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ESTIMATES FOR THE POISSON KERNEL ON HIGHER RANK NA GROUPS

BY

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Abstract. We obtain an estimate for the Poisson kernel for the class of second order left-invariant differential operators on higher rank NA groups.

The authors would like to dedicate this paper to the memory of Andrzej Hulanicki. As is clear from the bibliography, this work owes much to the influence of him and his co-workers. Indeed, this whole area of exploration was initiated by this group. The current work could not have been done without the foundation they laid.

However, our debt goes far beyond this. The second author was a student of Andrzej's student, Ewa Damek. The first author came to Poland for the first time in 1976 at Andrzej's invitation. Since then he has visited Poland regularly, at first to attend conferences, and later to do mathematics both with Andrzej and others. This collaboration has been one of the most rewarding experiences of his career. In the process Andrzej and his wife Barbara became good friends of his. He spent many memorable hours with them, both in Poland and elsewhere, sharing a good meal (cooked by Barbara) and discussing mathematics, life, etc. over a glass of good wine or vodka. Andrzej will be dearly missed.

1. Statement of the result. Let S be a semidirect product $S = N \rtimes A$ where N is a connected and simply connected nilpotent Lie group and A is isomorphic with $\mathbb{R}^k$. For $g \in S$ we let $n(g) = n$ and $a(g) = a$ denote the components of g in this product so that $g = (n, a)$.

We assume that there is a basis $X_1, \ldots, X_m$ for $\mathfrak{n}$ that diagonalizes the A-action. Let $\xi_1, \ldots, \xi_m \in \mathfrak{a}^* = \mathbb{R}^k$ be the corresponding roots, i.e., for every $H \in \mathfrak{a}$, $[H, X_j] = \xi_j(H)X_j$, $j = 1, \ldots, m$. As in [3], we assume that there is an element $H \in \mathbb{R}^k$ such that

$$\xi_j(H) > 0 \quad \text{for } 1 \le j \le m.$$

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Let, for $\alpha = (\alpha_1, \ldots, \alpha_k) \in \mathbb{R}^k$ and real d_j's,

$$\mathcal{L}_\alpha = \sum_{j=1}^{r} (e^{2\xi_j(a)} X_j^2 + d_j e^{\xi_j(a)} X_j) + \varDelta_\alpha, \tag{1.1}$$

where

$$\varDelta_\alpha = \sum_{i=1}^{k} (\partial_{a_i}^2 - 2\alpha_i \partial_{a_i}),$$

and $X_1, \ldots, X_r$ satisfy the Hörmander condition, i.e., they generate the Lie algebra $\mathfrak{n}$ of N.

Define

$$\gamma(\alpha) = 2 \min_{1\le j\le r} \frac{\xi_j(\alpha)}{\xi_j^2} \quad \text{and} \quad \rho_0 = \sum_{j=1}^{m} \xi_j, \tag{1.2}$$

where for the vector $x \in \mathbb{R}^k$ we write

$$x^2 = x \cdot x = \sum_{i=1}^{k} x_i^2.$$

Let

$$\chi(g) = \det(\mathrm{Ad}(g)) = e^{\rho_0 \cdot a}.$$

Let dx be left-invariant Haar measure on S. We have

$$\int_S f(xg)\, dx = \chi(g) \int_S f(x)\, dx.$$

We set

$$\begin{aligned} \overline{A^-} &= \{a \in \mathbb{R}^k : \xi_j(a) \le 0 \text{ for } 1 \le j \le r\}, \\ A^- &= \mathrm{Int}(\overline{A^-}), \qquad A^+ = -A^-. \end{aligned} \tag{1.3}$$

Remark 1.1. It is clear that we could have used all of the roots in defining A^- since from the Hörmander condition the span over $\mathbb{N}$ of ξ_j, $1 \le j \le r$, contains all of the roots.

If $\alpha \in A^+$ then there exists a *Poisson kernel* ν for $\mathcal{L}_\alpha$ [3]. That is, there is a C^∞ function ν on N such that every bounded $\mathcal{L}_\alpha$-harmonic function F on S may be written as a Poisson integral against a bounded function f on $S/A = N$,

$$F(g) = \int_{S/A} f(gx)\nu(x)\, dx = \int_N f(x)\check{\nu}^a(x^{-1}n)\, dx,$$

where

$$\check{\nu}^a(x) = \nu(a^{-1}x^{-1}a)\chi(a)^{-1}.$$

Conversely, the Poisson integral of any $f \in L^\infty(N)$ is a bounded $\mathcal{L}_\alpha$-harmonic function.

For $t \in \mathbb{R}^+$ and $\rho \in A^+$, let

$$\delta_t^\rho = \mathrm{Ad}((\log t)\rho)|_N.$$

Then $t \mapsto \delta_t^\rho$ is a one-parameter group of automorphisms of N for which the corresponding eigenvalues on $\mathfrak{n}$ are all positive. It is known [9] that then N has a δ_t^ρ-homogeneous norm: a non-negative continuous function $|\cdot|_\rho$ on N such that $|n|_\rho = 0$ if and only if $n = e$ and

$$|\delta_t^\rho n|_\rho = t|n|_\rho.$$

The main result is the following.

Theorem 1.2. *Let $\alpha \in A^+$. Assume that the rank* (*dimension of A*) *is $k > 1$. Let $V^+ \subset A^+$ be an open convex cone with vertex at* 0 *such that $\overline{V^+} \setminus \{0\} \subset A^+$. Then there exists a constant $c > 0$, depending only on the ξ_i's and V^+, such that for all $\rho \in V^+$ there exists C_ρ such that for all $x \in N$,*

$$\nu(x) \le C_\rho (1 + |x|_\rho)^{-c\rho_0(\rho)\gamma(\alpha)}. \tag{1.4}$$

Remark 1.3. The constant c above is one-fourth of the constant from Corollary 2.2.

We say that the operator $\mathcal{L}_\alpha$ has *independent coefficients* if the linear forms ξ_j's depend on disjoint sets of variables. In this situation we also say that $\mathcal{L}_\alpha$ is *independent*. For example, the following operator is independent:

$$e^{2(c_1a_1+c_2a_2)}X_1^2 + e^{2c_3a_3}X_2^2 + \sum_{i=1}^{3}(\partial_{a_i}^2 - 2\alpha_i\partial_{a_i}).$$

The proof of (1.4) can be simplified for independent operators without the first order term on N. We present the details of the proof for such operators in Sect. 8.

The outline of the rest of the paper is as follows. In Sect. 2 we clarify the dependence of the constant c from Theorem 1.2 on the cone V^+ and ξ_j's (Corollary 2.2). We also apply Theorem 1.2 to the Laplace–Beltrami operator for the product of two half-planes. In Sect. 3 we split the diffusion on S generated by $\mathcal{L}_\alpha$ into a skew product of diffusions on A and N, and we state the estimate for the "horizontal component" of the diffusion on N (Theorem 3.2, which we prove in Appendix A). In Sect. 4 we recall the construction of the Poisson kernel ν on N and its extension ν^a to S. In Sect. 5 we study some exponential functionals of Brownian motion. In Sect. 6 we prove the main estimate for ν^a. Combining this with the material from previous sections we get the required estimates for the Poisson kernel in Sect. 7 and 8.

