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ON THE BOUNDARY CONVERGENCE OF SOLUTIONS TO THE HERMITE–SCHRÖDINGER EQUATION

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Abstract. In the half-space $\mathbb{R}^d \times \mathbb{R}_+$, consider the Hermite–Schrödinger equation $i\partial u/\partial t = -\Delta u + |x|^2 u$, with given boundary values on $\mathbb{R}^d$. We prove a formula that links the solution of this problem to that of the classical Schrödinger equation. It shows that mixed norm estimates for the Hermite–Schrödinger equation can be obtained immediately from those known in the classical case. In one space dimension, we deduce sharp pointwise convergence results at the boundary by means of this link.

The authors dedicate this paper to the memory of Andrzej Hulanicki. Both of us knew Andrzej since the 1970's. Since then he has been like an invariant for us. We have enjoyed the high quality of his mathematics, his capacity of work, his ability to organize important mathematical events, his generosity when sharing ideas and his sympathy. All this, and even his age, seemed to be invariant during these decades.

1. Introduction and results. The solution of the classical, free Schrödinger equation in the half-space $\mathbb{R}^d \times \mathbb{R}_+$ with variables (x,t),

$$\begin{cases} i\dfrac{\partial u}{\partial t} = -\Delta u, \\ u(\cdot,0) = f, \end{cases}$$

can be written $u(x,t) = e^{it\Delta} f(x)$, for $f \in L^2(\mathbb{R}^d)$. For $p,q \in [1,\infty]$, one measures the size of u by means of the mixed norm

$$\|u\|_{L^q_t(\mathbb{R},L^p_x(\mathbb{R}^d))} = \Big(\int\limits_{\mathbb{R}} \Big(\int\limits_{\mathbb{R}^d} |u(x,t)|^p\,dx\Big)^{q/p} dt\Big)^{1/q},$$

with the obvious interpretation for p or $q=\infty$. The Strichartz estimate

$$\|u\|_{L^q_t(\mathbb{R},L^p_x(\mathbb{R}^d))} \le C_{d,p}\|f\|_{L^2(\mathbb{R}^d)} \tag{1}$$

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is known to hold if and only if

$$(2)\qquad \frac{d}{p}+\frac{2}{q}=\frac{d}{2}\quad\text{and}\quad \begin{cases} 2\le p\le\infty & \text{for } d=1\\ 2\le p<\infty & \text{for } d=2\\ 2\le p\le 2d/(d-2) & \text{for } d\ge 3.\end{cases}$$

This is due essentially to J. Ginibre and G. Velo [5]. M. Keel and T. Tao [6] obtained the endpoint results.

Results about the pointwise convergence of $u(x,t)$ at the boundary are also known, for f in Sobolev spaces

$$W^s(\mathbb{R}^d)=\{f\in L^2(\mathbb{R}^d):(I-\Delta)^{s/2}f\in L^2\}.$$

For $d=1$, L. Carleson [3] and B. Dahlberg and C. Kenig [4] have proved that $e^{it\Delta}f\to f$ a.e. as $t\to 0^+$ for all $f\in W^s(\mathbb{R})$ if and only if $s\ge 1/4$.

In this paper, we consider the same questions for the Hermite operator

$$H=-\Delta+|x|^2,\qquad x\in\mathbb{R}^d.$$

Thus u will be the solution $u(x,t)=e^{-itH}f(x)$ to the Hermite–Schrödinger equation in $\mathbb{R}^d\times\mathbb{R}_+$ with given boundary values,

$$(3)\qquad \begin{cases} i\dfrac{\partial u}{\partial t}=Hu,\\ u(\cdot,0)=f.\end{cases}$$

As in the classical case, the Strichartz estimate

$$(4)\qquad \|e^{-itH}f\|_{L^q_t((0,2\pi),L^p_x(\mathbb{R}^d))}\le C_{d,p,q}\|f\|_{L^2(\mathbb{R}^d)}$$

holds under the assumption (2); see H. Koch and D. Tataru [8]. Moreover, since the interval of integration in the t variable is now bounded, (4) remains true if the equality in (2) is replaced by the inequality $d/p+2/q\ge d/2$.

Our Lemma 1 in Section 2 gives an explicit relation between the two solution operators e^{-itH} and $e^{it\Delta}$. It makes it easy to prove the following result, which implies that the estimates (1) and (4) are actually equivalent when the equality in (2) holds.

THEOREM 1. *Let $1\le p,q\le\infty$, and assume that $d/p+2/q=d/2$. Then for $f\in L^2$,*

$$\|e^{-itH}f\|_{L^q_t((0,\pi/4),L^p_x(\mathbb{R}^d))}=\|e^{it\Delta}f\|_{L^q_t((0,\infty),L^p_x(\mathbb{R}^d))}.$$

As we shall see below, it does not matter whether the t interval in (4) is $(0,2\pi)$ or $(0,\pi/4)$; the two mixed norms obtained are proportional for real functions f.

