A SPECTRAL GAP PROPERTY FOR SUBGROUPS OF FINITE COVOLUME IN LIE GROUPS

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Dedicated to the memory of Andrzej Hulanicki

Abstract. Let \( G \) be a real Lie group and \( H \) a lattice or, more generally, a closed subgroup of finite covolume in \( G \). We show that the unitary representation \( \lambda_{G/H} \) of \( G \) on \( L^2(G/H) \) has a spectral gap, that is, the restriction of \( \lambda_{G/H} \) to the orthogonal complement of the constants in \( L^2(G/H) \) does not have almost invariant vectors. This answers a question of G. Margulis. We give an application to the spectral geometry of locally symmetric Riemannian spaces of infinite volume.

1. Introduction. Let \( G \) be a locally compact group. Recall that a unitary representation \((\pi, \mathcal{H})\) of \( G \) has almost invariant vectors if, for every compact subset \( Q \) of \( G \) and every \( \varepsilon > 0 \), there exists a unit vector \( \xi \in \mathcal{H} \) such that \( \sup_{x \in Q} \| \pi(x)\xi - \xi \| < \varepsilon \). If this holds, we also say that the trivial representation \( 1_G \) is weakly contained in \( \pi \) and write \( 1_G \trianglelefteq \pi \).

Let \( H \) be a closed subgroup of \( G \) for which there exists a non-zero \( G \)-invariant regular Borel measure \( \mu \) on \( G/H \) (see [BHV, Appendix B] for a criterion of the existence of such a measure). Denote by \( \lambda_{G/H} \) the unitary representation of \( G \) given by left translations on the Hilbert space \( L^2(G/H, \mu) \) of square integrable measurable functions on the homogeneous space \( G/H \). If \( \mu \) is finite, we say that \( H \) has finite covolume in \( G \). In this case, the space \( C^1_{G/H} \) of constant functions on \( G/H \) is contained in \( L^2(G/H, \mu) \) and is \( G \)-invariant, as also is its orthogonal complement

\[
L_0^2(G/H, \mu) = \left\{ \xi \in L^2(G/H, \mu) : \int_{G/H} \xi(x) \, d\mu(x) = 0 \right\}.
\]

In case \( \mu \) is infinite, we set \( L_0^2(G/H, \mu) = L^2(G/H, \mu) \).

Denote by \( \lambda_{G/H}^0 \) the restriction of \( \lambda_{G/H} \) to \( L_0^2(G/H, \mu) \) (in case \( \mu \) is infinite, \( \lambda_{G/H}^0 = \lambda_{G/H} \)). We say that \( \lambda_{G/H} \) (or \( L^2(G/H, \mu) \)) has a spectral gap if \( \lambda_{G/H} \) does not have almost invariant vectors.

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gap if \( \lambda_{G/H}^0 \) has no almost invariant vectors. In the terminology of Chapter III, (1.8)], \( H \) is called weakly cocompact.

By a Lie group we mean a locally compact group \( G \) whose connected component of the identity \( G^0 \) is open in \( G \) and is a real Lie group. We prove
the following result which has been conjectured in Chapter III, Remark (1.12)].

**Theorem 1.** Let \( G \) be a Lie group and \( H \) a closed subgroup with finite covolume in \( G \). Then the unitary representation \( \lambda_{G/H} \) on \( L^2(G/H) \) has a spectral gap.

It is a standard fact that \( L^2(G/H) \) has a spectral gap when \( H \) is co-compact in \( G \) (see Chapter III, Corollary (1.10)]). When \( G \) is a semisimple Lie group, the conclusion of Theorem 1 is an easy consequence of Lemma 3 in [Bekk98]. Our proof is by reduction to these two cases. The crucial tool for this reduction is Proposition (1.11) from Chapter III in [Marg91] (see Proposition 5 below). From Theorem 1 and again from that proposition, we obtain the following corollary.

**Corollary 2.** Let \( G \) be a second countable Lie group, \( H \) a closed subgroup with finite covolume in \( G \), and \( \sigma \) a unitary representation of \( H \). Let \( \pi = \text{Ind}_{G}^{G} \sigma \) be the representation of \( G \) induced from \( \sigma \). If \( 1_H \) is not weakly contained in \( \sigma \), then \( 1_G \) is not weakly contained in \( \pi \).

Observe that, by continuity of induction, the converse is also true: if \( 1_H \prec \sigma \), then \( 1_G \prec \pi \).