2. The cones V^+ and V^-. Let $\overline{A^-}$, A^- and $\mathrm{Int}(\overline{A^-})$ be as in (1.3) and set $A^+ = -A^-$. Let $V \subset A^+$ be an open convex cone with vertex at 0 such that $\bar{V} \setminus \{0\} \subset A^+$. The *open dual cone* V^* of V is defined by

$$V^* = \{\xi : \xi(a) > 0 \text{ for all } a \in \bar{V} \setminus \{0\}\}.$$

PROPOSITION 2.1. *There exists $\xi_0 \in V^*$ such that for $1 \le j \le r$,*

$$(\xi_j - \xi_0)(a) > 0 \quad \textit{for all } a \in \bar{V} \setminus \{0\}.$$

Proof. We take an arbitrary $\xi \in V^*$. Then

$$\lim_{c \to 0} (\xi_j - c\xi) = \xi_j.$$

But $\xi_j \in V^*$ and V^* is open, hence there exists c_0 such that for all $c < c_0$, and all $1 \le j \le r$, we have $\xi_j - c\xi \in V^*$. Therefore $c\xi$ with $c < c_0$ can be taken as ξ_0. ∎

COROLLARY 2.2. *There exist $c > 0$ such that $c\rho_0$, where $\rho_0 = \sum_{j=1}^m \xi_j$, can be taken as ξ_0 in Proposition 2.1.*

We define

$$V^- = -V$$

and, for obvious reasons, we often denote V by V^+, i.e.,

$$V^+ = V.$$

EXAMPLE 1. Consider the operator L of the form (1.1) with $\xi_1 = (1,0)$, $\xi_2 = (0,1)$ and $\alpha = (1/2, 1/2)$, i.e.,

$$L = e^{2a_1}\partial_x^2 + e^{2a_2}\partial_y^2 + \partial_{a_1}^2 - \partial_{a_1} + \partial_{a_2}^2 - \partial_{a_2}.$$

This is the Laplace–Beltrami operator for the product of two half-planes where each half-plane is identified with $\mathbb{R}^2$ using the map of $\mathbb{R}^2$ onto H^+ defined by $(x,t) \mapsto x + ie^t$. The Poisson kernel for this operator is

$$\nu(x,y) = \frac{1}{(1+x^2)(1+y^2)}.$$

In fact, to check this we note that in the multiplicative notation

$$\begin{aligned} L &= a^2\partial_x^2 + b^2\partial_y^2 + (a\partial_a)^2 - a\partial_a + (b\partial_b)^2 - b\partial_b \\ &= a^2\partial_x^2 + b^2\partial_y^2 + a^2\partial_a^2 + b^2\partial_b^2. \end{aligned}$$

Now it is enough to check that $LF = 0$, where

$$F(x,y,a,b) = a^{-1}b^{-1}\nu(a^{-1}x, b^{-1}y).$$

Clearly, $A^+ = \mathbb{R}^+ \times \mathbb{R}^+$. For a fixed $\varepsilon > 0$ let $V^+ \subset A^+$ be the open cone whose boundary is the union of the half-lines

$$a_2 = (2-\varepsilon)a_1 \quad \text{and} \quad a_2 = (2+\varepsilon)a_1, \qquad a_1 \ge 0.$$

In other words,

$$V^+ = \{(a_1,a_2) \mid a_1 > 0,\ a_2 = ba_1,\ 2-\varepsilon < b < 2+\varepsilon\}.$$

To find the constant c from Corollary 2.2, we note that for all $a \in \bar{V} \setminus \{0\}$,

$$(\xi_j - c\rho_0) \cdot a > 0 \quad \text{for } j = 1, 2$$

is equivalent to

$$(1 - c, -c) \cdot a > 0 \quad \text{and} \quad (-c, 1 - c) \cdot a > 0.$$

Using the relation $a_2 = ba_1$ with $2 - \varepsilon < b < 2 + \varepsilon$ we get

$$c < \frac{1}{b+1} \quad \text{and} \quad c < \frac{b}{b+1}.$$

Therefore, we need to have

$$c < \frac{1}{3+\varepsilon} \quad \text{and} \quad c < \frac{2-\varepsilon}{3+\varepsilon},$$

and consequently,

$$c < \frac{1}{3+\varepsilon}.$$

We can take

$$\rho = (1, 2) \in V.$$

The corresponding dilation and homogeneous norms are respectively

$$\delta_t^\rho(x, y) = \delta_t^{(1,2)}(x, y) = (tx, t^2 y), \qquad |(x, y)|_\rho = |x| + |y|^{1/2}.$$

We have $\gamma(\alpha) = 1$. Our Theorem 1.2 and Remark 1.3 say that

$$\nu(x, y) = \frac{1}{(1 + x^2)(1 + y^2)} \leq C_\rho (1 + |x| + |y|^{1/2})^{-\frac{1}{3+\varepsilon} \cdot \frac{1}{4} \cdot 3}.$$

If we let $\varepsilon \to 0$, we get

$$\nu(x, y) \leq C_\rho (1 + |x| + |y|^{1/2})^{-1/4}.$$

3. Disintegration of the diffusion into vertical and horizontal components. Let

$$\mathcal{L}_\alpha = \sum_{i=1}^{k} (\partial_{a_i}^2 - 2\alpha_i \partial_{a_i}) + \sum_{j=1}^{r} (e^{2\xi_j(a)} X_j^2 + d_j e^{\xi_j(a)} X_j) = \varDelta_\alpha + L_a. \tag{3.1}$$

3.1. Vertical component. The process σ_t in $\mathbb{R}^k$ generated by the operator

$$\varDelta_\alpha = \sum_{i=1}^{k} (\partial_{a_i}^2 - 2\alpha_i \partial_{a_i}),$$

i.e., the Brownian motion with drift, is called the *vertical component* of the diffusion generated by $\mathcal{L}_\alpha$.

3.2. Horizontal component. Let $C_\infty(N)$ be the space of continuous functions f on N for which $\lim_{x\to\infty} f(x)$ exists. For $X \in \mathfrak{n}$, we let $\tilde{X}$ denote the corresponding right-invariant vector field. For a multi-index $I = (i_1, \dots, i_m)$, $i_j \in \mathbb{Z}^+$, and a basis $X_1, \dots, X_m$ of the Lie algebra $\mathfrak{n}$ we write $X^I = X_1^{i_1} \dots X^{i_m}$. For $k = 1, 2, \dots, \infty$ we define

$$C^{(k,l)}(N) = \{f : \tilde{X}^I X^J f \in C_\infty(N) \text{ for every } |I| \le k \text{ and } |J| \le l\}$$

and

$$\begin{aligned} \|f\|^0_{(k,l)} &= \sup_{|I|=k,\,|J|=l} \|\tilde{X}^I X^J f\|_\infty, \\ \|f\|_{(k,l)} &= \sup_{|I|\le k,\,|J|\le l} \|\tilde{X}^I X^J f\|_\infty. \end{aligned} \tag{3.2}$$

In particular, $C^{(0,k)}(N)$ is a Banach space with the norm $\|f\|_{(0,2)}$.

For $a \in \mathbb{R}^k$, let

$$L_a = \sum_{j=1}^{r} (e^{2\xi_j(a)} X_j^2 + d_j e^{\xi_j(a)} X_j). \tag{3.3}$$

For a continuous function $\sigma : [0, \infty) \to \mathbb{R}^k$, we consider the operator

$$L_{\sigma_t} = \sum_{j=1}^{r} (e^{2\xi_j(\sigma_t)} X_j^2 + d_j e^{\xi_j(\sigma_t)} X_j). \tag{3.4}$$

Let $\{U^\sigma(s,t) : 0 \le s \le t\}$ be the (unique) family of bounded operators on $C_\infty(N)$ which satisfies

(i) $U^\sigma(s,s) = \mathrm{Id}$ for all $s \ge 0$,
(ii) $\lim_{h\to 0} U^\sigma(s, s+h)f = f$ in $C_\infty(N)$,
(iii) $U^\sigma(s,r)U^\sigma(r,t) = U^\sigma(s,t)$, $0 \le s \le r \le t$,
(iv) $\partial_s U^\sigma(s,t)f = -L_{\sigma_s} U^\sigma(s,t)f$ for every $f \in C^{(0,2)}$,
(v) $\partial_t U^\sigma(s,t)f = U^\sigma(s,t)L_{\sigma_t} f$ for every $f \in C^{(0,2)}$,
(vi) $U^\sigma(s,t) : C^{(0,2)} \to C^{(0,2)}$.

The operator $U^\sigma(s,t)$ is a convolution operator with a probability measure with a smooth density, i.e., $U^\sigma(s,t)f = f * p^\sigma(t,s)$. In particular, $U^\sigma(s,t)$ is left-invariant. By (iii), $p^\sigma(t,r) * p^\sigma(r,s) = p^\sigma(t,s)$ for $t > r > s$. Existence of $U^\sigma(s,t)$ follows from [15]. Notice that from the above properties it follows that

(vii) $U^{\sigma\circ\theta_u}(s,t) = U^\sigma(s+u, t+u)$, where $\sigma \circ \theta_u(s) = \sigma_{s+u}$ is the shift operator.

In fact, $V(s,t) := U^\sigma(s+u, t+u)$ satisfies (i)–(vi) with the operator $L_{\sigma_{t+u}}$. Hence, the result follows from the uniqueness of $U^{\sigma_{t+u}}(s,t)$.

The stochastic process (evolution) in N corresponding to the transition probabilities $p^\sigma(t,s)$ is called the *horizontal component* of the diffusion generated by $\mathcal{L}_\alpha$.

