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FEYNMAN–KAC FORMULA, λ-POISSON KERNELS AND λ-GREEN FUNCTIONS OF HALF-SPACES AND BALLS IN HYPERBOLIC SPACES

BY

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Dedicated to the memory of Andrzej Hulanicki

Abstract. We apply the Feynman–Kac formula to compute the λ-Poisson kernels and λ-Green functions for half-spaces or balls in hyperbolic spaces. We present known results in a unified way and also provide new formulas for the λ-Poisson kernels and λ-Green functions of half-spaces in $\mathbb{H}^n$ and for balls in real and complex hyperbolic spaces.

1. Introduction. In a series of papers ([BGS], [BM], [JG], [Ma], [Z]) the Poisson kernels and the Green functions in hyperbolic spaces were investigated. The authors of all those papers use the Feynman–Kac formula as the main tool in describing the distribution of a stopped multiplicative functional. In this paper we summarize these investigations and exhibit the main idea of the method. We also give several new results, namely formulas for the λ-Poisson kernels and the λ-Green functions of half-spaces or balls for hyperbolic Brownian motions. This complements the results obtained by Matsumoto [M].

In [BDH] the global Poisson kernel (for the whole space) and generators of the form of a Laplace–Beltrami operator plus some additional term of the first order were investigated for NA groups. The main example of NA group is complex hyperbolic space, realized as Siegel upper half-space. We believe that our method can be applied also in this context.

2. Preliminaries

A. Hypergeometric function and Bessel functions. The equation

$$z(1-z)y''(z)+(\gamma+(\alpha+\beta+1)z)y'(z)-\alpha\beta y(z)=0, \tag{2.1}$$

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where α, β and γ are constants independent of z, is called the *hypergeometric equation.* A solution of (2.1) which is regular at $z = 0$ is given by the *hypergeometric function* ${}_2F_1(\alpha, \beta; \gamma; z)$ defined by the hypergeometric series

$$ {}_2F_1(\alpha,\beta;\gamma;z) = \sum_{n=0}^{\infty} \frac{(\alpha)_n(\beta)_n}{(\gamma)_n n!}\, z^n, \qquad |z|<1, $$

whenever $\gamma \neq 0, -1, -2, \ldots$. Here $(\alpha)_n = \Gamma(\alpha+n)/\Gamma(\alpha)$ denotes the Pochhammer symbol. For analytic continuations of the hypergeometric series see [E, Chapter II]. Another solution of (2.1) is given by some modifications of ${}_2F_1$ and will be described later.

The *modified Bessel function* I_ϑ *of the first kind* is defined by (see, e.g., [E, 7.2.2 (12)])

$$ I_\vartheta(z) = \frac{z^\vartheta}{2^\vartheta} \sum_{k=0}^{\infty} \left(\frac{z}{2}\right)^{2k} \frac{1}{k!\,\Gamma(k+\vartheta+1)}, \qquad z \in \mathbb{C} \setminus (-\mathbb{R}_+), \tag{2.2} $$

where $\vartheta \in \mathbb{R}$ is not a negative integer. The *modified Bessel function of the second kind* is defined by (see [E, 7.2.2 (13) and 7.2.5 (36)])

$$ K_\vartheta(z) = \frac{\pi}{2\sin(\vartheta\pi)}\, [I_{-\vartheta}(z) - I_\vartheta(z)], \qquad \vartheta \notin \mathbb{Z}, \tag{2.3} $$

$$ K_n(z) = \lim_{\vartheta\to n} K_\vartheta(z) = (-1)^n\, \frac{1}{2} \left[\frac{\partial I_{-\vartheta}}{\partial\vartheta} - \frac{\partial I_\vartheta}{\partial\vartheta}\right]_{\vartheta=n}, \qquad n \in \mathbb{Z}. \tag{2.4} $$

The functions I_ϑ, K_ϑ are linearly independent solutions to the differential equation

$$ z^2\varphi''(z) + z\varphi'(z) - (z^2+\vartheta^2)\varphi(z) = 0, \tag{2.5} $$

which is known as the *modified Bessel equation* of order ϑ. The *Wronskian* of the pair $(I_\vartheta, K_\vartheta)$ is given by

$$ W(I_\vartheta, K_\vartheta)(z) = I_\vartheta(z)K_\vartheta'(z) - I_\vartheta'(z)K_\vartheta(z) = -1/z. \tag{2.6} $$

B. One-dimensional diffusion. Let us recall briefly some notions pertaining to linear diffusion [BS]. We consider the one-dimensional diffusion with generator of the form

$$ \mathcal{L}f(x) = \frac{1}{2}\, a^2(x)\, \frac{d^2f}{dx^2}(x) + b(x)\, \frac{df}{dx}(x) - c(x)f(x), \tag{2.7} $$

with regular coefficients $a(x)$, $b(x)$ and $c(x)$. Let $B(x)$ be any indefinite integral for $2b(x)/a^2(x)$. The basic characteristics of the diffusion (the speed measure density $m(x)$, the scale function $s(x)$ and the killing measure density $k(x)$) are given by

$$m(x) = \frac{2}{a^2(x)}\, e^{B(x)}, \tag{2.8}$$

$$s'(x) = e^{-B(x)}, \tag{2.9}$$

$$k(x) = \frac{2}{a^2(x)}\, c(x) e^{B(x)}. \tag{2.10}$$

The λ-*Green function* $G^{\lambda}_{a,b}(x,y)$ of the interval (a,b), $-\infty \le a < b \le \infty$, with respect to the speed measure $m(y)dy$ is given by

$$G^{\lambda}_{a,b}(x,y) = \begin{cases} c\cdot \phi_{\uparrow}(x)\psi_{\downarrow}(y), & a < x < y < b, \\ c\cdot \phi_{\uparrow}(y)\psi_{\downarrow}(x), & a < y < x < b, \end{cases} \tag{2.11}$$

where $\phi_{\uparrow}$ and $\psi_{\downarrow}$ are respectively increasing and decreasing solutions of the equation $\mathcal{L}u = \lambda u$, such that $\lim_{x\to a^+} \phi_{\uparrow}(x) = 0$ and $\lim_{x\to b^-} \psi_{\downarrow}(x) = 0$. The constant c is given by

$$c = \frac{s'(z)}{W(\phi_{\uparrow}, \psi_{\downarrow})(z)},$$

where $W(\phi_{\uparrow}, \psi_{\downarrow})$ is the Wronskian of the pair $(\phi_{\uparrow}, \psi_{\downarrow})$.

C. Hyperbolic spaces and hyperbolic Brownian motion. We investigate Brownian motion in three models of hyperbolic spaces: in $\mathbb{H}^n$ which, as a set, is a half-space of $\mathbb{R}^n$, in $\mathbb{D}^n$, being the unit ball in $\mathbb{R}^n$, and in the unit ball of $\mathbb{C}^n$. The last one is a model of complex hyperbolic space. In every model there is a specific Riemannian (hyperbolic) metric, which determines the Riemannian volume and the Laplace–Beltrami operator Δ_{LB}. Being the divergence of the gradient, Δ_{LB} is a second-order differential operator. By the general theory of diffusion processes on manifolds, Δ_{LB} is the generator of a diffusion process (X_t), which is called the *hyperbolic Brownian motion.* The transition probabilities $p(t;\, x,y)$ of the process (X_t) are solutions of the so-called heat equation $\partial p(t;\, x,y)/\partial t = \Delta_{\mathrm{LB}} p(t;\, x,y)$; their properties and estimates are well-known (see [D]).

D. Killed stochastic processes and their potential theory. Let D be a domain in a hyperbolic space and let (X_t) be a hyperbolic Brownian motion in this space, starting from a point $x \in D$. We set $\tau_D = \inf\{t > 0 : X_t \notin D\}$ and consider (X^D_t), the process killed on exiting D, defined as X_t for $t < \tau_D$ and ∂ for $t \ge \tau_D$. Here ∂ is some additional isolated state (cemetery). The explicit form of the transition densities of (X^D_t) are known only for a few classes of D. In general, such a density is given by the formula

$$p_D(t;x,y) = p(t;x,y) - \mathbb{E}^x[p(t-\tau_D; X_{\tau_D}, y) : t > \tau_D].$$

There are two notions, crucial in potential theory of stochastic processes: the Poisson kernel and the Green function of a domain D. Because diffusions have continuous trajectories, when exiting D, they hit the boundary of D. If $x \in D$ and $\lambda > 0$ define the λ-*harmonic measure* by setting, for a Borel set $A \subset \partial D$,

$$P_D^\lambda(x, A) = \mathbb{E}^x[e^{-\lambda\tau_D} 1_A(X_{\tau_D})]. \tag{2.12}$$

We will consider two types of domains D: half-spaces and balls. In both cases there exist natural measures on ∂D: the Lebesgue measure or the uniform surface measure on the sphere. The λ-*Poisson kernel* $P_D^\lambda(x, y)$ is the density of the λ-harmonic measure with respect to the corresponding measure on ∂D.

