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THREE RESULTS IN DUNKL ANALYSIS

BY

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In memory of Andrzej Hulanicki (1933–2008), a distinguished Polish mathematician, a guide and a friend, who has left many orphans in Wrocław and around the world. We miss you.

Abstract. We first establish a geometric Paley–Wiener theorem for the Dunkl transform in the crystallographic case. Next we obtain an optimal bound for the $L^p \to L^p$ norm of Dunkl translations in dimension 1. Finally, we describe more precisely the support of the distribution associated to Dunkl translations in higher dimension.

1. Introduction. Dunkl theory generalizes classical Fourier analysis on $\mathbb{R}^N$. It started twenty years ago with Dunkl's seminal work [5] and was further developed by several mathematicians. See for instance the surveys [14, 6] and the references cited therein.

In this setting, the Paley–Wiener theorem is known to hold for balls centered at the origin. In [8], a Paley–Wiener theorem was conjectured for convex neighborhoods of the origin, which are invariant under the underlying reflection group, and was partially proved. Our first result in Section 3 is a proof of this conjecture in the crystallographic case, following the third approach in [8].

Generalized translations were introduced in [13] and further studied in [18, 15, 17]. Apart from their abstract definition, we lack precise information, in particular about their integral representation

$$(\tau_x f)(y) = \int_{\mathbb{R}^N} f(z)\, d\gamma_{x,y}(z),$$

which was conjectured in [13] and established in few cases, for instance in dimension $N = 1$ or when f is radial. Our second result in Section 4 is an optimal bound for the integral

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$$\int_{\mathbb{R}} |\, d\gamma_{x,y}(z)|$$

in dimension $N = 1$, improving upon earlier results in [12, 17]. Our bound depends on the multiplicity $k \geq 0$ and tends from below to $\sqrt{2}$ as $k \to +\infty$. Our third result in Section 5 deals with the support of the distribution $\gamma_{x,y}$ in higher dimension, which we determine rather precisely in the crystallographic case.

2. Background. In this section, we recall some notations and results in Dunkl theory; for more details we refer to the articles [5, 7] and surveys [14, 6].

Let $G \subset \mathrm{O}(\mathbb{R}^N)$ be a finite reflection group associated to a reduced root system R, and $k : R \to [0, +\infty)$ a G-invariant function (called *multiplicity function*). Let R^+ be a positive root subsystem, Γ_+ the corresponding open positive chamber, $\overline{\Gamma_+}$ its closure, $\overline{\Gamma^+} = \sum_{\alpha \in R^+} \mathbb{R}_+ \alpha$ the dual cone, and denote by x_+ the intersection point of any orbit $G.x$ in $\mathbb{R}^N$ with $\overline{\Gamma_+}$.

The *Dunkl operators* T_ξ on $\mathbb{R}^N$ are the following k-deformations of the directional derivatives ∂_ξ by difference operators:

$$T_\xi f(x) = \partial_\xi f(x) + \sum_{\alpha \in R^+} k(\alpha) \langle \alpha, \xi \rangle \, \frac{f(x) - f(\sigma_\alpha . x)}{\langle \alpha, x \rangle},$$

where

$$\sigma_\alpha . x = x - \frac{\langle \alpha, x \rangle}{2 |\alpha|^2} \, \alpha$$

denotes the reflection with respect to the hyperplane orthogonal to α. The Dunkl operators are skewsymmetric with respect to the measure $w(x)\, dx$ with density

$$w(x) = \prod_{\alpha \in R^+} |\langle \alpha, x \rangle|^{2k(\alpha)}.$$

The operators ∂_ξ and T_ξ are intertwined by a Laplace-type operator

$$V f(x) = \int_{\mathbb{R}^N} f(y) \, d\mu_x(y) \tag{1}$$

associated to a family of compactly supported probability measures $\{\mu_x \mid x \in \mathbb{R}^N\}$. Specifically, μ_x is supported in the convex hull

$$C^x = \mathrm{co}(G.x).$$

For every $\lambda \in \mathbb{C}^N$, the simultaneous eigenfunction problem

$$T_\xi f = \langle \lambda, \xi \rangle f \quad \forall \xi \in \mathbb{R}^N$$

has a unique solution $f(x) = E(\lambda, x)$ such that $E(\lambda, 0) = 1$, which is given by

$$E(\lambda, x) = V(e^{\langle \lambda, \cdot \rangle})(x) = \int_{\mathbb{R}^N} e^{\langle \lambda, y \rangle} \, d\mu_x(y) \quad \forall x \in \mathbb{R}^N. \tag{2}$$

Furthermore, $\lambda \mapsto E(\lambda, x)$ extends to a holomorphic function on $\mathbb{C}^N$ and the following estimate holds:

$$|E(\lambda, x)| \le e^{\langle (\operatorname{Re} \lambda)_+, \, x_+ \rangle} \quad \forall \lambda \in \mathbb{C}^N, \, \forall x \in \mathbb{R}^N.$$

In dimension $N = 1$, these functions can be expressed in terms of Bessel functions. Specifically,

$$E(\lambda, x) = j_{k-1/2}(\lambda x) + \frac{\lambda x}{2k+1} \, j_{k+1/2}(\lambda x),$$

where

$$j_\nu(z) = \Gamma(\nu + 1) \sum_{n=0}^{+\infty} \frac{(-1)^n}{n! \Gamma(\nu + n + 1)} \left(\frac{z}{2} \right)^{2n}$$

are normalized Bessel functions.