In the case $d=1$, we shall also consider the almost everywhere convergence as $t\to 0^+$ of the solution $e^{-itH}f$, to the initial data. To state these results, we use both $W^s(\mathbb{R})$ and the Sobolev spaces associated to H, defined by

$$W^s_H(\mathbb{R}) = \{f \in L^2(\mathbb{R}) : H^{s/2} f \in L^2\}$$

with the obvious norm. These spaces have been introduced by S. Thangavelu [10]. We point out that there is a continuous inclusion $W^s_H \subset W^s$ (see B. Bongioanni and J. L. Torrea [2, Theorem 3(i)]).

Yajima [11] proved the a.e. convergence $e^{-itH} f \to f$ as $t \to 0^+$ for f in the intersection $W^s(\mathbb{R}) \cap L^1$, with $s > 1/2$. Then Bongioanni and Rogers [1] obtained the same convergence for $f \in W^s_H(\mathbb{R})$, with $s > 1/3$. The following result is sharp with respect to both types of Sobolev spaces.

Theorem 2. *Let $d = 1$.*

- (i) *Assume $f \in W^{1/4}(\mathbb{R})$. Then for a.a. $x \in \mathbb{R}$ the function $t \mapsto e^{-itH} f(x)$, $0 < t < \pi/8$, will, after modification on a null set, be continuous with limit $f(x)$ as $t \to 0^+$.*
- (ii) *If $s < 1/4$, there exists an $f \in W^s(\mathbb{R})$ such that for all x in a set of positive measure the function $t \mapsto e^{-itH} f(x)$ does not converge as $t \to 0^+$, even after any modification on a null set.*
- (iii) *The statements in* (i) *and* (ii) *also hold if the spaces $W^s(\mathbb{R})$ are replaced by $W^s_H(\mathbb{R})$.*

Bongioanni and Rogers obtained their convergence result via a sharp global maximal operator estimate from W^s_H into L^p. The relevant maximal function is

$$\mathcal{M} f(x) = \operatorname*{ess\,sup}_{0<t<\pi/8} |e^{-itH} f(x)|.$$

The proof of Theorem 2(i) is based on a local L^1 estimate for $\mathcal{M}$. The following result says that there is no global L^p estimate for $\mathcal{M}$ from W^s into L^p.

Theorem 3. *Let $d = 1$ and $s > 0$. The operator $\mathcal{M}$ does not map $W^s(\mathbb{R})$ boundedly into L^p nor into weak L^p for any $p < \infty$, and $\mathcal{M}$ maps $W^s(\mathbb{R})$ boundedly into L^∞ if and only if $s > 1/2$.*

By $c > 0$ and $C < \infty$ we denote many different constants.

2. Some key formulas; proof of Theorem 1. Let $h_n(x)$, $n \in \mathbb{N}_0$, denote the Hermite functions in $\mathbb{R}$, normalized in L^2. By Φ_μ, $\mu \in \mathbb{N}_0^d$, we denote the d-dimensional, normalized Hermite functions, which are simply the tensor products of the h_n. See further Thangavelu [9, Sect. 1.1].

The semigroup e^{-tH}, $t > 0$, generated by H can be defined also with a complex parameter z instead of t, for $\Re z > 0$. Moreover, for these z the operator e^{-zH} is given by integration against the kernel

$$K_z(x,y) = \sum_{\mu \in \mathbb{N}_0^d} e^{-(2|\mu|+d)z} \Phi_\mu(x) \Phi_\mu(y). \tag{5}$$

For real and for complex parameter values, this series can be summed. The sum is the well-known Mehler kernel, which can be found for instance in [9, equation (4.1.3), p. 85]. For $\Re z > 0$ one has

$$K_z(x,y) = \frac{1}{(2\pi \sinh 2z)^{d/2}} \exp\biggl(\frac{1}{2}\biggl(-\coth 2z\,(|x|^2+|y|^2) + \frac{2}{\sinh 2z}\,x\cdot y\biggr)\biggr).$$

This expression is well defined also for z on the imaginary axis, except at the multiples of $i\pi/2$. Indeed, for $t \in \mathbb{R}\setminus(\pi/2)\mathbb{Z}$ we get

$$\begin{aligned}(6)\qquad K_{it}(x,y) &= \frac{e^{-i\pi d/4}}{(2\pi \sin 2t)^{d/2}} \exp\biggl(\frac{i}{2}\biggl(\cot 2t\,(|x|^2+|y|^2) - \frac{2}{\sin 2t}\,x\cdot y\biggr)\biggr)\\ &= \frac{e^{-i\pi d/4}}{(2\pi \sin 2t)^{d/2}} \exp\biggl(\frac{i}{2}\biggl(\cot 2t\,\biggl|y - \frac{x}{\cos 2t}\biggr|^2 - \tan 2t\,|x|^2\biggr)\biggr).\end{aligned}$$

By analytic continuation from $\Re z > 0$, one sees that the argument of the quantity $(2\pi\sin 2t)^{d/2}$ occurring here should be chosen as $[2t/\pi]\pi d/2$. One can also check that integration against this kernel gives the solution of the problem (3), at least for test functions f. Since K_{it} is the kernel of e^{-itH}, we shall often write $K_{it}f$ instead of $e^{-itH}f$. Clearly, each operator e^{-itH} is bounded on L^2.