From the previous corollary we deduce a spectral gap result for some subgroups of \( G \) with infinite covolume.

Recall that a subgroup \( H \) of a topological group \( G \) is called co-amenable in \( G \) if there is a \( G \)-invariant mean on the space \( C^b(G/H) \) of bounded continuous functions on \( G/H \). When \( G \) is locally compact, this is equivalent to \( 1_G \prec \lambda_{G/H} \); this property has been extensively studied by Eymard [Eyma72] who calls it the amenability of the homogeneous space \( G/H \). Observe that a normal subgroup \( H \) in \( G \) is co-amenable in \( G \) if and only if the quotient group \( G/H \) is amenable.

**Corollary 3.** Let \( G \) be a second countable Lie group and \( H \) a closed subgroup with finite covolume in \( G \). Assume that \( L \) is not co-amenable in \( H \). Then \( \lambda_{G/L} \) (which is defined as \( \text{Ind}_{L}^{G} 1_L \) in case \( G/L \) has no \( G \)-invariant measure) does not weakly contain \( 1_G \).

Corollary 3 is a direct consequence of Corollary 2 since the representation \( \lambda_{G/L} \) on \( L^2(G/L) \) is equivalent to the induced representation \( \text{Ind}_{L}^{G} \lambda_{H/L} \).

Here is a reformulation of the previous corollary. Let \( G \) be a Lie group and \( H \) a closed subgroup with finite covolume in \( G \). If a subgroup \( L \) of \( H \) is co-amenable in \( G \), then \( L \) is co-amenable in \( H \). Observe that the finiteness
of the covolume of $H$ is essential, as examples in [MoPo03] and [Pest03] show. Observe also that the converse (if $L$ is co-amenable in $H$, then $L$ is co-amenable in $G$) is true for any topological group $G$ and any closed subgroup $H$ which is co-amenable in $G$ (see [Eyma72, p. 16]).

Using methods from [Leuz03] (see also [Broo86]), we obtain the following consequence for the spectral geometry of infinite coverings of locally symmetric Riemannian spaces of finite volume. Recall that a lattice in the locally compact group $G$ is a discrete subgroup of $G$ with finite covolume.

**Corollary 4.** Let $G$ be a semisimple Lie group with finite centre and maximal compact subgroup $K$ and let $\Gamma$ be a torsion-free lattice in $G$. Let $\tilde{V}$ be a covering of the locally symmetric space $V = K \backslash G/\Gamma$. Assume that the fundamental group of $\tilde{V}$ is not co-amenable in $\Gamma$.

(i) We have $h(\tilde{V}) > 0$ for the Cheeger constant $h(\tilde{V})$ of $\tilde{V}$.

(ii) We have $\lambda_0(\tilde{V}) > 0$, where $\lambda_0(\tilde{V})$ is the bottom of the $L^2$-spectrum of the Laplace–Beltrami operator on $\tilde{V}$.

There is in general no uniform bound for $h(\tilde{V})$ or $\lambda_0(\tilde{V})$ for all coverings $\tilde{V}$. However, it was shown in [Leuz03] that, when $G$ has Kazhdan’s Property (T), such a bound exists for every locally symmetric space $V = K \backslash G/\Gamma$. Observe also that if, in the previous corollary, the fundamental group of $\tilde{V}$ is co-amenable in $\Gamma$ and has infinite covolume, then $h(\tilde{V}) = \lambda_0(\tilde{V}) = 0$, as shown in [Broo81].

2. Proofs of Theorem 1 and Corollary 4. The following result of Margulis (Proposition (1.11) in Chapter III of [Marg91]) will be crucial.

**Proposition 5** ([Marg91]). Let $G$ be a second countable locally compact group, $H$ a closed subgroup of $G$ such that $G/H$ has a $G$-invariant measure, and $\sigma$ a unitary representation of $H$. Assume that $\lambda_{G/H}$ has a spectral gap and that $1_H$ is not weakly contained in $\sigma$. Then $1_G$ is not weakly contained in $\text{Ind}_H^G \sigma$.

In order to reduce the proof of Theorem 1 to the semisimple case, we will use the following proposition several times.