The *Dunkl transform* is defined on $L^1(\mathbb{R}^N, w(x)dx)$ by

$$\mathcal{D}f(\xi) = \frac{1}{c} \int_{\mathbb{R}^N} f(x) E(-i\xi, x) w(x) \, dx,$$

where

$$c = \int_{\mathbb{R}^N} e^{-|x|^2/2} w(x) \, dx.$$

We list some known properties of this transform:

(i) The Dunkl transform is a topological automorphism of the Schwartz space $\mathcal{S}(\mathbb{R}^N)$.

(ii) (*Plancherel Theorem*) The Dunkl transform extends to an isometric automorphism of $L^2(\mathbb{R}^N, w(x)dx)$.

(iii) (*Inversion formula*) For every $f \in \mathcal{S}(\mathbb{R}^N)$, and more generally for every $f \in L^1(\mathbb{R}^N, w(x)dx)$ such that $\mathcal{D}f \in L^1(\mathbb{R}^N, w(\xi)d\xi)$, we have

$$f(x) = \mathcal{D}^2 f(-x) \quad \forall x \in \mathbb{R}^N.$$

(iv) (*Paley–Wiener theorem*) The Dunkl transform is a linear isomorphism between the space of smooth functions f on $\mathbb{R}^N$ with $\operatorname{supp} f \subset \overline{B(0, R)}$ and the space of entire functions h on $\mathbb{C}^N$ such that

$$\sup_{\xi \in \mathbb{C}^N} (1 + |\xi|)^M e^{-R|\operatorname{Im} \xi|} \, |h(\xi)| < +\infty \quad \forall M \in \mathbb{N}. \tag{3}$$

3. A geometric Paley–Wiener theorem. In this section, we prove a geometric version of the Paley–Wiener theorem, which was looked for in [8, 18, 9], under the assumption that G is crystallographic. The proof consists merely in resuming the third approach in [8] and applying it to the convex sets considered in [1, 2, 3, 4] instead of the convex sets considered in [10]. Recall that the second family consists of the convex hulls

$$C^{\Lambda} = \operatorname{co}(G.\Lambda)$$

of G-orbits $G.\Lambda$ in $\mathbb{R}^N$, while the first family consists of the polar sets

$$C_{\Lambda} = \{x \in \mathbb{R}^N \mid \langle x, g.\Lambda\rangle \leq 1 \ \forall g \in G\}.$$

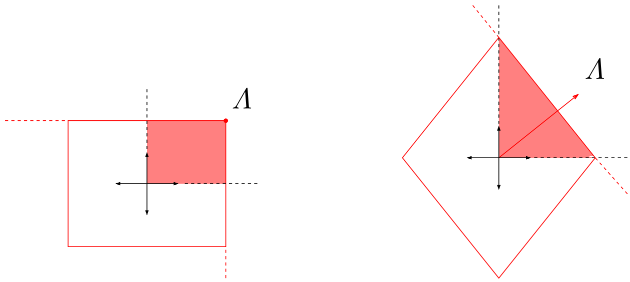

Fig. 1. The sets C^{Λ} and C_{Λ} for the root system $A_1 \times A_1$

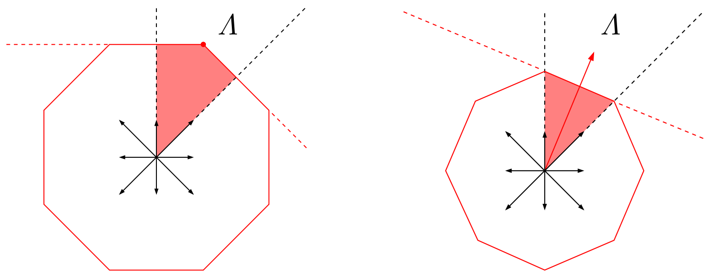

Fig. 2. The sets C^{Λ} and C_{Λ} for the root system B_2

Before stating the geometric Paley–Wiener theorem, let us make some remarks about the sets C^{Λ} and C_{Λ}. Firstly, they are convex, closed, G-invariant and the following inclusion holds:

$$C^{\Lambda} \subset |\Lambda|^2 C_{\Lambda}.$$

Secondly, we may always assume that $\Lambda = \Lambda_+$ belongs to the closed positive chamber $\overline{\Gamma_+}$ and, in this case, we have

$$C^{\Lambda} \cap \overline{\Gamma_+} = \overline{\Gamma_+} \cap (\Lambda - \overline{\Gamma^+}), \qquad C_{\Lambda} \cap \overline{\Gamma_+} = \{x \in \overline{\Gamma_+} \mid \langle \Lambda, x\rangle \leq 1\}.$$