The Hermite functions h_n are real-valued and have the same parity as the index n. From (5), it follows that $K_{\overline{z}}(x,y) = \overline{K_z(x,y)}$, and also that $K_{z+i\pi/2}(x,y) = e^{-i\pi d/2}K_z(-x,y)$. Here $\Re z > 0$, but if $t \in \mathbb{R}$ is not a multiple of $\pi/2$, we also conclude that

$$K_{-it}(x,y) = \overline{K_{it}(x,y)} \quad \text{and} \quad K_{i(t+\pi/2)}(x,y) = e^{-i\pi d/2}K_{it}(-x,y).$$

For real functions f, it follows that the $L^p(\mathbb{R}^n)$ norm of $e^{-itH}f$ is even and $\pi/2$-periodic as a function of t, and thus determined by its values for $0 < t < \pi/4$.

We shall compare the operators e^{-itH} and $e^{it\Delta}$ by finding a link between their kernels. Observe that $e^{it\Delta}$ is given by convolution with the standard Schrödinger kernel

$$L_{it}(x) = e^{-i\pi d/4}\,\frac{1}{(4\pi t)^{d/2}} \exp\biggl(i\,\frac{1}{4t}\,|x|^2\biggr).$$

Instead of $e^{it\Delta}f$, we shall often write $L_{it}f$.

Lemma 1. *For any $f \in L^2$ and any $v > 0$,*

$$K_{i(\arctan v)/2}f(x) = \exp(-iv|x|^2/2)(1+v^2)^{d/4}L_{iv/2}f(x\sqrt{1+v^2}).$$

Proof. For $0 < t < \pi/4$, we let $\tan 2t = v$ in (6) and get

$$
\begin{aligned}
&K_{i(\arctan v)/2}(x,y)\\
&\quad = e^{-i\pi d/4}\left(\frac{\sqrt{1+v^2}}{2\pi v}\right)^{d/2} \exp\left(-i\,\frac{v}{2}\,|x|^2\right)\exp\left(i\,\frac{1}{2v}\,|y-x\sqrt{1+v^2}|^2\right)\\
&\quad = \exp(-iv|x|^2/2)(1+v^2)^{d/4}L_{iv/2}(x\sqrt{1+v^2}-y).
\end{aligned}
$$

Integrating against $f(y)\,dy$, we obtain the desired equation when $f \in C_0^\infty$. The general case then follows by continuity in L^2. ∎

Proof of Theorem 1. Assuming $p, q < \infty$, we get

$$
\begin{aligned}
&\int_0^{\pi/4}\Big(\int_{\mathbb{R}^d} |K_{it}f(x)|^p\,dx\Big)^{q/p}\,dt\\
&\qquad = \int_0^\infty \Big(\int_{\mathbb{R}^d} |K_{i(\arctan v)/2}f(x)|^p\,dx\Big)^{q/p}\,\frac{1}{2}\,\frac{1}{1+v^2}\,dv\\
&\qquad = \frac{1}{2}\int_0^\infty \Big(\int_{\mathbb{R}^d} |(1+v^2)^{d/4}L_{iv/2}f(x\sqrt{1+v^2})|^p\,dx\Big)^{q/p}\,\frac{1}{1+v^2}\,dv\\
&\qquad = \frac{1}{2}\int_0^\infty \Big(\int_{\mathbb{R}^d} |L_{iv/2}f(x)|^p\,dx\Big)^{q/p}(1+v^2)^{-q(d/p+2/q-d/2)/2}\,dv\\
&\qquad = \int_0^\infty \Big(\int_{\mathbb{R}^d} |L_{iv}f(x)|^p\,dx\Big)^{q/p}\,dv.
\end{aligned}
$$

The cases when p or q is infinite are similar. ∎

3. Proof of Theorem 2. From now on, $d = 1$. In this section, we shall need the following estimate, which is based on Carleson's lemma in [3, p. 24]. It can also be seen as a limit case of a lemma due to Kenig and A. Ruiz [7, Lemma 2] (cf. (7) below), but we prefer to give a direct proof.

LEMMA 2. *Let a and b be real numbers with $(a,b) \neq (0,0)$. Then for any interval J,*

$$
\Big|\int_J e^{i(at+bt^2)}\,\frac{dt}{|t|^{1/2}}\Big| \le C\min(|a|^{-1/2}, |b|^{-1/4}),
$$

where C is an absolute constant. If J is unbounded, the integral here is interpreted as the limit of the integrals over bounded intervals increasing to J.

Proof. Assume first $b = 0$. By homogeneity, we need then only consider the case $a = 1$, which is easy.