The Green function $G_D(x, y)$ measures the time spent by a process at y, when starting from x. More precisely, $G_D(x, y) = \int_0^\infty p_D(t; x, y)\, dt$. We consider a more general object, namely, for $\lambda > 0$, the λ-*Green function* is defined by the following formula ($x, y \in D$):

$$G_D^\lambda(x, y) = \int_0^\infty e^{-\lambda t}\, p_D(t; x, y)\, dt. \tag{2.13}$$

3. Harmonic measures and Green functions of a half-space in $\mathbb{H}^n$. We start with the description of the half-space model of real hyperbolic space. Define $\mathbb{H}^n = \{x = (x_1, \ldots, x_{n-1}, x_n) = (\tilde{x}, x_n) : x_n > 0\} \subset \mathbb{R}^n$. The Riemannian distance in $\mathbb{H}^n$ is given by the formula

$$\cosh(d_{\mathbb{H}^n}(x, y)) = 1 + \frac{|x - y|^2}{2x_n y_n}, \tag{3.14}$$

and the canonical (hyperbolic) volume element is given by

$$dV_{\mathbb{H}^n}(x) = \frac{dx_1 \ldots dx_n}{x_n^n}.$$

We will denote the Laplace–Beltrami operator in $\mathbb{H}^n$ by $\Delta_{\mathbb{H}^n}$. It is given by

$$\Delta_{\mathbb{H}^n} = x_n^2 \sum_{k=1}^n \frac{\partial^2}{\partial x_k^2} - (n-2)x_n \frac{\partial}{\partial x_n}.$$

The heat kernel on $\mathbb{H}^n$ is a function of $d = d_{\mathbb{H}^n}(x, y)$ and is given by Gruet's formula (see [Gr])

$$k_n(t, d) = \frac{e^{-(n-1)^2 t/4}}{2^{(n+1)/2}\pi^{1+n/2} t^{1/2}}\, \Gamma\left(\frac{n+1}{2}\right) \int_0^\infty \frac{e^{(\pi^2 - r^2)/4t} \sinh(r) \sin(\pi r/2t)}{(\cosh(r) + \cosh(d))^{(n+1)/2}}\, dr.$$

The transition probability of the process (X_t) starting from $x \in \mathbb{H}^n$ is given by

$$P^x(X_t \in B) = \int_B k_n(t, d_{\mathbb{H}^n}(x, y))\, dV_{\mathbb{H}^n}(y), \qquad B \in \mathcal{B}(\mathbb{H}^n).$$

Consider the half-space $D_a = \{x \in \mathbb{H}^n : x_n > a\}$ for some fixed $a > 0$. Define

$$\tau_{D_a} = \inf\{s > 0 : X(s) \notin D_a\}.$$

Using the law of the iterated logarithm it is easy to check that for every $x_n > 0$,

$$\lim_{t\to\infty} X_n(t) = 0 \qquad P^{x_n}\text{-a.s.}$$

This means that $\tau_{D_a} < \infty$ a.s. Observe that $X_n(\tau_{D_a}) = a$ and consequently the function $P^\lambda_{D_a}(x,y)$ depends only on the starting point x of the process $X(t)$ and $\tilde{y} \in \mathbb{R}^{n-1}$, where $y = (\tilde{y}, a)$. We will use the notation $P^\lambda_{D_a}(x,\tilde{y})$ for the function $P^\lambda_{D_a}(x,(\tilde{y},a))$.

A. λ-Poisson kernel of a half-space. Now we will compute the Fourier transform of the λ-Poisson kernel of D_a.

Theorem 1 (Fourier transform of the λ-Poisson kernel). *For every x in D_a, $y \in \partial D_a$ and $u \in \mathbb{R}^{n-1}$ we have*

$$\int\limits_{\mathbb{R}^{n-1}} \exp(i\langle u,\tilde{y}\rangle) P^\lambda_{D_a}(x,\tilde{y})\, d\tilde{y} = \exp(i\langle u,\tilde{x}\rangle)\Big(\frac{x_n}{a}\Big)^\nu \frac{K_\mu(rx_n)}{K_\mu(ra)},$$

where $r = |u|$, $\nu = (n-1)/2$, $\mu = (\nu^2+\lambda)^{1/2}$ and K_μ is a modified Bessel function of the third kind.

Proof. In a natural way the process (X_t) can be decomposed into two parts: $\tilde{X}(t) = (X_1(t),\ldots,X_{n-1}(t))$ and $X_n(t)$. Observe that the process (X_t) reaches ∂D_a for the first time when $X_n(t)$ hits the point a.

In order to compute the Fourier transform of the distribution of $\tilde{X}(\tau_{D_a})$ we consider the family of processes

$$f_u(\tilde{X}(t)) = e^{i\langle u,\tilde{X}(t)\rangle} \qquad \text{for } u \in \mathbb{R}^{n-1},$$

and try to compute the characteristic function $\phi(u) = \mathbb{E}^x(e^{i\langle u,\tilde{X}(t)\rangle})$. The main tool here is martingale theory; in order to apply it we will make the process $f_u(\tilde{X}(t))$ a martingale. For that purpose let us define the multiplicative functional

$$V_t = \exp\Big[\int\limits_0^t q(X_n(s))\,ds\Big],$$

where q is a non-negative, locally bounded Borel function to be specified later, and consider the process

$$Z_t = e^{-\lambda t} f_u(\tilde{X}(t)) V_t = e^{-\lambda t} e^{i\langle u,\tilde{X}(t)\rangle} \exp\Big[\int\limits_0^t q(X_n(s))\,ds\Big].$$

The process V_t is of bounded variation, hence $dV_t = q(X_n(t))V_t\,dt$. Because $d\langle X_j, X_k\rangle = 2\delta_{jk}\,dt$, $j,k = 1,\dots,n-1$, the Itô formula gives

$$\begin{aligned} dZ_t &= -\lambda e^{-\lambda t} f_u(\tilde{X}(t))V_t\,dt + e^{-\lambda t} f_u(\tilde{X}(t))V_t \sum_{k=1}^{n-1} iu_k X_n(t) dB_k(t) \\ &\quad + \frac{1}{2}\Big(\sum_{k=1}^{n-1} i^2 u_k^2\Big) e^{-\lambda t} f_u(\tilde{X}(t))V_t X_n^2(t)\,2dt + e^{-\lambda t} f_u(\tilde{X}(t))V_t q(X_n(t))\,dt \\ &= e^{-\lambda t} f_u(\tilde{X}(t))V_t\Big[iX_n(t)\sum_{k=1}^{n-1} u_k dB_k(t) + \Big(q(X_n(t)) - \lambda - X_n^2(t)\sum_{k=1}^{n-1} u_k^2\Big)\Big]. \end{aligned}$$

The process Z_t, being a sum of $n-1$ stochastic integrals and a bounded variation term, is a martingale if that term vanishes for all t. This is achieved if we put

$$q(x_n) = |u|^2 x_n^2 + \lambda, \quad \text{where} \quad |u|^2 = \sum_{k=1}^{n-1} u_k^2.$$

Now we can compute the expectation $\mathbb{E}^x(h(X_n(t))\, e^{i\langle u,\tilde{X}(t)\rangle})$, where h is a bounded Borel function. We have

$$\begin{aligned} \mathbb{E}^x(h(X_n(t))e^{-\lambda t} e^{i\langle u,\tilde{X}(t)\rangle}) &= \mathbb{E}^x(h(X_n(t))e^{-\lambda t} e^{i\langle u,\tilde{X}(t)\rangle} V_t \cdot V_t^{-1}) \\ &= \mathbb{E}^x(h(X_n(t))Z_t \cdot V_t^{-1}). \end{aligned}$$

But we know that Z_t is a martingale defined by the stochastic equation

$$dZ_t = \Big(\sum_{k=1}^{n-1} iu_k X_n(t) dB_k(t)\Big) e^{-\lambda t} f_u(\tilde{X}(t))V_t.$$

This implies that $Z_t = Z_0 + Z_t^{(1)}$, where $Z_t^{(1)}$ is a martingale with $\mathbb{E}Z_t^{(1)} = 0$. Moreover, $Z_t^{(1)}$ is a stochastic integral with respect to the Brownian processes $B_1(t), \dots, B_{n-1}(t)$, which are independent of $B_n(t)$, and $X_n(t)$ depends only on $B_n(t)$. This implies that $\mathbb{E}^x(h(X_n(t))Z_t^{(1)} \cdot V_t^{-1}) = 0$, hence

$$\begin{aligned} \mathbb{E}^x(h(X_n(t))Z_t \cdot V_t^{-1}) &= \mathbb{E}^x(h(X_n(t))(Z_0 + Z_t^{(1)}) \cdot V_t^{-1}) \\ &= \mathbb{E}^x(h(X_n(t))Z_0 \cdot V_t^{-1}) = Z_0 \mathbb{E}^{x_n}(h(X_n(t)) \cdot V_t^{-1}). \end{aligned}$$

