_j \subseteq V_{j+1}$,
(2) $V_{j-1} = D(V_j)$,
(3) $\bigcap V_j = \{0\}$ and $\bigcup V_j$ is dense in $\mathcal{H}$,
(4) V_0 is invariant under all powers of T.

The collection $\{V_j\}$ is called a *generalized multiresolution analysis* (GMRA) relative to T and D, and the subspace V_0 is called the *core subspace*.

As some immediate observations about a GMRA $\{V_j\}$, we have the following: For each integer j, write W_j for the orthogonal complement of V_j in V_{j+1}. Note that $\mathcal{H} = \bigoplus_{j=-\infty}^{\infty} W_j$, and, because of (2) and (4), W_j is invariant under all powers of T for each $j \geq 0$. The spaces $\{W_j\}$ are usually called the *wavelet subspaces* of the GMRA.

Let $\{V_j\}$ be a GMRA relative to T and D. If ϕ is an element of V_0 for which the collection $\{T^n\phi\}$ is an orthonormal basis for V_0, we call ϕ a *scaling vector* for the GMRA. This is the traditional extra assumption that makes a generalized multiresolution analysis a classical *multiresolution analysis* (MRA), which was studied extensively some twenty years ago by Mallat [7], Meyer [8] and Daubechies [5]. Define $J : V_0 \to L^2(\mathbb{T}) \equiv L^2([0,1))$ by $J(T^n\phi) = e^{2\pi inx}$. Then J extends to a unitary operator from V_0 onto $L^2(\mathcal{T}) \equiv L^2([0,1),\lambda)$, where λ is Lebesgue measure, and $[J(T(v))](x) = e^{2\pi ix}[J(v)](x)$ for all $v \in V_0$. The following theorem (see the references cited above) is in some sense the beginning of multiresolution analysis theory.

THEOREM 2.2. *Set* $h = J(D(\phi))$. *Then*

$$|h(x)|^2 + |h(x+1/2)|^2 = 2. \tag{2.1}$$

(*This is the classical* quadrature mirror filter equation.)

The proof follows by computing the Fourier coefficients of both sides and determining that they agree. The function h is called a *low-pass filter* associated to this GMRA (MRA).

There were two remarkable discoveries that researchers found concerning these notions. First of all, if h is a Borel function satisfying the filter equation above, is smooth and nonzero in a large enough neighborhood of 0, and satisfies the so-called *low-pass condition* $h(0) = \sqrt{2}$, then a multiresolution analysis relative to the standard translation and dilation operators could be constructed (in $\mathcal{H} = L^2(\mathbb{R})$) from h alone, and h is a low-pass filter associated with this GMRA. In particular, a scaling vector $\phi \in L^2(\mathbb{R})$ could be constructed in terms of h. The second discovery was that there exist "complementary high-pass filters" g to h, and, given any such g, the function ψ determined by $\sqrt{2}\,\widehat{\psi}(2x) = g(x)\widehat{\phi}(x)$ was an orthonormal wavelet for $L^2(\mathbb{R})$. We will have more to say and clarify about this later.

What happens if there is no scaling vector ϕ? The hypotheses in the following theorem (for $\mathcal{H} = L^2(\mathbb{R})$) were introduced by Benedetto and Li in [4].

THEOREM 2.3. *Let* $\{V_j\}$ *be a GMRA relative to operators* T *and* D. *Suppose there is an element* $\eta \in V_0$ *for which the collection* $\{T^n\eta\}$ *forms a frame for* V_0. (*We call such an* η *a* frame scaling vector.) *Then there exists a Borel subset* S *of* $[0,1)$ *and an element* $\phi \in V_0$ *such that the assignment* $T^n\phi \mapsto e^{2\pi inx}\chi_S(x)$ *determines a unitary operator* $J : V_0 \to L^2(S,\lambda)$ *such that*

(1) $[J(T(v))](x) = e^{2\pi ix}[J(v)](x)$ *for all* $v \in V_0$.
(2) *If* h *is defined by* $h = J(D(\phi))$, *then*

$$|h(x)|^2 + |h(x+1/2)|^2 = 2\chi_S(2x). \tag{2.2}$$

(*An* augmented filter equation.)