When $b \neq 0$, we see by taking the conjugate that we may assume $b > 0$. Let $u = b^{1/2}t$ and $A = -ab^{-1/2}/2$. Then

$$\int_J e^{i(at+bt^2)} \frac{dt}{|t|^{1/2}} = b^{-1/4} \int_{J'} e^{i(-2Au+u^2)} \frac{du}{|u|^{1/2}} = e^{-iA^2} b^{-1/4} \int_{J'} e^{i(u-A)^2} \frac{du}{|u|^{1/2}}$$

for some interval J'. The lemma is equivalent to the following claim:

$$\left| \int_{J'} e^{i(t-A)^2} \frac{dt}{|t|^{1/2}} \right| \leq C \min(1, |A|^{-1/2}). \tag{7}$$

Without loss of generality, we may assume $A \geq 0$. Consider first the case $0 \leq A \leq 2$. Then we split the integral in (7) and integrate by parts in the second term, getting

$$\begin{aligned}
\left| \int_{J'} e^{i(t-A)^2} \frac{dt}{|t|^{1/2}} \right| &\leq \left| \int_{|t|<4} \chi_{J'} e^{i(t-A)^2} \frac{dt}{|t|^{1/2}} \right| + \left| \int_{|t|>4} \chi_{J'} e^{i(t-A)^2} \frac{dt}{|t|^{1/2}} \right| \\
&\leq \int_{|t|<4} \frac{dt}{|t|^{1/2}} + |\text{integrated terms}| + \left| \int_{|t|>4} \chi_{J'} e^{i(t-A)^2} \frac{d}{dt}\left(\frac{1}{2(t-A)|t|^{1/2}} \right) dt \right| \\
&\leq C + C + C \int_{|t|>4} \frac{dt}{|t|^{5/2}} \leq C.
\end{aligned}$$

Now let $A > 2$. We begin by observing that

$$\left| \int_{|t|<1/A} \chi_{J'} e^{i(t-A)^2} \frac{dt}{|t|^{1/2}} \right| \leq \int_{|t|<1/A} \frac{dt}{|t|^{1/2}} \leq CA^{-1/2}$$

and

$$\int_{|t-A|<1} \chi_{J'} \frac{dt}{|t|^{1/2}} \leq CA^{-1/2}.$$

In the remaining integral, taken over the set

$$\{t \in J' : |t| > 1/A \text{ and } |t-A| > 1\},$$

we integrate by parts, as above. The integrated terms will then be controlled by the values of $(t-A)^{-1}|t|^{-1/2}$ at a few points in the set

$$\{|t| \geq 1/A \text{ and } |t-A| \geq 1\},$$

and those values are all bounded by $CA^{-1/2}$. Thus we need only consider the integral

$$\Big|\int\limits_{\{t\in J':|t|>1/A \text{ and } |t-A|>1\}} e^{i(t-A)^2}\,\frac{d}{dt}\Big(\frac{1}{(t-A)|t|^{1/2}}\Big)\,dt\Big|$$
$$\le \int\limits_{\{|t|>1/A \text{ and } |t-A|>1\}} \frac{1}{(t-A)^2|t|^{1/2}}\,dt + \int\limits_{\{|t|>1/A \text{ and } |t-A|>1\}} \frac{1}{|t-A|\,|t|^{3/2}}\,dt$$
$$= I + II.$$

We split each of the integrals I and II thus defined into parts given by $|t| < A/2$ and $|t| > A/2$. For I we get

$$I = \Big(\int\limits_{\{1/A<|t|<A/2\}} + \int\limits_{\{|t-A|>1 \text{ and } |t|>A/2\}}\Big)\frac{1}{(t-A)^2|t|^{1/2}}\,dt$$
$$\le C\int\limits_{|t|<A/2} \frac{1}{A^2|t|^{1/2}}\,dt + C\int\limits_{|t-A|>1} \frac{1}{(t-A)^2A^{1/2}}\,dt \le CA^{-1/2},$$

and similarly

$$II = \Big(\int\limits_{\{1/A<|t|<A/2\}} + \int\limits_{\{|t-A|>1 \text{ and } |t|>A/2\}}\Big)\frac{1}{|t-A|\,|t|^{3/2}}\,dt$$
$$\le C\int\limits_{|t|>1/A} \frac{1}{A|t|^{3/2}}\,dt + \int\limits_{|t|>A/2} \frac{1}{|t|^{3/2}}\,dt \le CA^{-1/2}.$$

The claim is verified, and Lemma 2 is proved. ∎

The maximal function estimate in the next lemma will enable us to prove Theorem 2(i). For $f \in C_0^\infty$, the function $e^{-itH}f(x) = K_{it}f(x)$ is continuous in $(x,t) \in \mathbb{R}\times\overline{\mathbb{R}_+}$ if defined as $f(x)$ for $t = 0$, as is easily verified with Fourier transforms. In the definition of $\mathcal{M}f$, one can for $f \in C_0^\infty$ obviously replace the essential supremum by an ordinary supremum.

Lemma 3. *Let I be a bounded interval. Then for any $f \in C_0^\infty(\mathbb{R})$,*

$$\int\limits_I \mathcal{M}f(x)\,dx \le C\|f\|_{W^{1/4}}, \qquad C = C(I). \tag{8}$$

Before proving this lemma, we use it to prove Theorem 2(i). Given $f \in W^{1/4}$, we take a sequence $f_j \in C_0^\infty$, $j = 1,2,\ldots$, with $\|f_j - f\|_{W^{1/4}} < 2^{-j}$. Applying Lemma 3 to $f_j - f_{j+1}$, whose $W^{1/4}$ norm is less than 2^{1-j}, we get

$$\int\limits_I \sup_{0<t<\pi/8} |K_{it}f_j(x) - K_{it}f_{j+1}(x)|\,dx \le C2^{-j}. \tag{9}$$