Finally, for all bounded Borel functions h we have

$$\mathbb{E}^x(h(X_n(t))e^{-\lambda t} e^{i\langle u,\tilde{X}(t)\rangle}) = Z_0 \mathbb{E}^{x_n}\Big(h(X_n(t)) \exp\Big[-\int_0^t q(X_n(s))\,ds\Big]\Big).$$

Approximating τ_{D_a} by bounded stopping times (cf. [Z, p. 177]), we get

$$\mathbb{E}^{x_n}(e^{-\lambda \tau_{D_a}} e^{i\langle u,\tilde{X}(\tau_{D_a})\rangle}) = Z_0 \mathbb{E}^{x_n}\Big(\exp\Big[-\int_0^{\tau_{D_a}} q(X_n(s))\,ds\Big]\Big).$$

If $X_0 = (0, \dots, 0, x_n)$, then obviously $Z_0 = e^{i\langle u, \tilde{X}_0\rangle} = e^0 = 1$. The quantity

$$\phi(x_n) = \mathbb{E}^{x_n}\Big(\exp\Big[-\int_0^{\tau_{D_a}} q(X_n(s))\,ds\Big]\Big)$$

is called the *gauge* ([ChZ]); by the general Feynman–Kac theory for the Schrödinger equation, ϕ is a solution of the appropriate Schrödinger equation, based on the generator of the process $X_n(t)$ and the function q. In the case we consider, the generator of $X_n(t)$ is

$$\mathcal{L}f(x_n) = x_n^2\,\frac{d^2 f(x_n)}{dx_n^2} - (n-2)x_n\,\frac{df(x_n)}{dx_n}$$

and, as we have just found, $q(x_n) = |u|^2 x_n^2 + \lambda$. This means that $\phi(x_n)$ is a solution of

$$x^2\phi''(x) - (n-2)x\phi'(x) - (|u|^2x^2 + \lambda)\phi(x) = 0. \tag{3.15}$$

After substitution $\phi(x) = x^{(n-1)/2}\psi(x)$ we get

$$x^2\psi''(x) + x\psi'(x) - (|u|^2x^2 + \lambda + (n-1)^2/4)\psi(x) = 0.$$

Putting $\nu = (n-1)/2$ and $\mu = \sqrt{((n-1)/2)^2 + \lambda} = \sqrt{\nu^2 + \lambda}$, we get

$$x^2\psi''(x) + x\psi'(x) - (|u|^2x^2 + \mu^2)\psi(x) = 0.$$

After substituting $y = |u|x$ the above equation becomes precisely (2.5), the modified Bessel equation of order μ. Taking into account the general solution of (2.5) we infer that

$$\phi(x_n) = x_n^{\nu}(c_1 I_\mu(|u|x_n) + c_2 K_\mu(|u|x_n)).$$

Because q is non-negative for all $\lambda \ge 0$ and $x_n > 0$, the gauge function $\phi(x_n)$ is bounded for $x_n \to \infty$. The function $I_\mu(|u|x_n)$ is not bounded at infinity, hence $c_1 = 0$. Moreover, if the process starts from the point a, it is instantaneously killed, hence $\phi(a) = 1$ and this implies $c_2 = 1/(a^\nu K_\mu(|u|a))$. Thus

$$\phi(x_n) = \left(\frac{x_n}{a}\right)^{\nu}\frac{K_\mu(|u|x_n)}{K_\mu(|u|a)},$$

which gives the desired conclusion when the process starts from $(0, \dots, 0, x_n)$. If it starts from $(\tilde{x}, x_n)$, then

$$\phi(x_n) = \exp(i\langle u, \tilde{x}\rangle)\left(\frac{x_n}{a}\right)^{\nu}\frac{K_\mu(|u|x_n)}{K_\mu(|u|a)}. \ \blacksquare$$

B. λ-Green function for a half-space. In a similar way one can compute the λ-Green function of D_a. Observe that for every isometry I of $\mathbb{R}^{n-1}$ the function $\mathbb{H}^n \ni x \mapsto (I(\tilde{x}), x_n) \in \mathbb{H}^n$ is an isometry of $\mathbb{H}^n$. This is an easy consequence of the distance formula (3.14). The Laplace–Beltrami operator, the generator of (X_t), is invariant under isometries of $\mathbb{H}^n$. Moreover, τ_{D_a} depends only on X_n. Hence

$$G^\lambda_{D_a}(x,y) = G^\lambda_{D_a}((\tilde{x}+b, x_n), (\tilde{y}+b, y_n)), \qquad b \in \mathbb{R}^{n-1}.$$

This means that it is enough to consider $G^\lambda_{D_a}((\tilde{0}, x_n), y)$. Moreover, the function

$$\tilde{y} \mapsto G^\lambda_{D_a}((\tilde{0}, x_n), (\tilde{y}, y_n))$$

depends only on $|\tilde{y}|$. Consequently, its Fourier transform, taken at u, depends on $r = |u|$:

$$\widehat{(G^\lambda_{D_a})}_r(x_n, y_n) = \int\limits_{\mathbb{R}^{n-1}} \exp(i\langle u, \tilde{y}\rangle) G^\lambda_{D_a}((\tilde{0}, x_n), (\tilde{y}, y_n))\, d\tilde{y}, \qquad u \in \mathbb{R}^{n-1}.$$

We have the following formula for $\widehat{(G^\lambda_{D_a})}_r(x_n, y_n)$.

Theorem 2. *For every $r > 0$, $\lambda > 0$ and $x, y \in D_a$ we get*

$$\widehat{(G^\lambda_{D_a})}_r(x_n, y_n) = \begin{cases} (x_n y_n)^\nu K_\mu(r x_n)\left(I_\mu(r y_n) - K_\mu(r y_n)\dfrac{I_\mu(ra)}{K_\mu(ra)}\right), & x_n \geq y_n, \\ (x_n y_n)^\nu K_\mu(r y_n)\left(I_\mu(r x_n) - K_\mu(r x_n)\dfrac{I_\mu(ra)}{K_\mu(ra)}\right), & y_n > x_n, \end{cases}$$

where $\nu = (n-1)/2$ and $\mu = (\nu^2 + \lambda)^{1/2}$.

Proof. Being $(\mathcal{L} - q)$-harmonic, the Green function must satisfy equation (3.15). Moreover, from the general theory of the Feynman–Kac equation (see [ChZ]) and diffusions on the real line (see [BS]), we get

$$\widehat{(G^\lambda_{D_a})}_1(x_n, y_n) = \begin{cases} c \cdot \psi(x_n)\phi(y_n), & a < y_n \leq x_n, \\ c \cdot \phi(x_n)\psi(y_n), & a < x_n < y_n. \end{cases}$$

Here the functions ψ, ϕ are defined (up to a multiplicative constant) as solutions of the equation (3.15) such that $\psi(x_n)$ is decreasing, $\phi(y_n)$ is increasing and they satisfy the boundary conditions $\lim_{x_n \to \infty} \psi(x_n) = 0$, $\lim_{y_n \to a} \phi(y_n) = 0$. The constant c is given by the Wronskian of the pair (ψ, ϕ), the density of the speed measure $m(dx)$ and the function $a(x)$. Let $u(w) = w^\nu g(w)$, where $\nu = (n-1)/2$. We get

$$w^2 g''(w) + w g'(w) - (w^2 + \mu^2) g(w) = 0, \qquad \mu = (\nu^2 + \lambda)^{1/2}.$$

This is precisely (2.5), the modified Bessel equation of order μ. Taking into account the general solution of (3.15), we infer that

$$\psi(x_n) = x_n^\nu(c_1 I_\mu(x_n) + c_2 K_\mu(x_n)), \qquad \phi(y_n) = y_n^\nu(c_3 I_\mu(y_n) + c_4 K_\mu(y_n)).$$

From the boundary conditions we get $c_1 = 0$ and $c_4 = I_\mu(a)/K_\mu(a)$. Thus

$$\psi(x_n) = x_n^\nu K_\mu(x_n),$$
$$\phi(y_n) = y_n^\nu\bigg(I_\mu(y_n) - K_\mu(y_n)\,\frac{I_\mu(a)}{K_\mu(a)}\bigg).$$

Using (2.6) we calculate the Wronskian of the pair (ψ,ϕ):

$$\begin{aligned} W(\psi,\phi)(x) &= \psi(x)\phi'(x) - \psi'(x)\phi(x) \\ &= -x^{n-1}(K'_\mu(x)I_\mu(x) - K_\mu(x)I'_\mu(x)) = x^{n-2}. \end{aligned}$$

The density $m(x)$ of the speed measure $m(dx)$ for the diffusion with the generator $\mathcal{L}f(x_n) = x_n^2 f''(x_n) - (n-2)x_n f'(x_n)$ is given by the functions $a(x)$ and $b(x)$ (formula (2.8)). Thus

$$m(x) = x^{-2}e^{-(n-2)\log x} = x^{-n}.$$

The constant c is given by $c = \frac{1}{2}a^2(x)W(\psi,\phi)(x)m(x) = 1$, which ends the proof. ∎

Remark 3. The Fourier transform of the function $\tilde{y} \mapsto G^\lambda_{D_a}(x,(\tilde{y},y_n))$ for a general point $x \in D_a$ is given by

$$\widehat{(G^\lambda_{D_a})}_u(x,y_n) = \exp(i\langle u,\tilde{x}\rangle)\widehat{(G^\lambda_{D_a})}_{|u|}((\tilde{0},x_n),y_n).$$

4. Harmonic measures and Green functions of balls in real hyperbolic spaces. In [BM] the Poisson kernels for balls were computed. Now we will show how one can modify those computations to get formulas for the λ-Poisson kernels and λ-Green functions.