As in Theorem 2.2, this augmented filter equation can be proved by computing Fourier coefficients of both sides. For consistency, h is again called a low-pass filter associated to this GMRA. Whether such a generalized filter h is enough to construct a GMRA, as well as the existence of generalized complementary filters and wavelets, has furnished researchers considerable work. In general, the answers are both negative. Indeed, as we will see below, it is necessary that the subset S satisfy certain other conditions, and the filter h requires some additional assumptions as well. In fact, one needs to understand some extra details about GMRAs before these questions can adequately be addressed. We will say more on this later as well.

Suppose there is no scaling vector ϕ and no frame scaling vector η. Here is where measures other than Lebesgue measure can appear.

THEOREM 2.4. *Let $\{V_j\}$ be a GMRA relative to operators T and D. Suppose $\eta \in V_0$ is such that the closed span of the elements $T^n\eta$ is all of V_0. That is, η is a cyclic vector for the operator $T|_{V_0}$. Then there exists a finite Borel measure μ on $[0, 1)$, unique up to equivalence of measures, and a unitary map $J : V_0 \to L^2(\mu)$ satisfying*

(1) *$[J(T(v))](x) = e^{2\pi ix}[J(v)](x)$ for all $v \in V_0$ and μ-almost all $x \in [0, 1)$.*
(2) *If $\phi = J^{-1}(I)$, where I denotes the constant 1 function, and if $h = J(D(\phi))$, then the operator $J \circ D \circ J^{-1}$ coincides with the Ruelle operator S_h defined by*
$$[S_h(f)](x) = h(x)f(2x)$$
on $L^2(\mu)$.

Proof. We use the Spectral Theorem as applied to the unitary operator $T|_{V_0}$. The existence of the measure μ, the operator J, and conclusion (1) follow from that theorem.

By the definition of h, and the fact that $h \in L^2(\mu)$, we see that
$$[J(D(J^{-1}(I)))](x) = h(x) = h(x)I(2x) = [S_h(I)](x)$$
for μ-almost all x. Then
$$\begin{aligned}[J(D(J^{-1}(e_nI)))](x) &= [J(D(T^n(J^{-1}(I))))](x) = [J(T^{2n}(D(\phi)))](x)\\ &= e_{2n}(x)[J(D(\phi))](x) = h(x)e_n(2x)I(2x)\\ &= [S_h(e_nI)](x),\end{aligned}$$
where e_n denotes the exponential function $e^{2\pi inx}$. Hence, using the Weierstrass Approximation Theorem, we have
$$[J(D(J^{-1}(f)))](x) = [S_h(f)](x)$$
for every continuous (periodic) function f on $[0, 1) \equiv \mathbb{T}$. Conclusion (2) of Theorem 2.4 now follows for all L^2 functions by general integration theory techniques. ■

We will call a GMRA for which there is a cyclic vector η in the core subspace V_0 a *cyclic GMRA*.

REMARK 2.5. Theorem 2.4 clearly is a generalization of the first two theorems above, so that the operators $J\circ D\circ J^{-1}$ are given as Ruelle operators in those cases as well. This simply was not an important observation at first, the filter equation having been of primary interest. In the case of Theorem 2.4, the function h need not satisfy a standard filter equation. Nevertheless, we will continue to call h a low-pass filter associated to the GMRA.

We see from the definition of a GMRA that the operator $D|_{V_0}$ is a *pure isometry* on V_0, i.e., an isometry for which $\bigcap_n D^n(V_0) = \{0\}$. Therefore, the Ruelle operator S_h is a pure isometry on $L^2(\mu)$, i.e., $\bigcap_n \mathrm{range}(S_h^n) = \{0\}$.

It follows from direct calculations that any function h satisfying either of the filter equations, (2.1) or (2.2), has the property that the corresponding Ruelle operator S_h is an isometry. However, the filter equations by themselves do not guarantee that S_h is a pure isometry. Again, we will say more about that later.

We will see in the examples below that the two filter equations of Theorems 2.2 and 2.3 are not satisfied in general. However, we will derive a replacement equation.