3.3. Disintegration of the solution of a heat equation on $N \times \mathbb{R}^k$. Consider the operators $\mathcal{L}_\alpha$, Δ_α and L_a defined in (3.1). Let $U^\sigma(t,s)$ be the evolution generated by the operator L_{σ_t} defined in (3.4).

For $f \in C_c(N \times \mathbb{R}^k)$ and $t \geq 0$, we put

$$T_t f(x,a) = \mathbf{E}_a U^\sigma(0,t) f(x,\sigma_t) = \mathbf{E}_a p^\sigma(t,0) *_N f(x,\sigma_t), \tag{3.5}$$

where the expectation is taken with respect to the distribution of the process σ_t (Brownian motion with drift) in $\mathbb{R}^k$ with generator Δ_α. The operator $U^\sigma(0,t)$ acts on the first variable of the function f (as a convolution operator).

We have the following

THEOREM 3.1. *The family T_t defined in* (3.5) *is the semigroup of operators generated by $\mathcal{L}_\alpha$. That is,*

$$\partial_t T_t f = \mathcal{L}_\alpha T_t f \quad \textit{and} \quad \lim_{t\to 0} T_t f = f.$$

By now the proof of the above statement is standard and it goes along the lines of [6] with obvious changes. The idea of such a decomposition of the diffusion on the product of manifolds $N \times M$ generated by the skew-product $L = L_1(a) + L_2$ of the operators $L_1(a)$, $a \in M$, acting on N and L_2 acting on M goes back to [12, 13] (see also [16]).

3.4. Estimate for the evolution kernel $p^\sigma(t,0)$. Let $\sigma_t = w_t - 2\alpha t$ be the k-dimensional Brownian motion with drift -2α, $\alpha \in \mathbb{R}^k$. We define the functional

$$A^\sigma(s,t) = \int_s^t \max_{\substack{j=1,\dots,r\\ d=1,2}} e^{d\xi_j(\sigma_u)}\, du. \tag{3.6}$$

The following theorem is a generalization of the rank-one result [6, Theorem 4.1] to the higher rank setting. Here τ is a subadditive, δ_t-homogeneous norm on N which is smooth on $N \setminus \{e\}$ (see [10]).

THEOREM 3.2. *Let $K \subset N$ be closed and $e \notin K$. Then there exist constants C_1, C_2 and ν such that for every $x \in K$ and every t,*

$$p^\sigma(t,0)(x) \leq C_1 \Big(\int_0^t \chi(\sigma_u)^{2/\nu}\, du\Big)^{-\nu/2} \exp\Big(\frac{\tau(x)}{4} - \frac{\tau(x)^2}{C_2 A^\sigma(0,t)}\Big).$$

We give the proof of Theorem 3.2 in Appendix A.

4. The Poisson kernel ν. Let μ_t be the semigroup of probability measures on $S = N \rtimes A$ generated by $\mathcal{L}_\alpha$. It is known [4, 5] that

$$\lim_{t\to\infty} (\pi_N(\check{\mu}_t), f) = (\nu, f),$$

where π_N denotes the projection from S onto N, and $(\check{\mu}, f) = (\mu, \check{f})$, $\check{f}(x) = f(x^{-1})$. Let $a \in \mathbb{R}^k$ and let μ be a measure on N. We define

$$(\mu^a, f) = (\mu, f \circ \mathrm{Ad}(a)).$$

For $a \in \mathbb{R}^k$ we have

$$\nu^a(x) = \nu(a^{-1}xa)\chi(a)^{-1}, \tag{4.1}$$

where $\chi(b) = e^{\rho_0 \cdot b}$, $\rho_0 = \sum_{j=1}^m \xi_j$. It is an easy calculation to check that

$$\lim_{t\to\infty} (\pi_N(\check{\mu}_t)^a, f) = (\nu^a, f). \tag{4.2}$$

LEMMA 4.1. *We have*

$$(\pi_N(\check{\mu}_t)^a, f) = (\mathbf{E}_a \check{p}^\sigma(t, 0), f).$$

Proof. Let T_t be the semigroup of operators generated by $\mathcal{L}_\alpha$, i.e.,

$$T_t f(x, a) = f * \mu_t(x, a) = \int_S p_t(x, a; y, b) f(y, b)\chi(b)^{-1}\, dy\, db.$$

By Theorem 3.1,

$$T_t f(x, a) = \mathbf{E}_a \int_N f(xy^{-1}, \sigma_t) p^\sigma(t, 0)(y)\, dy.$$

Now we can write

$$\begin{aligned}
(\pi_N(\check{\mu}_t)^a, f) &= (\pi_N(\check{\mu}_t), f \circ \mathrm{Ad}(a)) = (\check{\mu}_t, f \circ \mathrm{Ad}(a) \circ \pi_N) \\
&= T_t(f \circ \mathrm{Ad}(a) \circ \pi_N)(e, 0) = \mathbf{E}_0 U^\sigma(0, t)(f \circ \mathrm{Ad}(a) \circ \pi_N)(e, 0) \\
&= \mathbf{E}_0 p^\sigma(t, 0) * (f \circ \mathrm{Ad}(a))(e) = \mathbf{E}_0 \int_N f(\mathrm{Ad}(a)y^{-1}) p^\sigma(t, 0)(y)\, dy \\
&= \mathbf{E}_0 \int_N f(\mathrm{Ad}(a)y) \check{p}^\sigma(t, 0)(y)\, dy \\
&= \mathbf{E}_0 \int_N f(x) \check{p}^\sigma(t, 0)(\mathrm{Ad}(-a)x)\chi(a)^{-1}\, dx \\
&= \mathbf{E}_0 \int_N f(x) \check{p}^{\sigma+a}(t, 0)(x)\, dx = \mathbf{E}_a \int_N f(x) \check{p}^\sigma(t, 0)(x)\, dx \\
&= (\mathbf{E}_a \check{p}^\sigma(t, 0), f). \quad \blacksquare
\end{aligned}$$

By (4.2) and Lemma 4.1 it follows that

$$(\nu^a, f) = \lim_{t\to\infty} (\pi_N(\check{\mu}_t)^a, f) = \lim_{t\to\infty} (\mathbf{E}_a \check{p}^\sigma(t, 0), f). \tag{4.3}$$

5. Some functionals of Brownian motion. Let w_s, $s \ge 0$, be the Brownian motion on $\mathbb{R}$ starting from $a \in \mathbb{R}$ and normalized so that

$$\mathbf{E}_a f(w_s) = \int_{\mathbb{R}} f(x+a)\,\frac{1}{\sqrt{4\pi s}}\,e^{-x^2/(4s)}\,dx. \tag{5.1}$$

Hence $\mathbf{E} w_s = a$ and $\operatorname{Var} w_s = 2s$.

For $d > 0$ and $\mu > 0$ we define the functional

$$I_{d,\mu} = \int_0^\infty e^{d(w_s - \mu s)} ds \tag{5.2}$$

which is called a *perpetual functional* in financial mathematics.

THEOREM 5.1 (Dufresne, [8]). *Let $w_0 = 0$. Then the functional $I_{2,\mu}$ is distributed as $(4\gamma_{\mu/2})^{-1}$, where $\gamma_{\mu/2}$ denotes a gamma random variable with parameter $\mu/2$, i.e., $\gamma_{\mu/2}$ has density $(1/\Gamma(\mu/2))x^{\mu/2-1}e^{-x}1_{[0,\infty)}(x)$.*

Many authors have been interested in this functional and the proof can be found in many places. See for example [7, 5] or the survey paper [14] and the references therein.

As a corollary from Theorem 5.1, by scaling the Brownian motion and changing the variable, we get the following

THEOREM 5.2. *Let $w_0 = a$. Then*

$$\mathbf{E}_a f(I_{d,\mu}) = c_{d,\mu} e^{\mu a} \int_0^\infty f(x) x^{-\mu/d} \exp\Big(-\frac{e^{da}}{d^2 x}\Big)\frac{dx}{x}.$$

The *inverse gamma density* (with respect to dx) is defined by

$$h_{\mu,\gamma} = C_{\mu,\gamma} x^{-\mu-1} e^{-\gamma/x} 1_{(0,\infty)}(x).$$

COROLLARY 5.3. *The random variable $I_{2,\mu}$ has the inverse gamma density $h_{\mu/2,1/4}$.*

We will also need the following lemma:

LEMMA 5.4. *Let $\sigma_u = w_u - 2\alpha u$ be the k-dimensional Brownian motion with a drift, $d > 0$, and let $\ell \in (\mathbb{R}^k)^*$ be such that $\ell(\alpha) > 0$. Then*

$$\mathbf{E}_a f\Big(\int_0^\infty e^{d\ell(\sigma_u)}\,du\Big) = c_{d,\ell,\alpha} e^{\gamma \ell(a)} \int_0^\infty f(u) u^{-\gamma/d} \exp\Big(-\frac{e^{d\ell(a)}}{d^2 \ell^2 u}\Big)\frac{du}{u},$$

where $\gamma = 2\ell(\alpha)/\ell^2$.