**Proposition 6.** Let $G$ be a separable locally compact group, and $H_1$ and $H_2$ be closed subgroups of $G$ with $H_1 \subset H_2$ and such that $G/H_2$ and $H_2/H_1$ have non-zero $G$-invariant and $H_2$-invariant regular Borel measures, respectively. Assume that the $H_2$-representation $\lambda_{H_2/H_1}$ on $L^2(H_2/H_1)$ and the $G$-representation $\lambda_{G/H_2}$ on $L^2(G/H_2)$ both have spectral gaps. Then the $G$-representation $\lambda_{G/H_1}$ on $L^2(G/H_1)$ has a spectral gap.

**Proof.** Recall that, for any closed subgroup $H$ of $G$, the representation $\lambda_{G/H}$ is equivalent to the representation $\text{Ind}_H^G 1_H$ induced by the identity.
representation $1_H$ of $H$. Hence, by transitivity of induction,
\[ \lambda_{G/H_1} = \text{Ind}^G_{H_1} 1_{H_1} = \text{Ind}^G_{H_2} (\text{Ind}^{H_2}_{H_1} 1_{H_1}) = \text{Ind}^G_{H_2} \lambda_{H_2/H_1}. \]

We have to consider three cases:

- **First case:** $H_1$ has finite covolume in $G$, that is, $H_1$ has finite covolume in $H_2$, and $H_2$ has finite covolume in $G$. Then
  \[ \lambda^0_{G/H_1} = \lambda^0_{G/H_2} \oplus \text{Ind}^G_{H_2} \lambda^0_{H_2/H_1}. \]
  By assumption, $\lambda^0_{H_2/H_1}$ and $\lambda^0_{G/H_2}$ do not weakly contain $1_{H_2}$ and $1_G$, respectively. It follows from Proposition 5 that $\text{Ind}^G_{H_2} \lambda^0_{H_2/H_1}$ does not weakly contain $1_G$. Hence, $\lambda^0_{G/H_1}$ does not weakly contain $1_G$.

- **Second case:** $H_1$ has finite covolume in $H_2$, and $H_2$ has infinite covolume in $G$. Then
  \[ \lambda_{G/H_1} = \lambda_{G/H_2} \oplus \text{Ind}^G_{H_2} \lambda^0_{H_2/H_1}. \]
  By assumption, $\lambda^0_{H_2/H_1}$ and $\lambda_{G/H_2}$ do not weakly contain $1_{H_2}$ and $1_G$. As above, using Proposition 5, we see that $\lambda_{G/H_1}$ does not weakly contain $1_G$.

- **Third case:** $H_1$ has infinite covolume in $H_2$. By assumption, $\lambda_{H_2/H_1}$ does not weakly contain $1_{H_2}$. By Proposition 5 again, it follows that $\lambda_{G/H_1} = \text{Ind}^G_{H_2} \lambda_{H_2/H_1}$ does not weakly contain $1_G$.

For the reduction of the proof of Theorem 1 to the case where $G$ is second countable, we will need the following lemma.

**Lemma 7.** Let $G$ be a locally compact group and $H$ a closed subgroup with finite covolume. The homogeneous space $G/H$ is $\sigma$-compact.

**Proof.** Let $\mu$ be the $G$-invariant regular probability measure on the Borel subsets of $G/H$. Choose an increasing sequence of compact subsets $K_n$ of $G/H$ with $\lim_n \mu(K_n) = 1$. The set $K = \bigcup_n K_n$ has $\mu$-measure 1 and is therefore dense in $G/H$. Let $U$ be a compact neighbourhood of $e$ in $G$. Then $UK = G/H$ and $UK = \bigcup_n UK_n$ is $\sigma$-compact.

**Proof of Theorem 1.** Through several steps the proof will be reduced to the case where $H$ is a lattice in $G$, and $G$ is a connected semisimple Lie group with trivial centre and without compact factors.

- **First step:** we can assume that $G$ is $\sigma$-compact and hence second-countable. Indeed, let $p : G \to G/H$ be the canonical projection. Since every compact subset of $G/H$ is the image under $p$ of some compact subset of $G$ (see [BHvL] Lemma B.1.1), it follows from Lemma 7 that there exists a $\sigma$-compact subset $K$ of $G$ such that $p(K) = G/H$. Let $L$ be the subgroup of $G$ generated by $K \cup U$ for a neighbourhood $U$ of $e$ in $G$. Then $L$ is a $\sigma$-compact open subgroup of $G$. We show that $L \cap H$ has a finite covolume in $L$, and that $\lambda_{G/H}$ has a spectral gap if $\lambda_{L/L\cap H}$ has a spectral gap.
Since \( LH \) is open in \( G \), the homogeneous space \( L/L \cap H \) can be identified as an \( L \)-space with \( LH/H \). Therefore \( L \cap H \) has finite covolume in \( L \). On the other hand, the restriction of \( \lambda_{G/H} \) to \( L \) is equivalent to the \( L \)-representation \( \lambda_{L/L \cap H} \), since \( LH/H = p(L) = G/H \). Hence, if \( \lambda_{L/L \cap H} \) has a spectral gap, then \( \lambda_{G/H} \) has a spectral gap.