It is our purpose in this paper to investigate what aspects of the classical MRA theory generalize to the non-Lebesgue cases. Many questions about these general multiresolution analyses need to be answered. For instance, exactly which measures μ can occur? Does every appropriate measure have a corresponding filter function h? Given a μ and a function h for which S_h is a pure isometry, is there a Hilbert space $\mathcal{H}$, operators T and D, and a GMRA $\{V_j\}$ for which this μ and h are the corresponding measure and filter arising from $\{V_j\}$? What about complementary filters and wavelets? And, what happens when the core subspace V_0 is not cyclic for the operator T? We give some initial answers in this first paper on the subject, and many other answers will be provided in a forthcoming article [2] by this author together with V. Furst, K. Merrill, and J. Packer.

3. The generalized filter equation. We continue to assume the hypotheses, and conclusions, of Theorem 2.4. Specifically, let μ be the measure and h the low-pass filter coming from the cyclic GMRA $\{V_j\}$.

Lemma 3.1. *Define measures μ_1 and μ_2 on $[0,1)$ as follows:*

$$\mu_1(E) = \mu(E/2) \quad \textit{and} \quad \mu_2(E) = \mu(E/2 + 1/2).$$

Then

(1) *We have*

$$\mu = |h(x/2)|^2\mu_1 + |h(x/2+1/2)|^2\mu_2. \tag{3.1}$$

(2) *In fact, if h is any Borel function on $[0,1)$, then the Ruelle operator S_h is an isometry on $L^2(\mu)$ if and only if h satisfies* (3.1).

Proof. We prove (2), from which (1) will follow, since by Theorem 2.4, S_h is an isometry for that h. For clarity we include the following integration formulas for any bounded Borel function g:

$$\int_0^1 g(x)\,d\mu_1(x) = \int_0^{1/2} g(2x)\,d\mu(x),$$
$$\int_0^1 g(x)\,d\mu_2(x) = \int_{1/2}^1 g(2x-1)\,d\mu(x).$$

Suppose h is a Borel function for which S_h is an isometry. Then for every $f \in L^2(\mu)$ we have

$$\begin{aligned}
\int_{[0,1)} |f(x)|^2\,d\mu(x) &= \int_{[0,1)} |[S_h(f)](x)|^2\,d\mu(x) = \int_{[0,1)} |h(x)|^2 |f(2x)|^2\,d\mu(x) \\
&= \int_{[0,1/2)} \left|h\left(\frac{1}{2}\,2x\right)\right|^2 |f(2x)|^2\,d\mu(x) \\
&\quad + \int_{[1/2,1)} \left|h\left(\frac{1}{2}\,(2x-1)+1\right)\right|^2 |f(2x)|^2\,d\mu(x) \\
&= \int_{[0,1)} |h(x/2)|^2 (f(x)|^2\,d\mu_1(x) + \int_{[0,1)} |h(x/2+1/2)|^2 |f(x)|^2\,d\mu_2(x) \\
&= \int_{[0,1)} |f(x)|^2 (|h(x/2)|^2\,d\mu_1(x) + \int_{[0,1)} |f(x)|^2 |h(x/2+1/2)|^2)\,d\mu_2(x),
\end{aligned}$$

which implies that

$$\mu = |h(x/2)|^2\,\mu_1 + |h(x/2+1/2)|^2\,\mu_2.$$

Of course this calculation can be reversed, showing that if h satisfies (3.1), then S_h is an isometry. ∎

We will call (3.1) the *measure-theoretic form* of the generalized filter equation relative to μ.

As a consequence of the generalized filter equation, we discover a necessary condition on a measure μ for it to arise from a GMRA in the above way. Indeed, we see that μ must be absolutely continuous with respect to μ^*, where μ^* is defined by

$$\mu^*(E) = \mu(E/2 \cup (E/2+1/2)).$$

Indeed, noting that $\mu^* = \mu_1 + \mu_2$, we see that if $\mu^*(E) = 0$, then clearly both $\mu_1(E)$ and $\mu_2(E)$ are 0, and so by (3.1), $\mu(E) = 0$ as well. Therefore, if $\mu(E) > 0$, then E must intersect in a measure-theoretically nontrivial way either $E/2$ or $E/2+1/2$. If μ is supported on a set S, then it follows that, up to sets of μ-measure 0, $S \subseteq 2S$. (We have discovered here a requirement on the set S of Theorem 2.3.) Hence, for example, no measure μ that is supported on the interval $[1/4, 1/2)$ arises from a cyclic GMRA.

Now, after introducing a few more items related to μ, we will be able to rephrase the measure-theoretic form of the generalized filter equation into an equation that h must satisfy.