Proof. Notice that $\ell(\sigma_u) = \ell(w_u) - 2\ell(\alpha)u$ is the 1-dimensional Brownian motion with (negative) drift. Moreover, $\mathbf{E}\ell(\sigma_u) = -2\ell(\alpha)u$ and $\operatorname{Var}\ell(\sigma_u) = 2\ell^2 u$. Therefore,

$$\mathbf{E}_a f\Big(\int_0^\infty e^{d\ell(\sigma_u)}\,du\Big) = \mathbf{E}_{\ell(a)} f\Big(\int_0^\infty e^{d(b_{\ell^2 u} - 2\ell(\alpha)u)}\,du\Big),$$

where b_u is the 1-dimensional Brownian motion with density normalized as in (5.1). Changing variables, the above expected value is equal to

$$\mathbf{E}_{\ell(a)} f\Big(\int_0^\infty e^{d(b_s - 2\ell(\alpha)s/\ell^2)} \frac{ds}{\ell^2}\Big),$$

and the result follows from Theorem 5.2. ∎

COROLLARY 5.5. *If $\ell(\alpha), d > 0$ then the functional $\int_0^\infty e^{d\ell(w_u - 2\alpha u)}\,du$ has the inverse gamma density $h_{2\ell(\alpha)/(d\ell^2),1/(d^2\ell^2)}$.*

6. Estimates for ν^a

THEOREM 6.1. *For all compact subsets $K \not\ni e$ of N and all $\rho \in V^+$ there exist constants $C = C(K) > 0$ and $c = c(V^+) > 0$ such that for all $x \in K$ and all $s < 0$,*

$$\nu^{s\rho}(x) \le C e^{c\rho_0(s\rho)\gamma(\alpha) - \rho_0(s\rho)}, \tag{6.1}$$

where

$$\gamma(\alpha) = 2 \min_{1\le j\le r} \frac{\xi_j(\alpha)}{\xi_j^2}.$$

Proof. By (4.3), Theorem 3.2 and the Cauchy–Schwarz inequality we get

$$\begin{aligned} \nu^{s\rho}(x) &\le C\mathbf{E}_{s\rho}\Big(\int_0^\infty \chi(\sigma_u)^{2/\nu}\,du\Big)^{-\nu/2} e^{-\beta/A^\sigma(0,\infty)} \\ &\le C\Big(\mathbf{E}_{s\rho}\Big(\int_0^\infty e^{(2/\nu)\rho_0(\sigma_u)}\,du\Big)^{-\nu}\Big)^{1/2}\Big(\mathbf{E}_{s\rho} e^{-2\beta/A^\sigma(0,\infty)}\Big)^{1/2} \end{aligned} \tag{6.2}$$

for some $\beta = \beta_K > 0$.

Consider the second term on the right in (6.2). The functional $A^\sigma(0,\infty)$ for the operator (1.1) is given by

$$A^\sigma =: A^\sigma(0,\infty) = \int_0^\infty \max_{\substack{j=1,\dots,r\\ d=1,2}} e^{d\xi_j(\sigma_u)}\,du,$$

and can be estimated as follows:

$$A^\sigma(0,\infty) \le \sum_{j=1}^r A_j^\sigma + \sum_{j=1}^r \tilde{A}_j^\sigma,$$

where

$$A_j^\sigma = \int_0^\infty e^{2\xi_j(\sigma_u)}\,du \quad \text{and} \quad \tilde{A}_j^\sigma = \int_0^\infty e^{\xi_j(\sigma_u)}\,du.$$

Therefore,

$$(6.3)\qquad \mathbf{E}_{s\rho}e^{-2\beta/A^\sigma} \le \mathbf{E}_{s\rho}\exp\Big(-2\beta/\Big(\sum_{j=1}^r A_j^\sigma + \sum_{j=1}^r \tilde{A}_j^\sigma\Big)\Big)$$
$$= \mathbf{E}_0\exp\Big(-2\beta/\Big(\sum_{j=1}^r e^{2\xi_j(s\rho)}A_j^\sigma + \sum_{j=1}^r e^{\xi_j(s\rho)}\tilde{A}_j^\sigma\Big)\Big)$$
$$\le \mathbf{E}_0\exp\Big(-2\beta/\Big(M(s\rho)\Big(\sum_{j=1}^r A_j^\sigma + \sum_{j=1}^r \tilde{A}_j^\sigma\Big)\Big)\Big),$$

where

$$M(a) = \max\{e^{2\xi_j(a)}, e^{\xi_j(a)} : j = 1,\dots,r\}.$$

We need the following

LEMMA 6.2. *For every $\beta > 0$ there exists a positive constant C such that for all positive real numbers m,*

$$\mathbf{E}_0\exp\Big(-\beta/\Big(m\Big(\sum_{j=1}^r A_j^\sigma + \sum_{j=1}^r \tilde{A}_j^\sigma\Big)\Big)\Big) \le C(m^{\gamma(\alpha)/2} \vee m^{\gamma(\alpha)}).$$

Proof of Lemma 6.2. Let

$$\Omega_1 = \Big\{1 \le m\Big(\sum_{j=1}^r A_j^\sigma + \sum_{j=1}^r \tilde{A}_j^\sigma\Big)\Big\},$$
$$\Omega_{0,n} = \Big\{\frac{1}{n+1} \le m\Big(\sum_{j=1}^r A_j^\sigma + \sum_{j=1}^r \tilde{A}_j^\sigma\Big) < \frac{1}{n}\Big\},\qquad n = 1,2,\dots.$$

Then

$$(6.4)\qquad \mathbf{E}_0\exp\Big(-\beta/\Big(m\Big(\sum_{j=1}^r A_j^\sigma + \sum_{j=1}^r \tilde{A}_j^\sigma\Big)\Big)\Big)$$
$$= \mathbf{E}_0 1_{\Omega_1}(\sigma)\exp\Big(-\beta/\Big(m\Big(\sum_{j=1}^r A_j^\sigma + \sum_{j=1}^r \tilde{A}_j^\sigma\Big)\Big)\Big)$$
$$+ \sum_{n=1}^\infty \mathbf{E}_0 1_{\Omega_{0,n}}(\sigma)\exp\Big(-\beta/\Big(m\Big(\sum_{j=1}^r A_j^\sigma + \sum_{j=1}^r \tilde{A}_j^\sigma\Big)\Big)\Big)$$
$$\le \mathbf{P}_0(\Omega_1) + \sum_{n=1}^\infty e^{-\beta n}\mathbf{P}_0(\Omega_{0,n}).$$

We estimate the probability

$$(6.5)\qquad \mathbf{P}_0(\Omega_1) \le \mathbf{P}_0\Big(\sum_{j=1}^r (A_j^\sigma + \tilde{A}_j^\sigma) \ge 1/m\Big)$$
$$\le \mathbf{P}_0(\text{there exists } j \text{ such that } A_j^\sigma + \tilde{A}_j^\sigma \ge (mr)^{-1})$$

$$\le \sum_{j=1}^{r} \mathbf{P}_0(A_j^\sigma + \tilde{A}_j^\sigma \ge (mr)^{-1})$$
$$\le \sum_{j=1}^{r} \mathbf{P}_0(A_j^\sigma \ge (2mr)^{-1}) + \sum_{j=1}^{r} \mathbf{P}_0(\tilde{A}_j^\sigma \ge (2mr)^{-1}).$$

It follows from the proof of Lemma 5.4 that for every j,

$$A_j^\sigma \stackrel{d}{=} \frac{1}{\xi_j^2} \int_0^\infty e^{2(b_s^j - \gamma_j s)}\, ds = \frac{1}{\xi_j^2} I_{2,\gamma_j}, \qquad \tilde{A}_j^\sigma \stackrel{d}{=} \frac{1}{\xi_j^2} I_{1,\gamma_j},$$

where

$$\gamma_j = \gamma_j(\alpha) = 2\xi_j(\alpha)/\xi_j^2,$$

I_{d,γ_j} is a perpetual functional defined in (5.2), b_s^j is a standard Brownian motion, and $\stackrel{d}{=}$ means having the same distribution.