By $\mathbb{D}^n$ we denote the ball model of n-dimensional hyperbolic space, that is, the unit ball in $\mathbb{R}^n$ equipped with the metric $d_{\mathbb{D}^n}(x,y)$ given by

$$\cosh(2d_{\mathbb{D}^n}(x,y)) = 1 + \frac{2|x-y|^2}{(1-|x|^2)(1-|y|^2)}, \qquad |x|<1,\ |y|<1.$$

The hyperbolic volume element in $\mathbb{D}^n$ is given by

$$dV_{\mathbb{D}^n}(x) = \frac{dx_1\ldots dx_n}{(1-|x|^2)^n}. \tag{4.16}$$

Let (X_t) be the hyperbolic Brownian motion in $\mathbb{D}^n$, that is, the diffusion generated by the Laplace–Beltrami operator $\Delta_{\mathbb{D}^n}$:

$$\Delta_{\mathbb{D}^n} = (1-|x|^2)^2\sum_{k=1}^n \frac{\partial^2}{\partial x_k^2} + 2(n-2)(1-|x|^2)\sum_{k=1}^n x_k\,\frac{\partial}{\partial x_k}.$$

As shown in [BM], the process (X_t) can be decomposed into radial and spherical parts: $R_t = |X_t|^2$, $\Phi_t = \cos\angle(X_0, X_t) = \frac{\langle X_t, X_0\rangle}{|X_t|\,|X_0|}$, where

$$\begin{cases} dR_t = 2(1-R_t)(\sqrt{R_t}\, dW_1(t) + ((n-4)R_t + n)dt), \\ d\Phi_t = (1-R_t)\Big(\dfrac{1-\Phi_t^2}{R_t}\Big)^{1/2} dW_2(t) - (n-1)\Big(\dfrac{(1-R_t)^2}{R_t}\Big)\Phi_t\, dt. \end{cases}$$

Here $W_1(t)$, $W_2(t)$ denote two independent one-dimensional Brownian motions such that $\mathbb{E}^0 W_i^2(t) = 2t$, $i = 1, 2$.

A. λ-Poisson kernel of a ball. Our task is to compute the λ-Poisson kernel of a ball $B_r = \{x : |x| < r\}$. To do this, we define $\tau_r = \inf\{t > 0 : X_t \notin B_r\}$ and compute the density of the measure $\mathbb{E}^x(e^{-\lambda\tau_{B_r}} 1_A(X_{\tau_{B_r}}))$, defined in (2.12).

When working on the unit sphere in $\mathbb{R}^n$, it is natural to consider the family of Gegenbauer polynomials $(C_k^\rho(x))_{k=0}^\infty$, where $\rho = (n-2)/2$. This family is an orthogonal basis in $L_2((-1,1), (1-x^2)^{(n-3)/2}dx)$, hence the coefficients of any function $f \in L_2((-1,1), (1-x^2)^{(n-3)/2}dx)$ with respect to this basis determine f uniquely.

We recall the orthogonality relations for Gegenbauer polynomials:

$$\int_{-1}^{1} C_k^\rho(x) C_l^\rho(x)(1-x^2)^{(n-3)/2}\, dx = \delta_{kl}\, \frac{\rho}{k+\rho} \cdot C_k^{(\rho)}(1) \cdot \frac{\omega_{n-1}}{\omega_{n-2}}, \tag{4.17}$$

where $C_k^\rho(1) = \Gamma(k+2\rho)/(k!\,\Gamma(2\rho))$ and $\omega_{n-1} = 2\pi^{n/2}/\Gamma(n/2)$ is the total mass of the associated $(n-1)$-dimensional spherical measure on the unit sphere in $\mathbb{R}^n$.

Consider the family of processes $Y_k(t) = C_k^\rho(\Phi_t)$. Our task is to compute $\mathbb{E}(e^{-\lambda\tau_r} C_k^\rho(\Phi_{\tau_r}))$, $k = 0, 1, 2, \ldots$. In order to do it, we examine the family of processes

$$Z_k(t) = C_k^\rho(\Phi_t)V_t = C_k^\rho(\Phi_t)\exp\Big(\int_0^t q(R_s)\, ds\Big)$$

and try to find q that makes Z_t a martingale. From the Itô formula we get

$$dZ_k(t) = \frac{1-R_t}{R_t^{1/2}}\,(1-\Phi_t^2)^{1/2} V_t (C_k^\rho)'(\Phi_t) dW_2(t) + \frac{(1-R_t)^2}{R_t}\, H(R_t, \Phi_t)dt,$$

where $H(R_t, \Phi_t)$ is equal to

$$(1-\Phi_t^2)(C_k^\rho)''(\Phi_t) - (n-1)\Phi_t (C_k^\rho)'(\Phi_t) + \frac{R_t}{(1-R_t)^2}\, q(R_t) C_k^\rho(\Phi_t).$$

The Gegenbauer polynomials satisfy the equation

$$(1-x^2)y''(x) - (n-1)xy'(x) + k(k+n-2)y(x) = 0,$$

and consequently the non-martingale part $H(R_t,\Phi_t)$ is equal to zero for all t whenever $q(R_t)=\frac{(1-R_t)^2}{R_t}k(k+n-2)$. Thus

$$Z_t = Z_0 + \int_0^t (C_k^\rho)'(\Phi_s)V_s(1-R_s)\left(\frac{1-\Phi_s^2}{R_s}\right)^{1/2} dW_2(s) = Z_0 + Z_t^{(1)}.$$

We start to evaluate the quantity $\mathbb{E}(e^{-\lambda\tau_D}C_k^\rho(\Phi_{\tau_{B_r}}))$ by computing the value of $\mathbb{E}(e^{-\lambda t}C_k^\rho(\Phi_t))$. We obtain

$$\begin{aligned}\mathbb{E}(e^{-\lambda t}C_k^\rho(\Phi_t)) &= \mathbb{E}(C_k^n(\Phi_t))V_t\cdot V_t^{-1}e^{-\lambda t}) = \mathbb{E}((Z_0+Z_t^{(1)})\cdot V_t^{-1}e^{-\lambda t})\\ &= \mathbb{E}(Z_0V_t^{-1}e^{-\lambda t}) + \mathbb{E}(Z_t^{(1)}V_t^{-1}e^{-\lambda t}) = \mathbb{E}(Z_0V_t^{-1}e^{-\lambda t}),\end{aligned}$$

because $Z_t^{(1)}$ is a stochastic integral with respect to $W_2(t)$, it has expectation zero, and the process V_t^{-1} depends only on $W_1(t)$. This means that we have to evaluate

$$\varphi(x) = \mathbb{E}^x(Z_0V_{\tau_{B_r}}^{-1}e^{-\lambda\tau_{B_r}}) = Z_0\cdot\mathbb{E}^x\Big(\exp\Big[\int_0^{\tau_{B_r}}(-q(R_s)-\lambda)\,ds\Big]\Big),$$

where $Z_0=C_k^n(1)$. But this is the gauge for the generator of the process (R_t) and potential $-(q+\lambda)$. From the general theory ([ChZ]), $\varphi(x)$ is a solution of the appropriate Schrödinger equation, based on the generator of the process (R_t) and the function $-(q+\lambda)$. In this case the generator of (R_t) is

$$\mathcal{L} = 4(1-x)^2x\,\frac{d^2}{dx^2} + 2(1-x)((n-4)x+n)\,\frac{d}{dx}$$

and, as we have found, $q(x)=\frac{(1-x)^2}{x}k(k+n-2)$. Observe that $x=R_t=|X_t|^2\in[0,1)$. This means that $\varphi(x)$ is a solution of the differential equation

$$4(1-x)^2xy''(x)+2(1-x)((n-4)x+n)y'(x)-(q(x)+\lambda)y(x)=0. \tag{4.18}$$

Making the substitution $\varphi(x)=(1-x)^{(n-1)/2-\mu/2}x^{k/2}f(x)$ in (4.18), where $\mu=\sqrt{(n-1)^2+\lambda}$, we get

$$\begin{aligned}(1-x)xf''(x)+\Big[k+\frac{n}{2}-x\Big(\frac{n}{2}+1-\mu+k\Big)\Big]f'(x)&\\ -\frac{1-\mu}{2}\Big(k+\frac{n-1}{2}-\frac{\mu}{2}\Big)f(x)&=0.\end{aligned}$$