Let μ be a finite Borel measure on $[0, 1)$ that is absolutely continuous with respect to the measure μ^* as defined above. If μ_1 and μ_2 are also defined as above, then both μ_1 and μ_2 are absolutely continuous with respect to μ^*. Write c_i for the Radon–Nikodym derivative of μ_i with respect to μ^*, and write ρ for the Radon–Nikodym derivative of μ with respect to μ^*. Note that $c_1(x) + c_2(x) = 1$ for μ^*-almost all x.

THEOREM 3.2. *Let the notation be as in the preceding paragraph. Let h be an element of $L^2(\mu)$. Then h satisfies* (3.1) *if and only if*

$$c_1(x)|h(x/2)|^2 + c_2(x)|h(x/2 + 1/2)|^2 = \rho(x) \tag{3.2}$$

for μ-almost all $x \in [0, 1)$.

Proof. Because the following sequence of equalities holds for every bounded Borel function f, it will follow that h satisfies (3.1) if and only if it satisfies (3.2), which proves the theorem.

$$\begin{aligned}\int f(x)\rho(x)\,d\mu^*(x) &= \int f(x)\,d\mu(x)\\ &= \int f(x)|h(x/2)|^2\,d\mu_1(x) + \int f(x)|h(x/2+1/2)|^2\,d\mu_2(x)\\ &= \int f(x)|h(x/2)|^2c_1(x)\,d\mu^*(x) + \int f(x)|h(x/2+1/2)|^2c_2(x)\,d\mu^*(x)\\ &= \int f(x)[c_1(x)|h(x/2)|^2 + c_2(x)|h(x/2+1/2)|^2]\,d\mu^*(x). \ \blacksquare\end{aligned}$$

We call (3.2) the *generalized filter equation* relative to μ.

Let us examine this generalized filter equation when μ is absolutely continuous with respect to Lebesgue measure λ.

THEOREM 3.3. *Suppose μ is absolutely continuous with respect to λ, and write $d\mu = \sigma(x)\,d\lambda$. Write S for the set where $\sigma(x) > 0$. If h satisfies the generalized filter equation relative to μ, then*

$$|h(x)|^2\sigma(x) + |h(x+1/2)|^2\sigma(x+1/2) = 2\sigma(2x)\chi_S(2x)$$

for λ-almost all $x \in [0, 1)$.

Proof. We assume that h satisfies the generalized filter equation, and we will use the measure-theoretic form (3.1). We will use the fact that Lebesgue measure λ on $\mathbb{T} \equiv [0, 1)$ is invariant under translation (by $1/2$) and also invariant under multiplication by 2. For each $f \in L^2(\mu)$, we have

$$
\begin{aligned}
\int_0^1 f(x)\chi_S(x)\sigma(x)\,dx &= \int_0^1 f(x)\,d\mu(x) \\
&= \int_0^1 f(x)|h(x/2)|^2\,d\mu_1(x) + \int_0^1 f(x)|h(x/2+1/2)|^2\,d\mu_2(x) \\
&= \int_0^{1/2} f(2x)|h(x)|^2\,d\mu(x) + \int_{1/2}^1 f(2x-1)|h(x)|^2\,d\mu(x) \\
&= \int_0^{1/2} f(2x)\chi_S(2x)|h(x)|^2\sigma(x)\,dx \\
&\quad + \int_{1/2}^1 f(2(x-1/2))\chi_S(2(x-1/2))|h(x)|^2\sigma(x)\,dx \\
&= \int_0^1 f(x)\chi_S(x)\,\frac{1}{2}\,|h(x/2)|^2\sigma(x/2)\,dx \\
&\quad + \int_0^1 f(x)\chi_S(x)\,\frac{1}{2}\,|h(x/2+1/2)|^2\sigma(x/2+1/2)\,dx \\
&= \int_0^1 f(x)\chi_S(x)\,\frac{1}{2}\,(|h(x/2)|^2\sigma(x/2) + |h(x/2+1/2)|^2\sigma(x/2+1/2))\,dx \\
&= \int_0^1 \left(f(x)\,\frac{1}{2}\,\frac{|h(x/2)|^2\sigma(x/2)}{\sigma(x)} + \frac{|h(x/2+1/2)|^2\sigma(x/2+1/2)}{\sigma(x)}\,\sigma(x)\right) dx,
\end{aligned}
$$