By the scaling property of the Brownian motion it follows that

$$I_{1,\mu} \stackrel{d}{=} 4 I_{2,2\mu}.$$

Therefore,

$$\mathbf{P}_0(\tilde{A}_j^\sigma \ge (2mr)^{-1}) = \mathbf{P}_0(4I_{2,2\gamma_j}/\xi_j^2 \ge (2mr)^{-1}) \le \mathbf{P}_0(I_{2,2\gamma} \ge \xi_j^2 (8mr)^{-1}),$$

where $\gamma = \gamma(\alpha) = \min_{1\le j\le r} \gamma_j(\alpha)$, and similarly,

$$\mathbf{P}_0(A_j^\sigma \ge (2mr)^{-1}) = \mathbf{P}_0(I_{2,\gamma_j} \ge \xi_j^2 (2mr)^{-1}) \le \mathbf{P}_0(I_{2,\gamma} \ge \xi_j^2 (2mr)^{-1}).$$

By Corollary 5.3 the random variable $I_{2,\gamma}$ has the inverse gamma distribution $h_{\gamma/2,1/4}(x) \sim x^{-\gamma/2-1}$ as $x \to \infty$. Therefore,

$$\mathbf{P}_0(I_{2,2\gamma} \ge \xi_j^2 (4mr)^{-1}) \le C \int_{\xi_j^2 (4mr)^{-1}}^{\infty} x^{-\gamma-1}\, dx \le C m^\gamma$$

and

$$\mathbf{P}_0(I_{2,\gamma} \ge \xi_j^2 (mr)^{-1}) \le C m^{\gamma/2}.$$

Consequently,

$$\mathbf{P}_0(\Omega_1) \le C(m^{\gamma/2} \vee m^\gamma). \tag{6.6}$$

Now we estimate $\mathbf{P}_0(\Omega_{0,n})$. In the same way as in (6.5) we get

$$\mathbf{P}_0(\Omega_{0,n}) \le \mathbf{P}_0\Big(\sum_{j=1}^{r} A_j^\sigma + \sum_{j=1}^{r} \tilde{A}_j^\sigma \ge \frac{1}{m(n+1)} \Big)$$
$$\le \sum_{j=1}^{r} \mathbf{P}_0\Big(A_j^\sigma \ge \frac{1}{2rm(n+1)} \Big) + \sum_{j=1}^{r} \mathbf{P}_0\Big(\tilde{A}_j^\sigma \ge \frac{1}{2rm(n+1)} \Big).$$

In order to estimate the above sums we repeat the previous calculation (after (6.5) with m replaced by $m(n+1)$) and we get

$$\mathbf{P}_0(\Omega_{0,n}) \le C(m^{\gamma/2} \vee m^{\gamma})(n+1)^{\gamma}.$$

Hence, we can sum the second series on the right in (6.4),

$$\sum_{n=1}^{\infty} e^{-\beta n}\mathbf{P}_0(\Omega_{0,n}) \le C(m^{\gamma/2}\vee m^{\gamma})\sum_{n=1}^{\infty} e^{-\beta n}(n+1)^{\gamma} \le C(m^{\gamma/2}\vee m^{\gamma}). \tag{6.7}$$

Now (6.4), (6.6) and (6.7) finish the proof. ∎

Since $\rho \in V^+$ and $s < 0$ it follows that $-s\rho \in V^+$. Therefore, by Proposition 2.1 and Corollary 2.2, there exists a positive constant c such that

$$\begin{aligned} M(s\rho) &= e^{\max\{2\xi_j(s\rho),\xi_j(s\rho):j=1,\dots,r\}} = e^{-\min\{2\xi_j(-s\rho),\xi_j(-s\rho):j=1,\dots,r\}} \\ &\le e^{-c\rho_0(-s\rho)} = e^{c\rho_0(s\rho)} < 1. \end{aligned} \tag{6.8}$$

By (6.8) and Lemma 6.2 we can continue estimating (6.3) as follows:

$$\mathbf{E}_{s\rho}e^{-2\beta/A^{\sigma}} \le \mathbf{E}_0 \exp\Big(-2\beta/\Big(e^{c\rho_0(s\rho)}\sum_{j=1}^{r}(A_j^{\sigma}+\tilde{A}_j^{\sigma})\Big)\Big) \le Ce^{(c/2)\rho_0(s\rho)\gamma(\alpha)}. \tag{6.9}$$

Finally, to estimate the first term on the right in (6.2) we notice that

$$\mathbf{E}_{s\rho}\Big(\int_0^{\infty} e^{(2/\nu)\rho_0(\sigma_u)}\,du\Big)^{-\nu} = e^{-2\rho_0(s\rho)}\mathbf{E}_0\Big(\int_0^{\infty} e^{(2/\nu)\rho_0(\sigma_u)}du\Big)^{-\nu}. \tag{6.10}$$

By Lemma 5.4,

$$\mathbf{E}_0\Big(\int_0^{\infty} e^{(2/\nu)\rho_0(\sigma_u)}\,du\Big)^{-\nu} \le C_{\alpha,\rho_0}. \tag{6.11}$$

Now, (6.2), (6.10), (6.11) finish the proof. ∎

7. Proof of Theorem 1.2. Using homogeneity this is an easy corollary from Theorem 6.1.

Proof of Theorem 1.2. By definition (4.1) of $\nu^{s\rho}$ and Theorem 6.1 we have, for x in a compact set $K \not\ni e$,

$$\nu((s\rho)^{-1}x(s\rho)) = e^{\rho_0(s\rho)}\nu^{s\rho}(x) \le Ce^{\rho_0(s\rho)}e^{c\rho_0(s\rho)\gamma(\alpha)-\rho_0(s\rho)} = Ce^{c\rho_0(s\rho)\gamma(\alpha)}.$$

Let $\delta_t^{\rho} = \mathrm{Ad}((\log t)\rho)$. Then $|\delta_t^{\rho}x|_{\rho} = t|x|_{\rho}$. Let $y = \delta^{\rho}_{\exp(-s)}x$ with $|x|_{\rho} = 1$ and $s < 0$. Then $|y|_{\rho} = e^{-s} > 1$, and using the above inequality for $K = \{x : |x|_{\rho} = 1\}$, we get

$$\nu(y) = \nu(\delta^{\rho}_{\exp(-s)}x) \le Ce^{c\rho_0(s\rho)\gamma(\alpha)} = C(e^{-sc\rho_0(\rho)\gamma(\alpha)})^{-1} = C|y|_{\rho}^{-c\rho_0(\rho)\gamma(\alpha)}.$$

Clearly, for y with $|y|_{\rho} \le 1$ we have $\nu(y) \le C_{\rho}$. ∎

8. Prooof of Theorem 1.2 for independent operators

8.1. Some probabilistic lemmas. Recall that the inverse gamma density (with respect to dx) is defined by

$$h_{\mu,\gamma}(x) = C_{\mu,\gamma} x^{-\mu-1} e^{-\gamma/x} 1_{(0,\infty)}(x).$$

LEMMA 8.1 ([1, Lemma 2]). *For all $n > 0$, the n-fold convolution of the inverse gamma density has asymptotic behavior* (1)

$$h^{*n}_{\mu,\gamma}(x) \sim n C_{\mu,\gamma} x^{-\mu-1}, \qquad x \to \infty.$$

LEMMA 8.2. *For every $n > 0$ there exist constants $C, c > 0$ such that for all $x \le 1$ we have*

$$h^{*n}_{\mu,\gamma}(x) \le C x^{-2\mu-1} e^{-\gamma/cx}.$$

Proof. We use induction with respect to n. First consider the convolution of two densities

$$h^{*2}_{\mu,\gamma}(x) = \int_0^x (x-t)^{-\mu-1} t^{-\mu-1} e^{-\gamma/(x-t)} e^{-\gamma/t} dt.$$