This is the hypergeometric equation with coefficients $\alpha=k+(n-1)/2-\mu/2$, $\beta=(1-\mu)/2$ and $\gamma=k+n/2$. The general solution of the equation is

$$c_1\cdot F_{n,k,\lambda}(x)+c_2\cdot G_{n,k,\lambda}(x),$$

where

$$F_{n,k,\lambda}(x) = {}_2F_1\Big(k+\frac{n-1}{2}-\frac{\mu}{2},\frac{1-\mu}{2};k+\frac{n}{2};x\Big).$$

The definition of $G_{n,k,\lambda}$ is more complicated. If $n \in 2\mathbb{N}+1$ or $n \in 2\mathbb{N}$ and $\mu = n-1$ and $k > 0$ then

$$G_{n,k,\lambda}(x) = x^{1-k-n/2}\,{}_2F_1\left(\frac{1-\mu}{2}, \frac{3}{2} - \frac{1}{2}\,n - \frac{1}{2}\,\mu - k; 2-k-\frac{n}{2}; x\right),$$

and if $n \in 2\mathbb{N}$ and $\mu = n-1$, $k = 0$ we have

$$G_{n,k,\lambda}(x) = \frac{n-2}{2} \sum_{i=0,\, i\neq(n-2)/2}^{n-2} \binom{n-2}{i} \frac{(-1)^{i+1}}{i-(n-2)/2}\, x^{i+1-1/n} + \frac{n-2}{2} \binom{n-2}{(n-2)/2} (-1)^{n/2} \log x.$$

In the case $n \in 2\mathbb{N}$ and μ is not an integer we have

$$G_{n,k,\lambda}(x) = {}_2F_1\left(k + \frac{n-1}{2} - \frac{\mu}{2}, \frac{1-\mu}{2}; 1-\mu; 1-x\right),$$

and if $n \in 2\mathbb{N}$ and $\mu > n-1$ and μ is an integer then

$$G_{n,k,\lambda}(x) = (1-x)^{\mu}\,{}_2F_1\left(\frac{1+\mu}{2}, k + \frac{n-1}{2} + \frac{1}{2}\mu; 1+\mu; 1-x\right).$$

In all cases the function $F_{n,k,\lambda}$ is bounded at 0 and $\lim_{x\to 0^+} x^{k/2} G_{n,k,\lambda}(x) = \infty$. We are looking for a solution φ that is bounded in the neighborhood of $x = 0$, so that

$$\varphi(x) = c_1 (1-x)^{(n-1)/2-\mu/2} x^{k/2} F_{n,k,\lambda}(x).$$

Moreover, if the process starts from the boundary of B_r (i.e. if (R_t) starts from r^2), we have $\tau_{B_r} = 0$, hence $\phi(r^2) = 1$, which implies

$$c_1 = \frac{1}{(1-r^2)^{(n-1)/2-\mu/2} r^k F_{n,k,\lambda}(r^2)}.$$

Finally, if the process starts from the point x inside B_r, then the number x in the above calculation is equal to $|x|^2$, so that

$$\mathbb{E}^x(e^{-\lambda\tau_{B_r}} C_k^{\rho}(\Phi_{\tau_{B_r}})) = C_k^{\rho}(1)\, \frac{(1-|x|^2)^{(n-1)/2-\mu/2} |x|^k F_{n,k,\lambda}(|x|^2)}{(1-r^2)^{(n-1)/2-\mu/2} r^k F_{n,k,\lambda}(r^2)}.$$

We have just proved the following representation for the λ-Poisson kernel of a ball in $\mathbb{D}^n$:

THEOREM 4 (λ-Poisson kernel for a ball in $\mathbb{D}^n$). *For every $x \in B_r$ and $y \in \partial B_r$ denote by θ the angle between x and y. Then*

$$P_{B_r}^{\lambda}(x,y) = C \sum_{k=0}^{\infty} \frac{k+\rho}{\rho} \left(\frac{1-|x|^2}{1-r^2}\right)^{(n-1)/2-\mu/2} \frac{|x|^k}{r^k}\, \frac{F_{n,k,\lambda}(|x|^2)}{F_{n,k,\lambda}(r^2)}\, C_k^{\rho}(\cos\theta),$$

where $C = \Gamma(n/2)/(2\pi^{n/2} r^{n-1})$.

Proof. It is easy to see that both sides have the same Gegenbauer coefficients, hence they are equal. The constant $C(k+\rho)/\rho$ appearing in the above formula is a consequence of the orthogonality relation (4.17) for the Gegenbauer polynomials. The proof of the convergence of the above series is precisely the same as the proof of the analogous result for the Poisson kernel in [BM] and is omitted. ∎

REMARK 5. If we put $\lambda = 0$, we get the formula of [BM].

B. λ-Green function of a ball. Let $(X_t^{B_r}, P_t^{B_r})$ be the hyperbolic Brownian motion in $\mathbb{D}^n$ killed at the boundary of the ball B_r. We want to find a formula for the λ-Green function of B_r, defined in (2.13).

We introduce the following definition. If $n \in 2\mathbb{N}+1$ or $n \in 2\mathbb{N}$ and $\mu = n-1$, then

$$a_{n,k,\lambda} = \frac{\Gamma((n-2)/2)}{4\pi^{n/2}};$$

if $n \in 2\mathbb{N}$, $\mu > n-1$ and μ is not an integer then

$$a_{n,k,\lambda} = -\frac{\Gamma(k+(n-1)/2-\mu/2)\Gamma(1/2-\mu/2)}{\Gamma(1-\mu)\Gamma(k+n/2-1)}\,\frac{\Gamma((n-2)/2)}{4\pi^{n/2}};$$

and finally for $n \in 2\mathbb{N}$, $\mu > n-1$ and μ an integer we put

$$a_{n,k,\lambda} = -\frac{\Gamma(k+(n-1)/2+\mu/2)\Gamma(1/2+\mu/2)}{\Gamma(1+\mu)\Gamma(k+n/2-1)}\,\frac{\Gamma((n-2)/2)}{4\pi^{n/2}}.$$

Due to the symmetry of the λ-Green function (with respect to the hyperbolic volume element (4.16)) it is enough to find the formula in the case $|x| < |y|$.

THEOREM 6 (λ-Green function formula). *For $|x| < |y| < r$,*

$$G_{B_r}^{\lambda}(x,y) = \sum_{k=0}^{\infty} a_{n,k,\lambda}\varphi_{n,k,\lambda}(|x|^2)\psi_{n,k,\lambda}(|y|^2)\, C_k^{\rho}(\cos\theta). \tag{4.19}$$

where

$$\varphi_{n,k,\lambda}(z) = (1-z)^{(n-1)/2-\mu/2} z^{k/2} F_{n,k,\lambda}(z),$$
$$\psi_{n,k,\lambda}(z) = (1-z)^{(n-1)/2-\mu/2} z^{k/2}\left(G_{n,k,\lambda}(z) - F_{n,k,\lambda}(z)\,\frac{G_{n,k,\lambda}(r^2)}{F_{n,k,\lambda}(r^2)}\right)$$

and θ denotes the angle between x and y.