Thirdly, on one hand, every G-invariant convex subset in $\mathbb{R}^N$ is a union of sets C^{Λ} while, on the other hand, every G-invariant closed convex subset in

$\mathbb{R}^N$ is an intersection of sets C_Λ. For instance,

$$\overline{B(0,R)} = \bigcup_{|\Lambda|=R} C^\Lambda = \bigcap_{|\Lambda|=R^{-1}} C_\Lambda.$$

Fourthly, we shall say that $\Lambda \in \overline{\Gamma_+}$ is *admissible* if the following equivalent conditions are satisfied:

(i) Λ has nonzero projections in each irreducible component of $(\mathbb{R}^N, R)$,
(ii) C^Λ is a neighborhood of the origin,
(iii) C_Λ is bounded.

In this case, we may consider the gauge

$$\chi_\Lambda(\xi) = \max_{x\in C_\Lambda} \langle x, \xi\rangle = \min\{r \in [0, +\infty) \mid \xi \in rC^\Lambda\}$$

on $\mathbb{R}^N$.

THEOREM 3.1. *Assume that $\Lambda \in \overline{\Gamma_+}$ is admissible. Then the Dunkl transform is a linear isomorphism between the space of smooth functions f on $\mathbb{R}^N$ with* $\operatorname{supp} f \subset C_\Lambda$ *and the space of entire functions h on $\mathbb{C}^N$ such that*

$$\sup_{\xi\in\mathbb{C}^N} (1 + |\xi|)^M e^{-\chi_\Lambda(\operatorname{Im}\xi)} |h(\xi)| < +\infty \qquad \forall M \in \mathbb{N}. \tag{4}$$

Proof. Following [8], this theorem is first proved in the trigonometric case, which explains the restriction to crystallographic groups, and next obtained in the rational case by passing to the limit. The proof in the trigonometric case is similar to the proof of the Paley–Wiener Theorem in [10, 11], and actually to the initial proof of Helgason for the spherical Fourier transform on symmetric spaces of the noncompact type. This was already observed in [16] and will be developed below for the reader's convenience. The limiting procedure is described thoroughly in [8] and needs no further explanation.

Thus assume that h is an entire function on $\mathbb{C}^N$ satisfying (4) and, by resuming the proof of [10, Theorem 8.6(2)], let us show that its inverse Cherednik transform

$$f(x) = \text{const} \cdot \int_{\mathbb{R}^N} h(\xi) \widetilde{E}(i\xi, x) \widetilde{w}(\xi)\, d\xi \tag{5}$$

vanishes outside C_Λ. Firstly, one may restrict by G-equivariance to $x = g_0.x_+$, where $x_+ \in \Gamma_+ \setminus C_\Lambda$ and g_0 denotes the longest element in G, which interchanges the chambers Γ_+ and $-\Gamma_+$. Secondly, by expanding

$$\Big\{ \prod_{\alpha\in R^+} (\langle\check{\alpha}, \xi\rangle - k_\alpha) \Big\} \widetilde{E}(\xi, x) = \sum_{g\in G} \sum_{q\in Q^+} \mathbf{c}(-g.\xi) \widetilde{E}_q(g, g.\xi) e^{\langle g.\xi + \varrho + q, x\rangle}$$

(5) becomes

$$f(x) = \text{const} \cdot \sum_{g \in G} \det g \sum_{q \in Q^+} f_{g,q}(x) e^{\langle \varrho + q, x \rangle},$$

where

$$f_{g,q}(x) = \int_{\mathbb{R}^N} h(g^{-1}.\xi) \widetilde{E}_q(g, i\xi) e^{i\langle \xi, x \rangle} \left\{ \prod_{\alpha \in R^+} \frac{\Gamma(i\langle \check{\alpha}, \xi \rangle + k_\alpha)}{\Gamma(i\langle \check{\alpha}, \xi \rangle + 1)} \right\} d\xi. \tag{6}$$

Thirdly, one shows that all expressions (6) vanish, by shifting the contour of integration from $\mathbb{R}^N$ to $\mathbb{R}^N + itg_0.\Lambda$ with $t > 0$, which produces an exponential factor e^{-ct} with $c = \langle \Lambda, x_+ \rangle - 1 > 0$, and by letting $t \to +\infty$. ■

Since every G-invariant convex compact neighborhood of the origin in $\mathbb{R}^N$ is the intersection of admissible sets C_Λ, Theorem 3.1 generalizes as follows.