Here the supremum can be taken over $0 \le t < \pi/8$, since each function $K_{it}f_j(x)$ is continuous in $\mathbb{R}\times[0,\pi/8)$ with the value $f_j(x)$ at $(x,0)$. The integrals in (9) have a finite sum over j, so that

$$\sum_{j=1}^{\infty} \sup_{0\le t<\pi/8} |K_{it}f_j(x) - K_{it}f_{j+1}(x)|$$

is finite for a.a. $x \in I$. But for any fixed x with this property, the functions $t \mapsto K_{it}f_j(x)$ will converge, uniformly in $0 \le t < \pi/8$, to a continuous function $u_x(t)$. On the other hand, $K_{it}f_j(x) \to K_{it}f(x)$ in $L^2(I \times (0, \pi/8))$, and $f_j \to f$ in $L^2(\mathbb{R})$. We conclude that for a.a. x, the function $t \mapsto K_{it}f(x)$ must coincide with the continuous function $u_x(t)$ for a.a. $t \in (0, \pi/8)$ and, moreover, $u_x(0) = f(x)$. This implies Theorem 2(i).

Proof of Lemma 3. Because of Lemma 1, one can replace $\mathcal{M}f(x)$ by

$$\sup_{0<v<1} |L_{iv/2}f(x\sqrt{1+v^2})|$$

when proving (8). It is clearly enough to show that for all $f \in C_0^\infty$,

$$\int_I \sup_{0<v<1} \Re_+ L_{iv/2}f(x\sqrt{1+v^2})\,dx \le C\|f\|_{W^{1/4}}, \tag{10}$$

where $\Re_+$ denotes the positive part of the real part.

We first compare the integrals over I of

$$\sup \Re_+ L_{iv/2}f(x\sqrt{1+v^2}) \quad \text{and} \quad \sup \Re L_{iv/2}f(x\sqrt{1+v^2}),$$

where both suprema are taken over $0 < v < 1$. They differ only on the set $M = \{x \in I : \sup \Re L_{iv/2}f(x\sqrt{1+v^2}) < 0\}$. Since $L_{iv/2}f(x\sqrt{1+v^2})$ converges pointwise to f as $v \to 0^+$, we have $\sup \Re L_{iv/2}f(x\sqrt{1+v^2}) \ge \Re f(x)$ for all x, and so

$$\begin{aligned}
&\int_I \sup \Re_+ L_{iv/2}f(x\sqrt{1+v^2})\,dx \\
&\qquad = \int_I \sup \Re L_{iv/2}f(x\sqrt{1+v^2})\,dx - \int_M \sup \Re L_{iv/2}f(x\sqrt{1+v^2})\,dx \\
&\qquad \le \int_I \sup \Re L_{iv/2}f(x\sqrt{1+v^2})\,dx + \int_M (-\Re f(x))\,dx \\
&\qquad \le \int_I \sup \Re L_{iv/2}f(x\sqrt{1+v^2})\,dx + C\|f\|_{W^{1/4}};
\end{aligned}$$

here the last step went via an L^2 estimate.

This means that we can replace $\Re_+$ by $\Re$ when we prove (10) for $f \in C_0^\infty$. We shall use the method of Kolmogorov–Seliverstov–Plessner, see also Carleson [3, Theorem, p. 24]. It is enough to let $v = v(x)$ be a measurable function of $x \in I$ with $0 < v(x) < 1$ and to prove that

$$\Re \int_I L_{iv(x)/2} f(x\sqrt{1+v(x)^2})\,dx \le C\|f\|_{W^{1/4}},$$

with $C = C(I)$ independent of $v(x)$ and f.

We define the Fourier transform by $\hat{h}(\xi) = \int_{\mathbb{R}} h(x)e^{-ix\xi}\,dx$ and observe that $\widehat{L_{it}}(\xi) = \exp(-it|\xi|^2)$. This leads to

$$\begin{aligned}
2\pi\Big|\int_I L_{iv(x)/2} f(x\sqrt{1+v(x)^2})\,dx\Big| &= \Big|\int_{-\infty}^{\infty} \hat{f}(\xi)\int_I e^{ix\xi\sqrt{1+v(x)^2}}e^{-iv(x)\xi^2/2}\,dx\,d\xi\Big| \\
&\le \Big(\int_{-\infty}^{\infty} |\hat{f}(\xi)|^2|\xi|^{1/2}\,d\xi\Big)^{1/2}\bigg(\int_{-\infty}^{\infty}\frac{d\xi}{|\xi|^{1/2}}\Big|\int_I e^{i(x\xi\sqrt{1+v(x)^2}-v(x)\xi^2/2)}\,dx\Big|^2\bigg)^{1/2}.
\end{aligned}$$