Proof. Similar arguments to those in the case of the λ-Poisson kernel show that the λ-Green function $G_{B_r}^{\lambda}(x,\cdot)$ as a function on the sphere $S_{|y|}$ (with radius $|y|$) is invariant under each orthogonal transformation U such that $Ux = x$ (see [BM]). Consequently, it is enough to find the values of

$$\widehat{(G^{\lambda}_{B_r})}_k(x,|y|) = \frac{1}{C^{\rho}_k(1)}\,\frac{1}{\omega_{n-1}|y|^{n-1}}\int\limits_{S_{|y|}} C^{\rho}_k(\cos\theta)G^{\lambda}_{B_r}(x,y)\,d\sigma_{|y|}(y), \qquad x \neq 0,$$

for all $k = 0,1,\ldots$. Observe that the function $x \mapsto \widehat{(G^{\lambda}_{B_r})}_k(x,|y|)$ depends only on $|x|$. Exactly the same arguments as in [BM] show that

$$\widehat{(G^{\lambda}_{B_r})}_k(|x|,|y|) = \frac{\Gamma(n/2)}{\pi^{n/2}|y|^{n-2}}\,G^{\lambda}_r(|x|^2,|y|^2),$$

where $G^{\lambda}_r(\eta,\xi)$ is the λ-Green function for the interval $[0,r^2]$ and the one-dimensional diffusion with generator

$$\mathcal{L}f(x) = \frac{1}{2}\,a^2(x)\,\frac{d^2f}{dx^2}(x) + b(x)\,\frac{df}{dx}(x) - c(x)f(x),$$

where $a(x) = 2(1-x)\sqrt{2x}$, $b(x) = 2(1-x)((n-4)x+n)$ and $c(x) = k(k+n-2)(1-x)^2/x$. The speed measure density, scale function and killing measure density are given by

$$\begin{aligned}
m(x) &= \frac{1}{4x(1-x)^2}\,\frac{x^{n/2}}{(1-x)^{n-2}} = \frac{1}{4}\,\frac{x^{n/2-1}}{(1-x)^n},\\
s'(x) &= \frac{(1-x)^{n-2}}{x^{n/2}},\\
k(x) &= \frac{1}{4}\left(k(k+n-2)\,\frac{(1-x)^2}{x}\right)\frac{x^{n/2-1}}{(1-x)^n}.
\end{aligned}$$

Using (2.11) we get

$$G^{\lambda}_r(\eta,\xi) = C_{n,k,\lambda}\cdot\varphi_{n,k,\lambda}(\eta)\psi_{n,k,\lambda}(\xi), \qquad 0<\eta<\xi<r^2,$$

where $\varphi_{n,k,\lambda}$ and $\psi_{n,k,\lambda}$ are positive solutions of the equation $\mathcal{L}u = \lambda u$ such that $\lim_{\eta\to 0}\varphi_{n,k,\lambda}(\eta) = 0$ and $\lim_{\xi\to r^2}\psi_{n,k,\lambda}(\xi) = 0$ (the boundary conditions are consequences of the character of the boundary points for the diffusion). The constant $C_{n,k,\lambda}$ is given by

$$C_{n,k,\lambda} = \frac{s'(z)}{W(\varphi_{n,k,\lambda},\psi_{n,k,\lambda})(z)},$$

where $W(\varphi_{n,k,\lambda},\psi_{n,k,\lambda})$ is the Wronskian of the pair $(\varphi_{n,k,\lambda},\psi_{n,k,\lambda})$.

The equation $\mathcal{L}u = \lambda u$ is just (4.18) and taking into account its general solution, we find that the functions

$$\begin{aligned}
\varphi_{n,k,\lambda}(\eta) &= (1-\eta)^{(n-1)/2-\mu/2}\eta^{k/2}F_{n,k,\lambda}(\eta),\\
\psi_{n,k,\lambda}(\xi) &= (1-\xi)^{(n-1)/2-\mu/2}\xi^{k/2}\left(G_{n,k,\lambda}(\xi) - F_{n,k,\lambda}(\xi)\,\frac{G_{n,k,\lambda}(r^2)}{F_{n,k,\lambda}(r^2)}\right)
\end{aligned}$$

are the solutions which satisfy the boundary conditions at 0 and r^2. From the general formula for the Wronskian for solutions of the hypergeometric equation we get

$$\begin{aligned}W(\varphi_{n,k,\lambda},\psi_{n,k,\lambda})(z) &= z^k(1-z)^{n-1-\mu}W(F_{n,k,\lambda},G_{n,k,\lambda})(z)\\ &= z^k(1-z)^{n-1-\mu}z^{-k-n/2}(1-z)^{\mu-1}\,\frac{k+\rho}{\rho}\,\frac{\Gamma(n/2)}{4\pi^{n/2}a_{n,k,\lambda}}\\ &= s'(z)\,\frac{k+\rho}{\rho}\,\frac{\Gamma(n/2)}{4\pi^{n/2}a_{n,k,\lambda}}.\end{aligned}$$

Finally, comparing the formulas for the volume element $dV_{\mathbb{D}^n}$ and the density of the speed measure $m(\xi)$, we arrive at

$$\widehat{(G^{\lambda}_{B_r})}_k(|x|,|y|) = \frac{\rho}{k+\rho}\,a_{n,k,\lambda}\varphi_{n,k,\lambda}(|x|^2)\psi_{n,k,\lambda}(|y|^2).$$

The formula (4.19) is now an easy consequence of the orthogonality relation (4.17) for Gegenbauer polynomials. The proof that the series is convergent is very similar to that in [BM] and is omitted. ∎

5. Harmonic measures and Green functions of balls in complex hyperbolic spaces. Using a similar method to the one in the preceding section, we can compute formulas for the λ-Poisson kernel and the λ-Green function of a ball in complex hyperbolic space. For $\lambda = 0$ it was done in [Z].

Consider $\mathbb{C}^n$ with the Hermitian product $\langle z,w\rangle = \sum_{j=1}^n z_j\bar{w}_j$ for $z,w\in\mathbb{C}^n$. Let B_1 be the unit ball in $\mathbb{C}^n$ equipped with the Bergman metric, induced by the form $h = -4\partial\bar{\partial}\log K(z)$ with $K(z) = 1-|z|^2$. This means that the metric is given by the matrix (h_{ij}), where

$$h_{ij} = \frac{(1-|z|^2)\delta_{ij} + \bar{z}_i z_j}{(1-|z|^2)^2} \qquad \text{for } i,j = 1,\ldots,n.$$

The unit ball for this metric is a model of complex hyperbolic space. The volume element is given by $dV_{\mathbb{C}^n}(z) = (1-|z|^2)^{-n-1}\,dz$ and $\Delta_{\mathbb{C}^n}$, the Laplace–Beltrami operator in this space, is (see [R])

$$\Delta_{\mathbb{C}^n} f(z) = 4(1-|z|^2)\sum_{j,k=1}^n(\delta_{j,k} - z_j\bar{z}_k)\,\frac{\partial^2 f(z)}{\partial z_j\partial\bar{z}_k}.$$

Denote by (X_t) the process generated by $\Delta_{\mathbb{C}^n}$. It is called the *complex hyperbolic Brownian motion.*

A. λ-Poisson kernel of a ball. In order to compute the Poisson kernel or the Green function, we decompose (X_t) (as in [Z]) into the radial part $r_t = |X_t|^2$ and "unitary spherical" part $Y_t = \langle X_t, X_0\rangle/(|X_t|\,|X_0|) = R_t e^{i\theta_t}$. Now for $0<r<1$ take $B_r = \{z\in\mathbb{C}^n : |z|<r\}$ and put $\tau_r = \inf\{t>0 : X_t\notin B_r\}$.

We are looking for $\mathbb{E}^x(e^{-\lambda\tau_r}X_{\tau_r})$. In this context it is convenient to consider the family of processes $(H_n^{p,q}(R_t e^{i\theta_t}))_{p,q=0,1,\ldots}$, where $(H_n^{p,q})$, $p,q = 0,1,\ldots$, is the family of unitary spherical harmonics, described by Koornwinder in [K] (cf. also [F]). This family forms an orthogonal basis in $L^2(\{x^2+y^2<1\},(1-x^2-y^2)^{n-2}dx\,dy)$.

The structure of unitary spherical harmonics is rather complicated, hence the computations we need would be very long. Therefore we will use the computations and results from [Z]: if we put

$$Q(x) = 4(1-x)\left[\frac{\frac{p+q}{2}\left(\frac{p+q}{2}+n-1\right)}{x} - \left(\frac{p-q}{2}\right)^2\right],$$

then the process

$$Z_t = H_n^{p,q}\left(\frac{\langle X_t, X_0\rangle}{|X_t|\,|X_0|}\right)\exp\Big(\int_0^t Q(r_s)\,ds\Big)$$

is a martingale. The generator of the process $r_t = |X_t|^2$ is

$$\mathcal{L} = 4x(1-x)^2\,\frac{d^2}{dx^2} + 4(1-x)(n-x)\,\frac{d}{dx}.$$

Now we compute the expectation just as for $\mathbb{D}^n$ in the previous section:

$$\begin{aligned}\mathbb{E}^x\left(e^{-\lambda t}H_n^{p,q}\left(\frac{\langle X_t, X_0\rangle}{|X_t|\,|X_0|}\right)\right) &= \mathbb{E}^x\Big(e^{-\lambda t}Z_t\exp\Big(-\int_0^t Q(r_s)\,ds\Big)\Big)\\ &= Z_0\mathbb{E}^x\Big(\exp\Big(-\int_0^t (Q(r_s)+\lambda)\,ds\Big)\Big).\end{aligned}$$

We see that this time the gauge

$$\phi(x) = \mathbb{E}^x\Big(\exp\Big(-\int_0^{\tau_r}(Q(r_s)+\lambda)\,ds\Big)\Big)$$

satisfies the equation

$$4x(1-x)^2\phi''(x) + 4(1-x)(n-x)\phi'(x) - (Q(x)+\lambda)\phi(x) = 0. \tag{5.20}$$