implying that

$$
|h(x/2)|^2\sigma(x/2) + |h(x/2+1/2)|^2\sigma(x/2+1/2) = 2\sigma(x)\chi_S(x)
$$

for λ-almost all x. Replacing x with $2x$ we have

$$
|h(x)|^2\sigma(x) + |h(x+1/2)|^2\sigma(x+1/2) = 2\sigma(2x)\chi_S(2x)
$$

for λ-almost all x. ■

REMARK 3.4. Note that if σ is constant on the set S, then h actually satisfies the augmented filter equation (2.2). In general, if h' is defined by

$$
h'(x) = h(x)\,\frac{\sqrt{\sigma(x)}}{\sqrt{\sigma(2x)}},
$$

where we are extending σ to be periodic with period 1, then h' satisfies (2.2), showing that, if μ is absolutely continuous with respect to Lebesgue measure, then there always exists a low-pass filter that satisfies the augmented filter equation. We will see an example below of a measure for which there is a low-pass filter satisfying (3.2), but no filter satisfying (2.2).

If μ is a measure on $[0,1)$ that is absolutely continuous with respect to μ^*, and if ρ denotes the Radon–Nikodym derivative of μ with respect to μ^*, then h, defined by $h(x) = \sqrt{\rho(2x)}$, clearly satisfies (3.2). As above, we are extending ρ to be 1-periodic. Note that if $\mu = \mu^*$, as actually happens in the classical case described in Theorem 2.2, then $\rho(x) \equiv 1$. Certainly the function $h(x) \equiv 1$ satisfies the generalized filter equation, and hence S_h is an isometry, but in this case S_h is not a pure isometry. Indeed, S_h has a nontrivial fixed vector, e.g., h itself, so that $\bigcap_n \operatorname{range}(S_h^n)$ is not $\{0\}$. Therefore, satisfying the generalized filter condition is not enough to guarantee that a function h is a low-pass filter for some cyclic GMRA. The question of when a Ruelle operator of the form S_h is actually a pure isometry is apparently very subtle; for instance, it has been studied in various contexts in [3] and [1]. The theorem below, which is proved in [1], tells all that is known, at least to this author.

THEOREM 3.5. *Suppose μ is Lebesgue measure restricted to a set S, and h satisfies the generalized filter equation, equivalently the augmented filter equation. Then S_h fails to be a pure isometry if and only if there exists a function f for which $|f(x)| \equiv 1$ and a complex number λ of modulus 1 for which*

$$h(x) = \lambda \frac{f(x)}{f(2x)}$$

for almost all $x \in [0,1)$.

4. Examples. We present next examples of cyclic GMRAs $\{V_j\}$ whose associated measures μ are not Lebesgue measure. After seeing these two examples, it should be clear to the reader that a wide variety of such examples can be constructed. We begin with a measure μ and a function h that satisfies the generalized filter equation relative to μ. As was proved earlier, this means that the Ruelle operator S_h on $L^2(\mu)$ is an isometry. If it is in fact a pure isometry, then the results of [6] and [3], which depend on the theory of direct limits of Hilbert spaces, imply the existence of such a GMRA. In our first example, we will actually make this construction concrete.

EXAMPLE 4.1. Let S be the countable set

$$\left\{\frac{1}{2}, \frac{1}{4}, \frac{3}{4}, \frac{1}{8}, \frac{7}{8}, \frac{1}{16}, \frac{15}{16}, \dots\right\}.$$

Define an atomic probability measure μ on S by $\mu(1/2) = 1/2$, $\mu(1/4) = \mu(3/4) = 1/8$, $\mu(1/8) = \mu(7/8) = 1/16$, $\mu(1/16) = \mu(15/16) = 1/32, \dots$

One sees directly that the measure μ^* is given by $\mu^*(0) = 1/2$, $\mu^*(1/2) = 1/4$, $\mu^*(1/4) = \mu^*(3/4) = 1/16$, $\mu^*(1/8) = \mu^*(7/8) = 1/32, \ldots$ It is clear then that the Radon–Nikodym derivative ρ of μ with respect to μ^* is constantly equal to 2 on S.