Changing variables $u = 1/t$ we get

$$h^{*2}_{\mu,\gamma}(x) = \int_{1/x}^{\infty} \frac{u^{\mu-1}}{(x-u^{-1})^{\mu+1}} e^{-\gamma u x/(x-u^{-1})} du.$$

Changing variables again, $ux = s$, we obtain

$$h^{*2}_{\mu,\gamma}(x) = x^{-2\mu-1} \int_1^{\infty} \frac{s^{2\mu}}{(s-1)^{\mu+1}} e^{-\gamma s/(x(1-s^{-1}))} ds.$$

By a direct calculation one can show that the integral above is bounded by $Ce^{-\gamma/(2x)}$. To estimate $h^{*(n+1)}_{\mu,\gamma}(x) = h^{*n}_{\mu,\gamma} * h_{\mu,\gamma}(x)$ we use the induction hypothesis and proceed similarly to the previous case. ∎

LEMMA 8.3. *Let w^j_s, $j = 1, \ldots, n$, be independent Brownian motions on $\mathbb{R}$. Let $I^j_{2,\mu}$ be the corresponding perpetual functionals defined in* (5.2). *Then there exists a constant $C > 0$ such that for every $t > 0$,*

$$\mathbf{P}_0\Big(\sum_{j=1}^n I^j_{2,\mu} \ge t\Big) \le C t^{-\mu}.$$

Proof. Since w^j_s are independent, the functionals $I^j_{2,\mu}$ are also independent. Moreover, if $w^j_0 = 0$, $j = 1, \ldots, n$, then by Corollary 5.3, they have

(1) In [1] the case $\gamma = 1/4$ is considered, but the proof given there clearly works for all γ.

inverse gamma distributions $h_{\mu/2,1/4}$. Hence, the distribution of $\sum_{j=1}^{n} I_{2,\mu}^{j}$ is $h^{*n}_{\mu/2,1/4}$. By Lemma 8.1, for t sufficiently large we can estimate

$$\mathbf{P}_0\Big(\sum_{j=1}^{n} I_{2,\mu}^{j} \geq t\Big) = \int_t^\infty h^{*n}_{\mu/2,1/4}(x)\,dx \leq C\int_t^\infty x^{-\mu/2-1}\,dx = Ct^{-\mu/2}. \ \blacksquare$$

LEMMA 8.4. *Let w_s^j be independent Brownian motions on $\mathbb{R}$. Let $I_{2,\mu}^{j}$ be the corresponding perpetual functionals defined in* (5.2). *Then for every $\beta > 0$ there exists a constant $C > 0$ such that for all positive real numbers m,*

$$\mathbf{E}_0 e^{-\beta/(m\sum_{j=1}^{n} I_{2,\mu}^{j})} \leq \begin{cases} 1 & \text{for } m \geq 1, \\ Cm^{\mu/2} & \text{for } m < 1. \end{cases}$$

Proof. By independence

$$\mathbf{E}_0 e^{-\beta/(m\sum_{j=1}^{n} I_{2,\mu}^{j})} = \int_0^\infty e^{-\beta/u} h^{*n}_{\mu/2,1/4}(u/m)m^{-1}\,du.$$

By Lemma 8.1 and Lemma 8.2, respectively, we have

$$\int_1^\infty e^{-\beta/u} h^{*n}_{\mu/2,1/4}(u/m)m^{-1}\,du \leq Cm^{\mu/2}\int_1^\infty e^{-\beta/u}u^{-\mu/2-1}\,du$$

and

$$\int_0^1 e^{-\beta/u} h^{*n}_{\mu/2,1/4}(u/m)m^{-1}\,du \leq Cm^{\mu}\int_0^1 e^{-\beta/u}u^{-\mu-1}\,du.$$

Hence the estimate follows. ■

8.2. Sketch of the proof of Theorem 1.2 for independent operators without first order N-part. We modify slightly the proof of Theorem 6.1 so that we do not need to use Lemma 6.2. Instead, we make use of Lemma 8.4. As a result we get the estimate for $\nu^{s\rho}$ of the form (6.1) but only for all s smaller than some $s_0 < 0$. Clearly, this is sufficient for the homogeneity argument in Sect. 7.

Since the operator is without first order N-part, the functional $A^\sigma(s,t)$ in Theorem 3.2 has simpler form

$$A^\sigma(s,t) = \int_s^t \max_{1\leq j\leq r} e^{2\xi_j(\sigma_u)}\,du.$$

Therefore,

$$A^\sigma(0,\infty) \leq \sum_{j=1}^{r} A_j^\sigma,$$

and (6.3) reads

$$\mathbf{E}_{s\rho} e^{-2\beta/A^\sigma} \le \mathbf{E}_0 \exp\Big(-2\beta/\Big(M(s\rho)\sum_{j=1}^r A_j^\sigma\Big)\Big). \tag{8.1}$$

By the independence of $\mathcal{L}_\alpha$ the random variables A_j^σ are independent. Moreover,

$$A_j^\sigma \stackrel{d}{=} \frac{1}{\xi_j^2} I_{2,\gamma_j},$$

where b_s^j, $j = 1, \dots, n$, appearing in I_{2,γ_j} are independent Brownian motions.

Let

$$\varXi = \max_{1\le j\le r} \frac{1}{\xi_j^2}.$$

Then the right side of (8.1) can be estimated by

$$\mathbf{E}_0 \exp\Big(-2\beta/\Big(\varXi e^{c\rho_0(s\rho)} \sum_{j=1}^r I_{2,\gamma}^j\Big)\Big), \tag{8.2}$$

where $\gamma = \min_{1\le j\le r}\gamma_j$, and $I_{2,\gamma}^j$ are independent random variables with the same distribution as $I_{2,\gamma}$.

We choose $s_0 < 0$ so that $\varXi e^{c\rho_0(s_0\rho)} < 1$. Then, by Lemma 8.4, it follows from (8.2) that for all $s < s_0$,

$$\mathbf{E}_0 \exp\Big(-2\beta/\Big(M(s\rho)\sum_{j=1}^r A_j^\sigma\Big)\Big) \le Ce^{(c/2)\rho_0(s\rho)\gamma},$$

providing the desired estimate for (8.1). The remainder of the proof proceeds as before.

Appendix A. Proof of Theorem 3.2. We follow the proof of [6, Theorem 4.1]. Let notation be as in Sect. 3.2. The $L^2(N)$-adjoint of L_a is

$$\tilde{L}_a = \sum_{j=1}^r (e^{2\xi_j(a)} X_j^2 - d_j e^{\xi_j(a)} X_j).$$

Let $\tilde{U}^\sigma(t,s)$, $t \ge s$, be the *fundamental solution* for $\tilde{L}_{\sigma_t} - \partial_t$, i.e., the (unique) family of bounded operators $\tilde{U}^\sigma(t,s)$, $t \ge s \ge 0$, on $C_\infty(N)$ with the following properties (analogous to (i)–(vi) in Sect. 3.2):

(1) $\tilde{U}^\sigma(s,s) = \mathrm{Id}$,
(2) $\lim_{h\to 0} \tilde{U}^\sigma(s+h,s)f = f$ in $C_\infty(N)$,
(3) $\tilde{U}^\sigma(t,r)\tilde{U}^\sigma(r,s) = \tilde{U}^\sigma(t,s)$, $t \ge r \ge s \ge 0$,
(4) $\partial_s \tilde{U}^\sigma(t,s)f = -\tilde{U}^\sigma(t,s)\tilde{L}_{\sigma_s} f$ for every $f \in C^{(0,2)}$,
(5) $\partial_t \tilde{U}^\sigma(t,s)f = \tilde{L}_{\sigma_t}\tilde{U}^\sigma(t,s)f$ for every $f \in C^{(0,2)}$,
(6) $\tilde{U}^\sigma(t,s) : C^{(0,2)} \to C^{(0,2)}$.

The following is a simple consequence of the uniqueness of $U^\sigma(s,t)$ (which is the fundamental solution for $L_{\sigma_s} + \partial_s$).

Lemma A.1. *For all* $f \in C_\infty(N)$ *and* $t \geq s$,

$$\tilde{U}^\sigma(t,s)f = f * \tilde{p}^\sigma(s,t), \quad \text{where} \quad \tilde{p}^\sigma(s,t)(x) = p^\sigma(t,s)(x^{-1}).$$

Also, for all $f \in L^1(N)$ *and* $g \in L^\infty(N)$,

$$(U^\sigma(s,t)f, g) = (f, \tilde{U}^\sigma(t,s)g). \tag{A.1}$$

In the following proposition, the notation is as in, and above, (3.2).