Making the substitution $\phi(x) = (1-x)^{n/2-\mu/2}x^{(p+q)/2}y(x)$, where $\mu = \sqrt{n^2+\lambda}$, we get

$$\begin{aligned}x(1-x)y''(x) &+ (n+p+q-(n+p+q+1-\mu)x)y'(x)\\ &+ \frac{1}{4}[2(n+p+q)\mu - 2(n+p+q)n - 4pq - \lambda]y(x) = 0.\end{aligned}$$

This is the hypergeometric equation (2.1) with $\alpha = p+(n-\mu)/2$, $\beta = q+(n-\mu)/2$ and $\gamma = n+p+q$. Here the roles of the coefficients p and q are symmetric, hence we always assume that $p \le q$. The general solution can be written ([E, 2.2.2 and 2.3.1]) as

$$c_1 \cdot F_{n,p,q,\lambda}(x) + c_2 \cdot G_{n,p,q,\lambda}(x),$$

where

$$F_{n,p,q,\lambda}(x) = {}_2F_1\left(p + \frac{n-\mu}{2}, q + \frac{n-\mu}{2}; n+p+q; x\right).$$

The definition of $G_{n,p,q,\lambda}(x)$ is more complicated: if $(n-\mu)/2 \notin \mathbb{Z}$ and $1-\mu \neq 0, -1, -2, \ldots$ then ([E, p. 75, formula (7)])

$$G_{n,p,q,\lambda}(x) = {}_2F_1\left(p + \frac{n-\mu}{2}, q + \frac{n-\mu}{2}; 1-\mu; 1-x\right);$$

if $(n-\mu)/2 \notin \mathbb{Z}$ and $1-\mu = 0, -1, -2, \ldots$ then ([E, p. 75, formula (8)])

$$G_{n,p,q,\lambda}(x) = (1-x)^{\mu}\, {}_2F_1\left(p + \frac{n+\mu}{2}, q + \frac{n+\mu}{2}; 1+\mu; 1-x\right);$$

if $(n-\mu)/2 \in \mathbb{Z}$ and $p + (n-\mu)/2 = p-1, \ldots, 1$ then ([E, p. 72, case 20])

$$G_{n,p,q,\lambda}(x) = (-x)^{-q-(n-\mu)/2}\, {}_2F_1\left(q + \frac{n-\mu}{2}, 1-p-\frac{n+\mu}{2}; q-p+1; x^{-1}\right);$$

and finally if $(n-\mu)/2 \in \mathbb{Z}$ and $p + (n-\mu)/2 = 0, -1, \ldots$ then ([E, p. 73, case 23])

$$G_{n,p,q,\lambda}(x) = (1-x)^{\mu}\, {}_2F_1\left(q + \frac{n+\mu}{2}, p + \frac{n+\mu}{2}; 1+\mu; 1-x\right),$$

which is the same formula as in the second case.

In all cases $F_{n,p,q,\lambda}$ is bounded at 0 and $\lim_{x\to 0^+} x^{(p+q)/2} G_{n,p,q,\lambda}(x) = \infty$. Investigating the Poisson kernel, we are looking for a solution ϕ that is bounded in the neighborhood of $x = 0$, so that

$$\phi(x) = c_1 x^{(p+q)/2}\,(1-x)^{\frac{n-\sqrt{n^2+\lambda}}{2}} F_{n,p,q,\lambda}(x).$$

Condition $\phi(r) = 1$ gives c_1 and we finally get, for the process (X_t) with $X_0 = x$,

$$\mathbb{E}^x\left(e^{-\lambda\tau_r} H_n^{p,q}\left(\frac{\langle X_{\tau_r}, X_0\rangle}{|X_{\tau_r}|\,|X_0|}\right)\right) = H_n^{p,q}(1)\left(\frac{|x|}{r}\right)^{p+q}\left(\frac{1-|x|^2}{1-r^2}\right)^{\frac{n-\sqrt{n^2+\lambda}}{2}} \frac{F_{n,p,q,\lambda}(|x|^2)}{F_{n,p,q,\lambda}(r^2)}.$$

Knowing the coefficients with respect to the orthogonal basis $(H_n^{p,q})$, we can write down the series expansion of the function.

THEOREM 7 (λ-Poisson kernel for a ball in complex hyperbolic space). *For every $x \in B_r$, $x \neq 0$, and $y \in \partial B_r$,*

$$P^{\lambda}_{B_r}(x,y) = \frac{1}{r^{2n-1}} \sum_{p,q=0}^{\infty} \left(\frac{|x|}{r}\right)^{p+q} \left(\frac{1-|x|^2}{1-r^2}\right)^{\frac{n-\sqrt{n^2+\lambda}}{2}} \frac{F_{p,q,n}(|x|^2)}{F_{p,q,n}(r^2)}\, H^{p,q}_n\left(\frac{\langle y,x\rangle}{|y|\,|x|}\right).$$

The proof of the convergence of the above series is precisely the same as the proof of the analogous result for the Poisson kernel in [Z] and is omitted.

REMARK 8. 1. If $x = 0$ then, by the unitary invariance of the process, the variable X_{τ_r} is uniformly distributed on the sphere ∂B_r.

2. If we put $\lambda = 0$ in the above formula, we get the result of [Z].

B. λ-Green function of a ball. We continue the calculations from the previous section, using the analogous method to compute the λ-Green function in $\mathbb{D}^n$. This time we compute the coefficients of the expansion of the λ-Green function with respect to the orthogonal basis, consisting of $(H^n_{p,q})$.

Recall that $\mu = \sqrt{n^2+\lambda}$. Denote by $\omega_{2n-1} = 2\pi^n/\Gamma(n)$ the area of the unit sphere in $\mathbb{R}^{2n}$ and define $b_{n,p,q,\lambda}$ to be

$$\begin{cases} -\dfrac{\Gamma\big(p+\frac{n-\mu}{2}\big)\Gamma\big(q+\frac{n-\mu}{2}\big)}{\omega_{2n-1}\Gamma(1-\mu)\Gamma(n+p+q)} & \text{if } \frac{n-\mu}{2}\notin\mathbb{Z} \text{ and } 1-\mu\neq 0,-1,\ldots,\\[2ex] -\dfrac{\Gamma\big(q+\frac{n+\mu}{2}\big)\Gamma\big(p+\frac{n+\mu}{2}\big)}{\omega_{2n-1}\Gamma(1+\mu)\Gamma(n+p+q)} & \begin{array}{l}\text{if } \frac{n-\mu}{2}\notin\mathbb{Z} \text{ and } 1-\mu=0,-1,\ldots,\\ \text{or } \frac{n-\mu}{2}\in\mathbb{Z} \text{ and } \frac{2p+n-\mu}{2}=-1,-2,\ldots,\end{array}\\[2ex] \dfrac{(-1)^{n+p+q}\Gamma\big(q+\frac{n-\mu}{2}\big)\Gamma\big(q+\frac{n+\mu}{2}\big)}{\omega_{2n-1}\Gamma(1+q-p)\Gamma(n+p+q)} & \text{if } \frac{n-\mu}{2}\in\mathbb{Z} \text{ and } \frac{2p+n-\mu}{2}=p-1,\ldots,0. \end{cases}$$

We have the following formula.

THEOREM 9 (λ-Green function formula for B_r). *For $0<|x|<|y|<r$,*

$$G^{\lambda}_{B_r}(x,y) = \frac{1}{2r^{2n-1}} \sum_{p,q=0}^{\infty} b_{n,p,q,\lambda}\varphi_{n,p,q,\lambda}(|x|)\psi_{n,p,q,\lambda}(|y|)H^{p,q}_n\left(\frac{\langle x,y\rangle}{|x|\,|y|}\right),$$

where

$$\begin{aligned} \varphi_{n,p,q,\lambda}(|x|) &= (1-|x|^2)^{(n-\mu)/2}|x|^{p+q}F_{n,p,q,\lambda}(|x|^2),\\ \psi_{n,p,q,\lambda}(|y|) &= (1-|y|^2)^{(n-\mu)/2}|y|^{p+q}\\ &\qquad\times\left(G_{n,p,q,\lambda}(|y|^2) - F_{n,p,q,\lambda}(|y|^2)\,\frac{G_{n,p,q,\lambda}(r^2)}{F_{n,p,q,\lambda}(r^2)}\right). \end{aligned}$$

Proof. The Laplace–Beltrami operator $\Delta_{\mathbb{C}^n}$ commutes with unitary transformations, hence the complex hyperbolic Brownian motion generated by $\Delta_{\mathbb{C}^n}$ and starting at x is invariant under each unitary transformation U such that $Ux = x$ (see [Z]). This implies that the λ-Green function $G^{\lambda}_{B_r}(x,\cdot)$, as a function on the sphere $S_{|y|}$ with radius $|y|$, is invariant under all unitary transformations U with $Ux = x$.