Here the first factor is controlled by the norm of f in $W^{1/4}$. Thus Lemma 3 will follow if we prove that the second factor is bounded by some C. To this end, we write

$$\begin{aligned}
(11)\qquad &\int_{-\infty}^{\infty}\frac{d\xi}{|\xi|^{1/2}}\Big|\int_I e^{i(x\xi\sqrt{1+v(x)^2}-v(x)\xi^2/2)}\,dx\Big|^2 \\
&= \int_{-\infty}^{\infty}\frac{d\xi}{|\xi|^{1/2}}\iint_{I\times I} e^{i(x\,\xi\sqrt{1+v(x)^2}-v(x)\xi^2/2)}e^{-i(y\xi\sqrt{1+v(y)^2}-v(y)\xi^2/2)}\,dx\,dy \\
&= \iint_{I\times I} dx\,dy \int_{-\infty}^{\infty}\frac{e^{ia\xi+ib\xi^2}}{|\xi|^{1/2}}\,d\xi,
\end{aligned}$$

where $a = x\sqrt{1+v(x)^2} - y\sqrt{1+v(y)^2}$ and $b = (v(y)-v(x))/2$. Observe that

$$|\sqrt{1+v(x)^2} - \sqrt{1+v(y)^2}| = \frac{|v(x)+v(y)|\,|v(x)-v(y)|}{\sqrt{1+v(x)^2}+\sqrt{1+v(y)^2}} \le |v(x)-v(y)|.$$

In order to bound the last inner integral in (11), we shall distinguish between two cases.

Case 1: $|y|\,|b| < |x-y|/4$. Then we have

$$\begin{aligned}
|a| &= |(x-y)\sqrt{1+v(x)^2} + y(\sqrt{1+v(x)^2} - \sqrt{1+v(y)^2})| \\
&> |x-y| - |y(\sqrt{1+v(x)^2} - \sqrt{1+v(y)^2})| \\
&\ge |x-y| - |y|\,|v(x)-v(y)| = |x-y| - |y|\,|2b| \ge |x-y|/2,
\end{aligned}$$

and Lemma 2 implies

$$\int_{-\infty}^{\infty}\frac{e^{ia\xi+ib\xi^2}}{|\xi|^{1/2}}\,d\xi \le C|a|^{-1/2} \le C\,\frac{1}{|x-y|^{1/2}}.$$

CASE 2: $|y|\,|b| \geq |x-y|/4$. By using again Lemma 2, we conclude

$$\int_{-\infty}^{\infty} e^{ia\xi+ib\xi^2} \frac{d\xi}{|\xi|^{1/2}} \leq Cb^{-1/4} \leq C\,\frac{|y|^{1/4}}{|x-y|^{1/4}}.$$

Summing up, for the iterated integral in (11) we get

$$\iint_{I\times I} dx\,dy \int_{-\infty}^{\infty} \frac{e^{ia\xi+ib\xi^2}}{|\xi|^{1/2}}\,d\xi \leq C \iint_{I\times I} \left(\frac{1}{|x-y|^{1/2}} + \frac{|y|^{1/4}}{|x-y|^{1/4}}\right) dx\,dy \leq C(I),$$

and the proof of Lemma 3 is complete. ■

Next, we prove Theorem 2(ii). Because of Lemma 1, it is sufficient to fix $s < 1/4$ and construct a $\varphi \in W^s$ for which the functions $L_{iv/2}\varphi(x\sqrt{1+v^2})$ diverge for x in a set of positive measure, as $v \to 0$ and v avoids any given null set. The method is taken from [4], though we prefer to make the construction more explicit.

Choose a nonzero $f \in C_0^\infty$ supported in $\mathbb{R}_- = \{x : x < 0\}$ and consider the functions $f_t(y) = f(y/t)e^{2iy/t^2}$ for small $t > 0$. Their Fourier transforms are $\widehat{f_t}(\xi) = t\hat{f}(t\xi - 2/t)$, and one finds that

$$\|f_t\|_{W^s} \leq Ct^{1/2-2s} \tag{12}$$

for $t < 1$. Except for constant factors, $(1+v^2)^{1/4}L_{iv/2}f_t(x\sqrt{1+v^2})$ is given by

$$(1+v^2)^{1/4}v^{-1/2}\int_{\mathbb{R}} \exp\left(i\,\frac{1}{2v}\,(x\sqrt{1+v^2}-y)^2\right) f_t(y)\,dy. \tag{13}$$

Here we choose $v = v(x,t) = xt^2/\sqrt{4-x^2t^4}$ for $0 < x < 1$, which implies

$$v(x,t)/\sqrt{1+v(x,t)^2} = xt^2/2. \tag{14}$$

Expanding the square in (13) and using (14), we find that the expression (13) for this value of v and $0 < x < 1$ equals $\sqrt{2/x}$ times

$$\begin{aligned}
\frac{1}{t}\int_{\mathbb{R}} \exp\left(i\,\frac{2v(x,t)}{t^4}\right) \exp\left(-i\,\frac{2y}{t^2}\right) \exp\left(i\,\frac{y^2}{2v(x,t)}\right) f\left(\frac{y}{t}\right) \exp\left(2i\,\frac{y}{t^2}\right) dy \\
= \frac{1}{t}\exp\left(i\,\frac{2v(x,t)}{t^4}\right) \int_{\mathbb{R}} \exp\left(i\,\frac{y^2}{2v(x,t)}\right) f\left(\frac{y}{t}\right) dy \\
= \exp\left(i\,\frac{2v(x,t)}{t^4}\right) \int_{\mathbb{R}} \exp\left(iy^2\,\frac{\sqrt{1-x^2t^4/4}}{x}\right) f(y)\,dy \\
= \exp\left(i\,\frac{2v(x,t)}{t^4}\right) \Phi\left(\frac{x}{\sqrt{1-x^2t^4/4}}\right),
\end{aligned}$$