Proposition A.2. *Let* $f \in L^1(N)$, $f \geq 0$. *Then for* $g \in C^{(0,2)}(N)$,

$$(U^\sigma(s,t)f, e^g) \leq (f, e^g)\exp(C\mathbf{a}(g)A^\sigma(s,t)),$$

where C *is independent of* f, g, s, *and* t *and where*

$$\mathbf{a}(g) = \max\{\|g\|^0_{(0,1)}, \|g\|^0_{(0,2)} + (\|g\|^0_{(0,1)})^2\},$$

$$A^\sigma(s,t) = \int_s^t \max_{\substack{j=1,\ldots,r\\ d=1,2}} e^{d\xi_j(\sigma_u)}\,du.$$

Proof. Set $f_s = U^\sigma(s,t)f$, $t \geq s$. Since $f \in L^1(N)$ and $e^g \in L^\infty(N)$, the following is a consequence of (A.1):

$$m(s) \equiv (f_s, e^g) = (U^\sigma(s,t)f, e^g) = (f, \tilde{U}^\sigma(t,s)e^g).$$

Then, by property (4) and (A.1),

$$\begin{aligned} -\partial_s m(s) &= (f, -\partial_s\tilde{U}^\sigma(t,s)e^g) = (f, \tilde{U}^\sigma(t,s)\tilde{L}_{\sigma_s}e^g) \\ &= (f_s, \tilde{L}_{\sigma_s}e^g) \\ &= \sum_{j=1}^r e^{2\xi_j(\sigma_s)}(f_s, X_j^2 e^g) - \sum_{j=1}^r d_j e^{\xi_j(\sigma_s)}(f_s, X_j e^g). \end{aligned} \tag{A.2}$$

Also

$$|X_j e^g| = |(X_j g)|e^g \leq \mathbf{a}(g)e^g, \qquad |X_j^2 e^g| = |(X_j^2 g + (X_j g)^2)|e^g \leq \mathbf{a}(g)e^g.$$

Thus, since $f_s > 0$ and $f_t = f$, we may continue (A.2) as

$$-\partial_s m(s) \leq C\mathbf{a}(g) \max_{1\leq j\leq r,\, d=1,2} e^{d\xi_j(\sigma_s)}m(s),$$

$$-\partial_s \ln(m(s)) \leq C\mathbf{a}(g) \max_{1\leq j\leq r,\, d=1,2} e^{d\xi_j(\sigma_s)},$$

$$-\int_s^t \partial_u \ln(m(u))\,du \leq C\mathbf{a}(g)\int_s^t \max_{1\leq j\leq r,\, d=1,2} e^{d\xi_j(\sigma_u)}\,du,$$

$$\begin{aligned}\ln m(s) - \ln m(t) &\le C\mathbf{a}(g)A^\sigma(s,t),\\ m(s) &\le (f, e^g)\exp(C\mathbf{a}(g)A^\sigma(s,t))\end{aligned}$$

as claimed. (Note that $m(t) = (f, e^g)$.) ∎

We note that N is a homogeneous group. In fact, if $A_0 \in A^+$ satisfies $\xi_j(A_0) \ge 1$ for all j then

$$\delta_t = \mathrm{Ad}(\exp((\log t)A_0))|_N$$

is a dilation.

According to [10], there is a subadditive, δ_t-homogeneous norm τ on N which is smooth on $N \setminus \{e\}$. From homogeneity, on $N \setminus \{e\}$,

$$X_j(\tau)\circ\delta_t = t^{1-\xi_j(A_0)}(X_j\tau).$$

It follows that for all multi-indices $I \ne 0$, $X^I\tau$ is uniformly bounded on $N \setminus B_r(e)$ for all $r > 0$, where $B_r(e)$ is the τ-ball of radius r. For $\varepsilon > 0$, let

$$\tau_\varepsilon = \frac{\tau}{1+\varepsilon\tau}.$$

LEMMA A.3. *Let $I \ne 0$. For $1 \ge \varepsilon > 0$, τ_ε is a bounded subadditive function on N for which*

$$|X^I\tau_\varepsilon(x)| \le C_{r,I}$$

for all $x \in N \setminus B_r(e)$, where $C_{r,I}$ does not depend on ε.

Proof. The subadditivity is easily seen. To prove the boundedness, note that

$$\begin{aligned}X_i\tau_\varepsilon &= (1+\varepsilon\tau)^{-2}X_i\tau,\\ |X_i\tau_\varepsilon| &\le |X_i\tau|,\\ X_jX_i\tau_\varepsilon &= -2\varepsilon(1+\varepsilon\tau)^{-3}X_j\tau X_i\tau + (1+\varepsilon\tau)^{-2}X_jX_i\tau,\\ |X_jX_i\tau_\varepsilon| &\le 2|X_j\tau|\,|X_i\tau| + |X_jX_i\tau|.\end{aligned}$$

The general case is proved similarly. ∎

THEOREM A.4. *For all $r > 0$ there is a constant C_r such that*

$$(p^\sigma(t,s), e^{\alpha\tau}) \le e^{2\alpha r}\exp(C_r(\alpha+\alpha^2)A^\sigma(s,t))$$

for all $\alpha > 0$ and $t > s$.

Proof. Let $0 \le \phi \in C_c^\infty(N)$, $\operatorname{supp}\phi \subset B_r(e)$, and $\int\phi = 1$. Let $\eta_\varepsilon(x) = \tau_\varepsilon * \phi(x)$. The following lemma is similar to results in [11]. (See also (3.5) of [2].)

LEMMA A.5. *There exists a positive constant C, independent of ε, such that*

$$\|\eta_\varepsilon\|^0_{(0,i)} \le C \quad \textit{for } i = 1, 2.$$

Proof of Lemma A.5. Let $\psi \in C_{\mathfrak{c}}^{\infty}(N)$ be non-negative, supported in $B_{2r}(e)$, and equal to 1 on $B_r(e)$. Let $\tilde{\psi} = 1 - \psi$. Then

$$\eta_\varepsilon = (\tau_\varepsilon \tilde{\psi}) * \phi + (\tau_\varepsilon \psi) * \phi.$$

The second term on the right is clearly bounded independently of ε since $\tau_\varepsilon \le \tau$ and

$$X^I((\tau_\varepsilon \psi) * \phi) = (\tau_\varepsilon \psi) * X^I \phi.$$

For the first term let $\tilde{\tau}_\varepsilon = \tau_\varepsilon \tilde{\psi}$.

From Lemma A.3, $X^I \tilde{\tau}_\varepsilon$ is uniformly bounded independently of ε. Furthermore, for $|I| \neq 0$,

$$\begin{aligned} X^I(\tilde{\tau}_\varepsilon * \phi)(x) &= \int\limits_{B_r(e)} (\mathrm{Ad}(y) X^I) \tilde{\tau}_\varepsilon (xy^{-1}) \phi(y)\, dy \\ &= \sum_{|J|=|I|} \int\limits_{B_r(e)} q_J(y) X^J \tilde{\tau}_\varepsilon (xy^{-1}) \phi(y)\, dy, \end{aligned}$$

where the q_J are polynomials in the roots. Our result follows since the q_J are uniformly bounded on $B_r(e)$. ∎

Notice that since τ_ε is subadditive and $\tau_\varepsilon \le \tau$, we have

$$\begin{aligned} \tau_\varepsilon(x) &= \int\limits_{B_r(e)} \tau_\varepsilon(x) \phi(y)\, dy \\ &\le \int\limits_{B_r(e)} (\tau_\varepsilon(xy^{-1}) + \tau_\varepsilon(y)) \phi(y)\, dy \le \eta_\varepsilon(x) + r, \\ \eta_\varepsilon(e) &= \int\limits_{B_r(e)} \tau_\varepsilon(y^{-1}) \phi(y)\, dy \le r. \end{aligned}$$

We apply Proposition A.2 with $g = \alpha \eta_\varepsilon$. Note that, by Lemma A.5,

$$\mathbf{a}(g) = \max\{\|g\|^0_{(0,1)}, \|g\|^0_{(0,2)} + (\|g\|^0_{(0,1)})^2\} \le C(\alpha + \alpha^2).$$

Hence,

$$(U^\sigma(t,s) f, e^{\alpha \eta_\varepsilon}) \le (f, e^{\alpha \eta_\varepsilon}) \exp(C(\alpha + \alpha^2) A^\sigma(s,t)).$$