Also, the measures μ_1 and μ_2 are given by $\mu_1(1/2) = 1/8$, $\mu_1(1/4) = 1/16$, $\mu_1(1/8) = 1/32, \ldots$, and $\mu_2(0) = 1/2$, $\mu_2(1/2) = 1/8$, $\mu_2(3/4) = 1/16, \ldots$. It follows that the two Radon–Nikodym derivatives c_1 and c_2 are given by $c_1(1/2) = 1/2$ and $c_1(1/2^n) = 1$ for all $n \geq 2$, and $c_2(0) = 1$, $c_2(1/2) = 1/2$, and $c_2(1 - 1/2^n) = 1$ for all $n \geq 2$.

Define h on $[0, 1)$ to be $\sqrt{2}$ times the indicator function of $S \setminus \{1/2\}$. We wish to show that μ and h come from a cyclic GMRA. Set

$$[S_h(f)](x) = h(x)f(2x).$$

PROPOSITION 4.2. *The Ruelle operator S_h is a pure isometry on $L^2(\mu)$.*

Proof. A direct computation shows that S_h is an isometry. Given the expressions above for c_1, c_2 and ρ, one could also easily prove this by verifying that h satisfies the generalized filter equation.

It is routine to see that the range of S_h^n, for $n > 1$, is supported on the subset of S containing the points

$$\frac{1}{2^{n+1}}, \frac{1}{2^{n+2}}, \ldots \quad \text{and} \quad 1 - \frac{1}{2^{n+1}}, 1 - \frac{1}{2^{n+2}}, \ldots,$$

from which it follows that $\bigcap_n \operatorname{range}(S_h^n) = \{0\}$. ∎

Now we construct a Hilbert space $\mathcal{H}$, unitary operators T and D, and a GMRA $\{V_j\}$. Set $K_0 = L^2(\mu)$, and, for $n \geq 1$, set $K_n = \mathbb{C}^2$. Let $\mathcal{H} = \bigoplus_{n=0}^{\infty} K_n$. We denote the elements of $\mathcal{H}$ as sequences

$$F = \{f_0, \vec{f}_1, \vec{f}_2, \ldots\}.$$

Let T be the operator on $\mathcal{H}$ given by

$$[T(F)]_n(x) = \begin{cases} e^{2\pi i x} f_0(x), & n = 0, \\ \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \times \vec{f}_1, & n = 1, \\ \vec{f}_n, & n \geq 2. \end{cases}$$

Define an operator D on $\mathcal{H}$ by

$$[D(F)]_n = \vec{f}_{n+1}$$

if $n \geq 1$, and

$$[D(F)]_0(x) = \begin{cases} (\vec{f}_1)_1, & x = 1/2, \\ h(x)f_0(2x) + \sqrt{2}(\vec{f}_1)_2, & x = 1/4, \\ h(x)f_0(2x) - \sqrt{2}(\vec{f}_1)_2, & x = 3/4, \\ h(x)f_0(2x), & \text{otherwise.} \end{cases}$$

THEOREM 4.3. *$DT = T^2 D$.*

Though relatively tiresome, the proof follows directly.

THEOREM 4.4. *For $j \geq 0$, set $V_j = \bigoplus_{n=0}^{j} K_n$, and for $j < 0$ set $V_j = S_h^{-j}(K_0)$. Then the collection $\{V_j\}$ is a cyclic GMRA relative to T and D, and the measure associated to the spectral decomposition of $T|_{V_0}$ is the given μ. Moreover, the given function h is a low-pass filter associated to this GMRA.*

Proof. Most of the properties of a GMRA follow immediately. For instance, since the operator $D|_{V_0}$ coincides with the Ruelle operator S_h, and S_h is a pure isometry on $V_0 \equiv K_0 \equiv L^2(\mu)$, it follows that $\bigcap_j V_j = \{0\}$. Also, since $V_0 = L^2(\mu)$, we may take as the unitary operator J of Theorem 2.4 the identity operator. The fact that μ is the measure arising from that theorem and h is the low-pass filter then follows directly. ∎

REMARK 4.5. The structure of this GMRA is remarkably different from the classical examples. For one thing, the "wavelet" subspaces $W_j = V_{j+1} \ominus V_j$, for $j \geq 0$, are all 2-dimensional, whereas in the classical examples they are infinite-dimensional. To make sense of wavelet bases in this example would require some further development, which we choose to leave for later. Secondly, the "translation operator" T has infinitely many fixed vectors, namely the elements of the W_j's for $j > 0$. The classical translation operators are direct sums of two-sided shifts, and hence have no fixed vectors. Such are the consequences of abstracting classical phenomena.