- **Second step:** we can assume that \( G \) is connected. Indeed, let \( G^0 \) be the connected component of the identity of \( G \). We show that \( G^0 \cap H \) has a finite covolume in \( G^0 \), and that \( \lambda_{G/H} \) has a spectral gap if \( \lambda_{G^0/G^0 \cap H} \) has a spectral gap.

  The subgroup \( G^0H \) is open in \( G \) and has finite covolume in \( G \) as it contains \( H \). It follows that \( G^0H \) has finite index in \( G \) since \( G/G^0H \) is discrete. Hence \( \lambda_{G/G^0H} \) has a spectral gap.

  On the other hand, since \( G^0H \) is closed in \( G \), the homogeneous space \( G^0/G^0 \cap H \) can be identified as a \( G^0 \)-space with \( G^0H/H \). Therefore \( G^0 \cap H \) has finite covolume in \( G^0 \). The restriction of \( \lambda_{G^0H/H} \) to \( G^0 \) is equivalent to the \( G^0 \)-representation \( \lambda_{G^0/G^0 \cap H} \). Suppose now that \( \lambda_{G^0/G^0 \cap H} \) has a spectral gap. Then the \( G^0H/H \)-representation \( \lambda_{G^0H/H} \) has a spectral gap, since \( L^2(G^0H/H) \cong L^2(G^0/G^0 \cap H) \) as \( G^0 \)-representations. An application of Proposition 6 with \( H_1 = H \) and \( H_2 = G^0H \) shows that \( \lambda_{G/H} \) has a spectral gap. Hence, we can assume that \( G \) is connected.

- **Third step:** we can assume that \( H \) is a lattice in \( G \). Indeed, let \( H^0 \) be the connected component of the identity of \( H \) and let \( N_G(H^0) \) be the normalizer of \( H^0 \) in \( G \). Observe that \( N_G(H^0) \) contains \( H \). By [Wang 76, Theorem 3.8], \( N_G(H^0) \) is cocompact in \( G \). Hence, \( \lambda_{G/N_G(H^0)} \) has a spectral gap. It follows from the previous proposition that \( \lambda_{G/H} \) has a spectral gap if \( \lambda_{N_G(H^0)/H} \) has a spectral gap.

  The second step applied to the Lie group \( N/H \) shows that \( \lambda_{N/H} \) has a spectral gap if \( \lambda_{N^0/N^0 \cap H} \) has a spectral gap. Observe that \( N^0 \cap H \) is a lattice in the connected Lie group \( N^0 \), since \( H \) is discrete and \( H \) has finite covolume in \( N_G(H^0) \).

  This shows that we can assume that \( H \) is a lattice in the connected Lie group \( G \).
• Fourth step: we can assume that $G$ is a connected semisimple Lie group with no compact factors. Indeed, let $G = SR$ be a Levi decomposition of $G$, with $R$ the solvable radical of $G$, and $S$ a semisimple subgroup. Let $C$ be the maximal compact normal subgroup of $S$. It is proved in \cite[Theorem B, p. 21]{Wang70} that $HCR$ is closed in $G$ and that $HCR/H$ is compact. Hence, by the previous proposition, $\lambda_{G/H}$ has a spectral gap if $\lambda_{G/HCR}$ has a spectral gap.

The quotient $\overline{G} = G/CR$ is a connected semisimple Lie group with no compact factors. Moreover, $\overline{\Pi} = HCR/CR$ is a lattice in $\overline{G}$ since $HCR/CR \cong H/H \cap CR$ is discrete and since $HCR$ has finite covolume in $G$. Observe that $\lambda_{G/HCR}$ is equivalent to $\lambda_{\overline{G}/\overline{\Pi}}$ as a $G$-representation.