Let $(\widehat{G^\lambda_{B_r}})_{p,q}$ denote the coefficient of the expansion of $G^\lambda_{B_r}$ with respect to the basis $(H^n_{p,q})$. We will compute the values of

$$\widehat{(G^\lambda_{B_r})}_{p,q}(x,|y|) = \frac{H^{p,q}_n(1)}{\omega_{2n-1}|y|^{2n-1}} \int\limits_{S_{|y|}} H^{p,q}_n\left(\frac{\langle x,y\rangle}{|x|\,|y|}\right) G^\lambda_{B_r}(x,y)\,d\sigma_{|y|}(y), \qquad x \neq 0,$$

for all $p,q = 0,1,\dots$. Observe that the function $x \mapsto \widehat{(G^\lambda_{B_r})}_{p,q}(x,|y|)$ depends only on $|x|$. Exactly the same arguments as in [Z] show that

$$\widehat{(G^\lambda_{B_r})}_{p,q}(|x|,|y|) = \frac{1}{\omega_{2n-1}|y|^{2n-1}}\, G^\lambda_r(|x|^2,|y|^2),$$

where G^λ_r is the λ-Green function for the interval $[0,r]$ of the one-dimensional diffusion with generator

$$\mathcal{L}f(x) = \frac{1}{2}\,a^2(x)\,\frac{d^2 f}{dx^2}(x) + b(x)\,\frac{df}{dx}(x) - c(x)f(x),$$

where

$$\begin{aligned} a(x) &= 2(1-x)\sqrt{2x}, \qquad b(x) = 4(1-x)(n-x),\\ c(x) &= 4(1-x)\left(\frac{\frac{p+q}{2}\left(\frac{p+q}{2}+n-1\right)}{x} - \frac{(p-q)^2}{4}\right). \end{aligned}$$

The basic characteristics of this diffusion are (cf. (2.8)–(2.10))

$$\begin{aligned} m(x) &= \frac{2}{a^2(x)}\,e^{B(x)} = \frac{1}{4x(1-x)^2}\,\frac{x^n}{(1-x)^{n-1}} = \frac{1}{4}\,\frac{x^{n-1}}{(1-x)^{n+1}},\\ s'(x) &= e^{-B(x)} = \frac{(1-x)^{n-1}}{x^n},\\ k(x) &= \frac{2}{a^2(x)}\,c(x)\,e^{B(x)} = \left[\frac{\frac{p+q}{2}\left(\frac{p+q}{2}+n-1\right)}{x} - \left(\frac{p-q}{2}\right)^2\right]\frac{x^{n-1}}{(1-x)^n}. \end{aligned}$$

The general theory of one-dimensional diffusions says that the λ-Green function for the interval $[0,r]$ is of the form

$$C_{n,p,q,\lambda}\cdot\varphi_{n,p,q,\lambda}(|x|)\psi_{n,p,q,\lambda}(|y|), \qquad |x| < |y|,$$

where $\varphi_{n,p,q,\lambda}$ and $\psi_{n,p,q,\lambda}$ are positive and monotone solutions of the equation $\mathcal{L}u = \lambda u$.

The equation $\mathcal{L}u = \lambda u$ is just (5.20) and taking into account its general solution, we get the formulas for $\varphi_{n,p,q,\lambda}(|x|)$ and $\psi_{n,p,q,\lambda}(|y|)$ as in the statement.

Finally, using the formula for the Wronskian of solutions to the hypergeometric equation ([L, p. 84]), we get

$$\begin{aligned}W(\varphi_{n,p,q,\lambda},\psi_{n,p,q,\lambda})(z) &= z^{p+q}(1-z)^{n-\mu}W(F_{n,p,q,\lambda},G_{n,p,q,\lambda})(z)\\ &= z^{p+q}(1-z)^{n-\mu}z^{-n-p-q}(1-z)^{\mu-1}C_{n,p,q} = s'(z)C_{n,p,q},\end{aligned}$$

where $C_{n,p,q}$ equals

$$\begin{cases} \dfrac{-\Gamma(1-\mu)\Gamma(n+p+q)}{\Gamma\big(p+\frac{n-\mu}{2}\big)\Gamma\big(q+\frac{n-\mu}{2}\big)} & \text{if } \frac{n-\mu}{2}\notin\mathbb{Z},\ 1-\mu\neq 0,-1,\ldots,\\ \dfrac{-\Gamma(1+\mu)\Gamma(n+p+q)}{\Gamma\big(q+\frac{n+\mu}{2}\big)\Gamma\big(p+\frac{n+\mu}{2}\big)} & \text{if } \frac{n-\mu}{2}\notin\mathbb{Z},\ 1-\mu=0,-1,\ldots,\\ & \text{or } \frac{n-\mu}{2}\in\mathbb{Z},\ \frac{2p+n-\mu}{2}=-1,-2,\ldots,\\ \dfrac{(-1)^{n+p+q}\Gamma(1+q-p)\Gamma(n+p+q)}{\Gamma\big(q+\frac{n-\mu}{2}\big)\Gamma\big(q+\frac{n+\mu}{2}\big)} & \text{if } \frac{n-\mu}{2}\in\mathbb{Z},\ \frac{2p+n-\mu}{2}=p-1,\ldots,0.\end{cases}$$

Comparing the formulas for the volume element $dV_{\mathbb{C}^n}$ and the density of the speed measure $m(x)$, we arrive at

$$\widehat{(G^\lambda_{B_r})}_{p,q}(|x|,|y|) = \frac{H^{p,q}_n(1)}{2C_{n,p,q,\lambda}}\,\varphi_{n,p,q,\lambda}(|x|^2)\psi_{n,p,q,\lambda}(|y|^2).$$

Hence the theorem follows, because the set of unitary spherical harmonics $(H^{p,q}_n)$, $p,q=0,1,2,\ldots$, is an orthogonal basis in $L^2(\{x^2+y^2<1\},(1-x^2-y^2)^{n-2}dx\,dy)$. The proof that the series is convergent is very similar to that given in [Z]. ∎

6. General case. Summarizing the above examples we can exhibit the main idea of calculations. Suppose we are given a diffusion $(X_t)_{t\geq 0}$ with values in a subset of $\mathbb{R}^n$. Suppose also that D is a regular domain in $\mathbb{R}^n$ and $\tau_D=\inf\{t>0 : X_t\notin D\}$ is finite with probability one, that is, $P^x(\tau_D<\infty)=1$. Consider the process $(X^D_t)_{t\geq 0}$ killed upon exiting D.

Assumptions.

A. The generator $\mathcal{L}$ of the diffusion (X_t) commutes with a group G of mappings of D, hence the distribution of the process starting from a point $x_0\in D$ is invariant with respect to all those mappings from G that leave x_0 invariant.

B. (X_t) can be decomposed into two parts: a process r_t such that $\tau_D=\inf\{t>0 : r_t=a\}$ for some constant $a\in\mathbb{R}$ and the second part, called $(\tilde{X}_t)$, invariant with respect to all those transformations from G that leave x_0 unchanged.

C. It is possible to use harmonic analysis on ∂D, i.e. there exists a set of functions $(f_u)_{u\in I}$ such that $f_u:\partial D\to\mathbb{C}$ and the family $(\mathbb{E}(f_u(\tilde{X}_t)))_{u\in I}$ describes a distribution of $\tilde{X}_t$ in a unique way.

D. There exists a multiplicative functional $V_t = \exp(\int_0^t q(r_s)\,ds)$ such that the process $Z_t = f_u(\tilde{X}_t)\,V_t$ is a martingale.

If all these assumptions hold then we can find an ordinary differential equation describing the gauge, so we can compute $\mathbb{E}^x(e^{-\lambda\tau_D} f_u(\tilde{X}_{\tau_D}))$.

Observe that all the above assumptions were satisfied in the examples discussed in Sections 3, 4 and 5. If the domain D is a half-space or a ball, the group G consists of automorphisms of the half-space (inner translations and rotations) or the ball (orthogonal or unitary transformations).

The method can also be applied in non-hyperbolic contexts. The authors of [JG] computed the Poisson kernel for balls in the case of an Ornstein–Uhlenbeck process. Using our method it is possible to describe the λ-Poisson kernel for such a process.

As we mentioned in the Introduction, the authors of [BDH] investigated the global Poisson kernel (on the whole space) for NA groups and generators of the form of a Laplace–Beltrami operator plus some additional term of the first order. Complex hyperbolic space, realized as Siegel upper half-space, is the main example of such NA groups. It is an interesting question whether the above methods can help to compute the λ-Poisson kernel or λ-Green function for a "half-space" of the Siegel domain, that is, for $D = \{z \in \mathbb{C}^n : \Im z_1 - \sum_{k=2}^n |z_k|^2 > a\}$, for given $a > 0$.

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