Finally, note that

$$|h(x)|^2 + |h(x+1/2)|^2 = \begin{cases} 0, & x = 1/2, \\ 4, & x = 1/4 \text{ or } 3/4, \\ 2, & \text{otherwise.} \end{cases}$$

That is, h does not satisfy the augmented filter equation (2.2). Of course, as we have noted, h does satisfy the generalized filter equation (3.2). Moreover, we have

PROPOSITION 4.6. *There is no h that satisfies both* (3.2) *and* (2.2) *relative to this measure μ.*

Proof. By way of contradiction, suppose h satisfies the generalized filter equation relative to μ, and also

$$|h(x)|^2 + |h(x+1/2)|^2 = 2\chi_S(2x)$$

for μ-almost all x. Then we must have $h(1/2) = 0$. We must also have $|h(1/4)|^2 + |h(3/4)|^2 = 2$. Moreover, for $n \geq 3$, we must have $|h(1/2^n)|^2 =$

$|h(1-1/2^n)|^2 = 2$. Hence, since S_h is an isometry,

$$
\begin{aligned}
1 &= \|I\|^2_{L^2(\mu)} = \|S_h(I)\|^2_{L^2(\mu)} = \|h\|^2_{L^2(\mu)} \\
&= \sum_{x\in S} |h(x)|^2\mu(x) = \left|h\left(\frac{1}{2}\right)\right|^2 \frac{1}{2} + \left|h\left(\frac{1}{4}\right)\right|^2 \frac{1}{8} + \left|h\left(\frac{3}{4}\right)\right|^2 \frac{1}{8} \\
&\quad + \sum_{n=3}^{\infty} \left(\left|h\left(\frac{1}{2^n}\right)\right|^2 + \left|h\left(1-\frac{1}{2^n}\right)\right|^2 \right) \frac{1}{2^{n+1}} \\
&= 0 + \frac{1}{4} + \sum_{n=3}^{\infty} \frac{4}{2^{n+1}} = \frac{1}{4} + \frac{1}{2} < 1,
\end{aligned}
$$

which is a contradiction. ■

EXAMPLE 4.7. Let ν be an arbitrary Borel probability measure supported on the interval $[1/4, 1/2)$. For each $n \geq 1$, define the probability measure ν_n supported on the interval $[1/2^{n+1}, 1/2^n)$ by $\nu_n(E) = \nu(2^{n-1}E)$. For each $n \geq 1$, let ρ_n be a positive function on the interval $[1/2^{n+1}, 1/2^n)$ for which the measure $\rho_n\, d\nu_n$ is a probability measure. Finally, set

$$\mu = \sum_{n=1}^{\infty} \frac{1}{2^n}\, \rho_n\, d\nu_n.$$

Let h be defined on $[0,1)$ by

$$h = \sum_{n=2}^{\infty} \frac{\sqrt{2}}{\sqrt{\rho_n}}\, \chi_{[1/2^{n+1}, 1/2^n)}.$$

PROPOSITION 4.8. *The Ruelle operator S_h is a pure isometry on $L^2(\mu)$.*

Proof. That S_h is an isometry follows from a direct computation. Similar to the previous example, we see that the range of the operator S_h^n comprises functions that are supported on the interval $[0, 1/2^{n+1})$, which implies that $\bigcap_n \operatorname{range}(S_h^n) = \{0\}$. ■

Using the results from [6] and [3], we know that there is a cyclic GMRA whose associated measure is this μ and whose associated generalized low-pass filter is this h. From the forthcoming work [2], we could give a concrete construction, but we defer that somewhat elaborate description to that paper.

As for the filter equation, note that

$$|h(x)|^2 + |h(x+1/2)|^2 = \begin{cases} 0, & x \in [1/4, 1/2), \\ 2/\rho_n(x), & x \in [1/2^{n+1}, 1/2^n),\ n \geq 2. \end{cases} \tag{4.1}$$

Given the arbitrariness of the ρ_n's, we see just how far away from satisfying the classical filter equation such an h can be.

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