with $\Phi(z) = \int_{\mathbb{R}} f(y)\exp(iy^2/z)\,dy$. The function Φ is holomorphic in $\mathbb{C}\setminus\{0\}$ and not identically 0. Thus there exists an interval $I \subset (1/2, 1)$ such that $|\Phi(z)| > c$ for some constant $c > 0$ when $z \in I$. We can then find a subinterval $I' \subset I$ and an $\varepsilon > 0$ for which $x \in I'$ and $0 < t < \varepsilon$ imply $x/\sqrt{1 - x^2t^4/4} \in I$ and thus $|\Phi(x/\sqrt{1 - x^2t^4/4})| > c$.

To summarize the above, we have shown that for some $c > 0$,

$$|L_{iv(x,t)/2} f_t(x\sqrt{1 + v(x,t)^2})| > c, \tag{15}$$

when $t < \varepsilon$ and $x \in I'$. By continuity, one gets a stronger version of this inequality: it will remain valid if $v(x,t)$ is replaced by any number in a sufficiently small neighborhood of $v(x,t)$, a neighborhood which may depend on x and t.

We shall choose $\varphi = \sum_{j=1}^{\infty} j f_{t_j}$, where the numbers $t_j \in (0, \varepsilon)$ will be defined recursively. In particular, they shall satisfy $\sum_j j t_j^{1/2-2s} < \infty$, which implies $\varphi \in W^s$ because of (12). Then

$$L_{iv/2}\varphi(x\sqrt{1 + v^2}) = \sum_{j=1}^{\infty} j L_{iv/2} f_{t_j}(x\sqrt{1 + v^2}).$$

Now consider $x \in I'$ and any $k = 1, 2, \ldots$. Our idea is to make sure that for v close to $v(x, t_k)$, the term with $j = k$ is dominating in the above sum. More precisely, we shall have

$$|L_{iv/2} f_{t_j}(x\sqrt{1 + v^2})| < 2^{-j}, \qquad j \neq k, \tag{16}$$

for $x \in I'$ and $1/2 < v/v(x, t_k) < 2$. Combining this with (15) and its stronger version, we see that for $x \in I'$ and v close to $v(x, t_k)$,

$$|L_{iv/2}\varphi(x\sqrt{1 + v^2})| \geq ck - \sum_{j \neq k} j2^{-j}.$$

The right-hand side here tends to $+\infty$ with k, and divergence will follow once we have established (16).

In the recursive construction of the t_j, we start with any $t_1 \in (0, \varepsilon)$. Assume now $t_1, \ldots, t_{J-1}$ chosen so that (16) holds when $j, k < J$. Then we must find t_J so that, when $x \in I'$,

$$|L_{iv/2} f_{t_j}(x\sqrt{1 + v^2})| < 2^{-j}, \qquad j = 1, \ldots, J - 1, \tag{17}$$

for $1/2 < v/v(x, t_J) < 2$, and

$$|L_{iv/2} f_{t_J}(x\sqrt{1 + v^2})| < 2^{-J} \tag{18}$$

for $1/2 < v/v(x, t_k) < 2$, $k = 1, \ldots, J - 1$. Aiming at (17), we observe that each f_{t_j} is a C_0^∞ function and so $L_{is} f_{t_j} \to f_{t_j}$ uniformly in $\mathbb{R}$ as $s \to 0^+$. Now $v(x,t) \to 0$ as $t \to 0$, and $I' \subset (1/2, 1)$ but the f_{t_j} are supported in $\mathbb{R}_-$.

This means that (17) will hold for the indicated values of x and v, if t_J is chosen small enough.

To obtain (18), we simply estimate $L_{iv/2}f_{t_J}$ by the supremum norm of the kernel $L_{iv/2}$ times the L^1 norm of f_{t_J}. This product is $Cv^{-1/2}t_J$, and (18) follows if t_J is small. The recursive construction and the proof of Theorem 2(ii) are complete.

Proof of Theorem 2(iii). The analog of part (i) for W^s_H is obvious, since $W^s_H \subset W^s$; see [2, Theorem 3(i)]. As for part (ii), observe that the function φ constructed above is in W^s and has compact support. But then φ is also in W^s_H, as proved in [2, Theorem 3(iii)].