• Fifth step: we can assume that $G$ has trivial centre. Indeed, let $Z$ be the centre of $G$. It is known that $ZH$ is discrete (and hence closed) in $G$ (see \cite[Chapter V, Corollary 5.17]{Ragh72}). Hence, $ZH/H$ is finite and $\lambda_{ZH/H}$ has a spectral gap.

By the previous proposition, $\lambda_{G/H}$ has a spectral gap if $\lambda_{G/ZH}$ has a spectral gap. Now, $\overline{G} = G/Z$ is a connected semisimple Lie group with no compact factors and with trivial centre, $\overline{\Pi} = ZH/Z$ is a lattice in $\overline{G}$, and $\lambda_{G/ZH}$ is equivalent to $\lambda_{\overline{G}/\overline{\Pi}}$.

• Sixth step: by the previous steps, we can assume that $H$ is a lattice in a connected semisimple Lie group $G$ with no compact factors and with trivial centre. In this case, the claim was proved in Lemma 3 of \cite{Bekk98}. This completes the proof of Theorem 1.

Proof of Corollary 4.

The proof is identical with the proof of Theorems 3 and 4 in \cite{Leuz03}; we give a brief outline of the arguments. Let $\Lambda$ be the fundamental group of $\tilde{V}$. First, it suffices to prove claims (i) and (ii) for $G/\Gamma$ instead of $K \backslash G/\Gamma$ (see Section 4 in \cite{Leuz03}). So we assume that $\tilde{V} = G/\Lambda$.

Equip $G$ with a right invariant Riemannian metric and $G/\Lambda$ with the induced Riemannian metric. Observe that $G/\Lambda$ has infinite volume, since $\Lambda$ is of infinite index in $\Gamma$. Claim (ii) is a consequence of (i), by Cheeger’s inequality $\frac{1}{4} h(G/\Lambda)^2 \leq \lambda_0(G/\Lambda)$. Recall that the Cheeger constant $h(G/\Lambda)$ of $G/\Lambda$ is the infimum over all numbers $A(\partial \Omega)/V(\Omega)$, where $\Omega$ is an open submanifold of $G/\Lambda$ with compact closure and smooth boundary $\partial \Omega$, and where $V(\Omega)$ and $A(\partial \Omega)$ are the Lebesgue measures of $\Omega$ and $\partial \Omega$.

To prove claim (i), we proceed exactly as in \cite{Leuz03}. By Corollary 3, there exists a compact neighbourhood $H$ of the identity in $G$ and a constant $\varepsilon > 0$ such that

$$
(*) \quad \varepsilon \| \xi \| \leq \sup_{h \in H} \| \lambda_{G/\Lambda}(h) \xi - \xi \| \quad \text{for all } \xi \in L^2(G/\Lambda).
$$
Let $\Omega$ be an open submanifold of $G/\Lambda$ with compact closure and smooth boundary $\partial \Omega$. By [Leuz03, Proposition 1], we can find an open subset $\tilde{\Omega}$ of $G/\Lambda$ with compact closure and smooth boundary such that, for all $h \in H$,

$$(**)
V(U_{|h|}(\partial \Omega)) \leq CV(\tilde{\Omega}) \frac{A(\partial \Omega)}{V(\Omega)},$$

where the constant $C > 0$ only depends on $H$. Here, $|h|$ denotes the distance $d_G(e,g)$ of $h$ to the group unit and, for a subset $S$ of $G/\Lambda$, $U_r(S)$ is the tubular neighbourhood

$$U_r(S) = \{ x \in G/\Lambda : d_G(x,S) \leq r \}.$$ 

Inequality (*) applied to the characteristic function $\chi_{\tilde{\Omega}}$ of $\tilde{\Omega}$ shows that there exists $h \in H$ such that

$$\varepsilon^2 V(\tilde{\Omega}) \leq \| \lambda_{G/\Lambda}(h) \chi_{\tilde{\Omega}} - \chi_{\tilde{\Omega}} \|^2 = V(X),$$

where

$$X = \{ x \in G/\Lambda : x \in \tilde{\Omega}, hx \notin \partial \tilde{\Omega} \} \cup \{ x \in G/\Lambda : x \notin \tilde{\Omega}, hx \in \partial \tilde{\Omega} \}.$$ 

One checks that $X \subset U_{|h|}(\partial \Omega)$. It follows from (*) and (**) that

$$\frac{\varepsilon^2}{C} \leq \frac{A(\partial \Omega)}{V(\Omega)}.$$ 

Hence, $0 < \varepsilon^2/C \leq h(G/\Lambda).$ 

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