Letting f run through an approximate identity, using $\eta_\varepsilon(x) \ge \tau_\varepsilon(x) - r$, and letting $\varepsilon \to 0$, yields

$$\begin{aligned} (p^\sigma(s,t), e^{\alpha \eta_\varepsilon}) &\le e^{\alpha r} \exp(C(\alpha + \alpha^2) A^\sigma(s,t)), \\ (p^\sigma(s,t), e^{\alpha(\tau_\varepsilon - r)}) &\le e^{\alpha r} \exp(C(\alpha + \alpha^2) A^\sigma(s,t)), \\ (p^\sigma(s,t), e^{\alpha \tau}) &\le e^{2\alpha r} \exp(C(\alpha + \alpha^2) A^\sigma(s,t)), \end{aligned}$$

proving our theorem. ∎

PROPOSITION A.6. *There exist positive constants C and ν such that for every $t > s \ge 0$,*

$$\|p^\sigma(t,s)\|_\infty \le C\Big(\int_s^t \chi(\sigma_u)^{2/\nu}\,du\Big)^{-\nu/2}.$$

Proof. Let L_0 be as defined in (3.3). From the Nash inequality ([17])

$$\|f\|_2^{2+4/\nu} \le -C(L_0 f, f)\|f\|_1^{4/\nu}$$

($\nu \in \mathbb{R}^+$is arbitrary such that $d \le \nu \le D$, where d and D denote the local dimension and the dimension at infinity of N respectively) applied to $f \circ \mathrm{Ad}(a)$ it follows that

$$\chi(a)^{2/\nu}\|f\|_2^{2+4/\nu} \le -C(L_a f, f)\|f\|_1^{4/\nu}.$$

For a function $0 \le f \in C_c^\infty(N)$ such that $\int f = 1$ we define

$$f_s(x) = f * p^\sigma(t,s)(x), \qquad h_s(x) = \|f_s\|_2^2.$$

From this, $\|f\|_1 = 1$, and the fact that $\partial_s f_s = -L_{\sigma_s} f_s$, using the Nash inequality, we can write

$$\begin{aligned}-\partial_s h_s &= -\partial_s(f_s, f_s) = 2(L_{\sigma_s} f_s, f_s)\\ &\le -2C^{-1}\chi(\sigma_s)^{2/\nu}\|f_s\|_2^{2(1+2/\nu)} = -C\chi(\sigma_s)^{2/\nu}h_s^{1+2/\nu}.\end{aligned}$$

We solve this differential inequality finding

$$\partial_s(h_s^{-2/\nu}) \le -C\chi(\sigma_s)^{2/\nu},$$

$$h_t^{-2/\nu} - h_s^{-2/\nu} \le -C\int_s^t \chi(\sigma_r)^{2/\nu}\,dr.$$

Hence

$$h_s \le C\Big(\int_s^t \chi(\sigma_r)^{2/\nu}\,dr\Big)^{-\nu/2}.$$

Replacing f with $f/\|f\|_1$ shows

$$\|f * p^\sigma(t,s)\|_2^2 \le C\Big(\int_s^t \chi(\sigma_r)^{2/\nu}\,dr\Big)^{-\nu/2}\|f\|_1^2,$$

$$\|p^\sigma(t,s)\|_2^2 \le C\Big(\int_s^t \chi(\sigma_r)^{\frac{2}{\nu}}\,dr\Big)^{-\nu/2}.$$

Hence, for $s < u < t$,

$$\begin{aligned}\|p^\sigma(t,s)\|_\infty &= \|p^\sigma(t,u) * p^\sigma(u,s)\|_\infty \le \|p^\sigma(t,u)\|_2\|p^\sigma(u,s)\|_2\\ &\le C\Big(\int_u^t \chi(\sigma_r)^{2/\nu}dr\Big)^{-\nu/4}\Big(\int_s^u \chi(\sigma_r)^{2/\nu}dr\Big)^{-\nu/4}.\end{aligned}$$

We choose u so that $\int_u^t = \int_s^u$, concluding that

$$\|p^\sigma(t,s)\|_\infty \le C\Big(\int_s^t \chi(\sigma_r)^{2/\nu} dr\Big)^{-\nu/2}. \ \blacksquare$$

Proof of Theorem 3.2. By the subadditivity of τ, the property $p^\sigma(t,r) * p^\sigma(r,s) = p^\sigma(t,s)$, Proposition A.6, and Theorem A.4 we have

$$\begin{aligned}
p^\sigma(0,t,x)e^{\alpha\tau(x)} &= p^\sigma(0,s) * p^\sigma(s,t)(x)e^{\alpha\tau(x)}\\
&\le \|p^\sigma(0,s)(\cdot)e^{\alpha\tau(\cdot)}\|_2\, \|p^\sigma(s,t)(\cdot)e^{\alpha\tau(\cdot)}\|_2\\
&\le \|p^\sigma(0,s)\|_\infty^{1/2}\|p^\sigma(s,t)\|_\infty^{1/2}(p^\sigma(0,s),e^{2\alpha\tau})^{1/2}(p^\sigma(s,t),e^{2\alpha\tau})^{1/2}\\
&\le C'\Big(\int_0^s \chi(\sigma_u)^{2/\nu}\,du\Big)^{-\nu/4}\Big(\int_s^t \chi(\sigma_u)^{2/\nu}\,du\Big)^{-\nu/4}\\
&\qquad\cdot e^{4\alpha r}\exp(C(\alpha+\alpha^2)A^\sigma(0,s))\exp(C(\alpha+\alpha^2)A^\sigma(s,t))\\
&= C'\Big(\int_0^s \chi(\sigma_u)^{2/\nu}\,du\Big)^{-\nu/4}\Big(\int_s^t \chi(\sigma_u)^{2/\nu}\,du\Big)^{-\nu/4}\\
&\qquad\cdot e^{4\alpha r}\exp(C(\alpha+\alpha^2)A^\sigma(0,t)).
\end{aligned}$$

Now choosing s analogously to u in the proof of Proposition A.6 we get

$$p^\sigma(0,t,x)e^{\alpha\tau(x)} \le C'e^{4\alpha r}\exp(C(\alpha+\alpha^2)A^\sigma(0,t))\Big(\int_0^t \chi(\sigma_u)^{2/\nu}\,du\Big)^{-\nu/2},$$

$$p^\sigma(0,t,x) \le C'e^{[4r-\tau(x)+CA^\sigma(0,t)]\alpha+CA^\sigma(0,t)\alpha^2}\Big(\int_0^t \chi(\sigma_u)^{2/\nu}\,du\Big)^{-\nu/2}.$$

Now, let $K \subset N$ and $e \notin K$. Choose $r < \inf_{x\in K}\tau(x)/16$ and $\varepsilon = 1/(4C_r)$ where C_r is as in Theorem A.4. Let

$$\alpha = \varepsilon\tau(x)/A^\sigma(0,t).$$

Then

$$\begin{aligned}
&C_r(\alpha+\alpha^2)A^\sigma(0,t) + 4\alpha r - \alpha\tau(x)\\
&\quad= C_r\Big(\frac{\varepsilon\tau(x)}{A^\sigma(0,t)} + \frac{\varepsilon^2\tau(x)^2}{A^\sigma(0,t)^2}\Big)A^\sigma(0,t) + \frac{4r\varepsilon\tau(x)}{A^\sigma(0,t)} - \frac{\varepsilon\tau(x)^2}{A^\sigma(0,t)}\\
&\quad= C_r\varepsilon\tau(x) + \frac{C_r\varepsilon^2\tau(x)^2 + 4\varepsilon r\tau(x) - \varepsilon\tau(x)^2}{A^\sigma(0,t)}\\
&\quad= \frac{\tau(x)}{4} + \frac{2C_r\varepsilon^2\tau(x)^2 + 8\varepsilon r\tau(x) - \varepsilon\tau(x)^2}{2A^\sigma(0,t)} - \frac{\varepsilon\tau(x)^2}{2A^\sigma(0,t)}.
\end{aligned}$$

The middle term is ≤ 0 since

$$2C_r\varepsilon^2\tau(x)^2 = \frac{\varepsilon\tau(x)^2}{2} \quad \text{and} \quad 8\varepsilon r\tau(x) \leq \frac{8\varepsilon\tau(x)^2}{16},$$

and the proof is finished. ∎

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