Theorem 2 is completely proved. ∎

4. Proof of Theorem 3. Lemma 1 implies that $\mathcal{M}f(x)$ can be estimated from below by a positive constant times

$$\operatorname*{ess\,sup}_{0<v<1} |L_{iv/2}f(x\sqrt{1+v^2})|. \tag{19}$$

We first consider the case $p < \infty$. Fix a large $x_0 > 0$ and choose a function $0 \le \tau \in C_0^\infty$ with $\operatorname{supp}\tau \subset (-1,1)$. Let f be given by $\hat{f}(\xi) = 2\pi e^{-ix_0\xi}\tau(\xi)$, and define $v(x) \in (0,1)$ by $x\sqrt{1+v(x)^2} = x_0$ for $x_0/\sqrt{2} < x < x_0$. Then for these x,

$$\begin{aligned} L_{iv(x)/2}f(x\sqrt{1+v(x)^2}) &= \frac{1}{2\pi}\int_{\mathbb{R}} e^{-iv(x)\xi^2/2}e^{ix\xi\sqrt{1+v(x)^2}}\hat{f}(\xi)\,d\xi \\ &= \int_{\mathbb{R}} e^{-iv(x)\xi^2/2}e^{ix_0\xi}e^{-ix_0\xi}\,\tau(\xi)\,d\xi = \int e^{-iv(x)\xi^2/2}\tau(\xi)\,d\xi. \end{aligned}$$

For $\xi \in \operatorname{supp}\tau$ one has $0 < v(x)\xi^2/2 < 1/2$, and so

$$\Re(L_{iv(x)/2}f(x\sqrt{1+v(x)^2})) > \cos\frac{1}{2}\int\tau > 0, \qquad x_0/\sqrt{2} < x < x_0. \tag{20}$$

By continuity, this holds also if the value of $v(x)$ is slightly modified. Thus $\|\mathcal{M}f\|_p \ge cx_0^{1/p}$ for some $c > 0$, and the weak L^p quasinorm of $\mathcal{M}f$ satisfies the same inequality. But

$$\|f\|^2_{W^s} = \int|\hat{f}(\xi)|^2(1+|\xi|^2)^s\,d\xi = 4\pi^2\int\tau(\xi)^2(1+|\xi|^2)^s\,d\xi$$

is independent of x_0. Finally, let $x_0 \to +\infty$ to get the desired unboundedness.

For $p = \infty$ we first assume that $s > 1/2$. Hölder's inequality then implies that $\|\hat{f}\|_{L^1} \le C\|f\|_{W^s}$. Thus for any x and any v one can estimate

$$L_{iv/2}f(x\sqrt{1+v^2}) = \frac{1}{2\pi}\int_{\mathbb{R}} e^{-iv\xi^2/2}e^{ix\xi\sqrt{1+v^2}}\hat{f}(\xi)\,d\xi$$

by means of the W^s norm of f, as required.

To find a counterexample for $p=\infty$ and $s\le 1/2$, we modify the above construction by taking now $0\le\tau\in C^\infty$ supported in $\mathbb{R}_+$ and such that $\tau(\xi)=\xi^{-1}(\log\xi)^{-2/3}$ for $\xi>2$. As before, $\hat f(\xi)=2\pi e^{-i x_0\xi}\tau(\xi)$, but $x_0>0$ is now fixed. One easily verifies that $f\in W^s$. The choice of $v(x)$ is again given by $x\sqrt{1+v(x)^2}=x_0$, but now only when x is in the interval

$$I=\left(\frac{x_0}{\sqrt{1+v_0^2}},\,\frac{x_0}{\sqrt{1+v_0^2/4}}\right),$$

for some small v_0. Then $v_0/2<v(x)<v_0$, and for almost all $x\in I$ we conclude essentially as before that

$$L_{iv(x)/2}f(x\sqrt{1+v(x)^2})=\int e^{-iv(x)\xi^2/2}\tau(\xi)\,d\xi.$$

Notice that since this is now obtained via a truncation of f at $+\infty$ and an L^2 limit, the integral here should be evaluated as $\lim_{R\to+\infty}\int_0^R$. Part of this integral can be estimated as above: indeed,

$$\Re\Bigl(\int_0^{1/\sqrt{v_0}} e^{-iv(x)\xi^2/2}\tau(\xi)\,d\xi\Bigr)>\cos\frac12\int_0^{1/\sqrt{v_0}}\tau(\xi)\,d\xi\ge c\Bigl(\log\frac{1}{\sqrt{v_0}}\Bigr)^{1/3}.$$

In the remaining part, we integrate by parts and get

$$\Bigl|\int_{1/\sqrt{v_0}}^{\infty} e^{-iv(x)\xi^2/2}\tau(\xi)\,d\xi\Bigr|\le\frac{1}{v(x)}\,\frac{\tau(1/\sqrt{v_0})}{1/\sqrt{v_0}}+\frac{1}{v(x)}\int_{1/\sqrt{v_0}}^{\infty}\Bigl|\frac{d}{d\xi}\frac{\tau(\xi)}{\xi}\Bigr|\,d\xi.$$

The last integral equals $\tau(1/\sqrt{v_0})\sqrt{v_0}$, because the derivative in the integrand is negative here. Since $v(x)>v_0/2$, each term of the above right-hand side is at most $2\log(1/\sqrt{v_0})^{-2/3}$.

Summing up, we see that

$$|L_{iv(x)/2}f(x\sqrt{1+v(x)^2})|\ge c\Bigl(\log\frac{1}{v_0}\Bigr)^{1/3}$$

for a.a. $x\in I$, also after a slight modification of $v(x)$. Letting $v_0\to 0$, we conclude that the essential supremum in (19) is not in L^∞ for this f, which ends the proof. ∎

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