COROLLARY 3.2 (Geometric Paley–Wiener Theorem). *Let C be a G-invariant convex compact neighborhood of the origin in $\mathbb{R}^N$ and $\chi(\xi) = \max_{x \in C} \langle x, \xi \rangle$ the dual gauge. Then the Dunkl transform is a linear isomorphism between the space $\mathcal{C}^\infty_C(\mathbb{R}^N)$ of smooth functions f on $\mathbb{R}^N$ with $\operatorname{supp} f \subset C$ and the space $\mathcal{H}_\chi(\mathbb{C}^N)$ of entire functions h on $\mathbb{C}^N$ such that*

$$\sup_{\xi \in \mathbb{C}^N} (1 + |\xi|)^M e^{-\chi(\operatorname{Im} \xi)} |h(\xi)| < +\infty \quad \forall M \in \mathbb{N}.$$

REMARK 3.3. Notice that the Dunkl transform $\mathcal{D}$ always maps $\mathcal{C}^\infty_C(\mathbb{R}^N)$ into $\mathcal{H}_\chi(\mathbb{C}^N)$ and that the assumption that G is crystallographic is only used to prove that $\mathcal{D}$ is onto.

4. L^p bounds for generalized translations in dimension 1. *Dunkl translations* are defined on $\mathcal{S}(\mathbb{R}^N)$ by

$$(\tau_x f)(y) = \frac{1}{c} \int_{\mathbb{R}^N} \mathcal{D}f(\xi)\, E(i\xi, x) E(i\xi, y) w(\xi)\, d\xi \quad \forall x, y \in \mathbb{R}^N.$$

They have an explicit integral representation [12] in dimension $N = 1$:

$$(\tau_x f)(y) = \int_{\mathbb{R}} f(z)\, d\gamma_{x,y}(z),$$

where

$$d\gamma_{x,y}(z) = \begin{cases} \gamma(x,y,z)|z|^{2k}\, dz & \text{if } x, y \in \mathbb{R}^*, \\ d\delta_y(z) & \text{if } x = 0, \\ d\delta_x(z) & \text{if } y = 0, \end{cases} \tag{7}$$

is a signed measure such that $\int_{\mathbb{R}} d\gamma_{x,y}(z) = 1$. Specifically,

$$\gamma(x,y,z) = d\sigma(x,y,z)\rho(|x|,|y|,|z|)\mathbb{1}_{I_{|x|,|y|}}(|z|) \qquad \forall x,y,z \in \mathbb{R}^*,$$

where

$$\begin{aligned} d &= \frac{\Gamma(k+1/2)}{\sqrt{\pi}\Gamma(k)}, \\ \sigma(x,y,z) &= 1 - \frac{x^2+y^2-z^2}{2xy} + \frac{z^2+y^2-x^2}{2zy} + \frac{x^2+z^2-y^2}{2xz} \\ &= \frac{(z+x+y)(z+x-y)(z-x+y)}{2xyz} \qquad \forall x,y,z \in \mathbb{R}^*, \\ \rho(a,b,c) &= \frac{\{c^2-(a-b)^2\}^{k-1}\{(a+b)^2-c^2\}^{k-1}}{(2abc)^{2k-1}} \\ &= \frac{(2b^2c^2+2a^2c^2+2a^2b^2-a^4-b^4-c^4)^{k-1}}{(2abc)^{2k-1}} \qquad \forall a,b,c > 0, \end{aligned}$$

and $I_{a,b}$ denotes the interval $[|a-b|, a+b]$. Notice the symmetries

$$\gamma(x,y,z) = \begin{cases} \gamma(y,x,z), \\ \gamma(-x,-y,-z), \\ \gamma(-z,y,-x) = \gamma(x,-z,-y). \end{cases} \tag{8}$$

PROPOSITION 4.1. *The following inequality holds, for every $x, y \in \mathbb{R}$:*

$$\int_{\mathbb{R}} |d\gamma_{x,y}(z)| \le A_k = \sqrt{2}\,\frac{\{\Gamma(k+1/2)\}^2}{\Gamma(k+1/4)\Gamma(k+3/4)}. \tag{9}$$

Actually there is equality if $x = y \in \mathbb{R}^*$. Moreover $A_k \overset{\le}{\to} \sqrt{2}$ as $k \to +\infty$.

REMARK 4.2. This result improves earlier bounds obtained in [12] and [17], which were respectively 4 and 3.

Proof. Let $x, y \in \mathbb{R}^*$.

CASE 1: $xy < 0$. Then $\big||x|-|y|\big| = |x+y|$ and $|x|+|y| = |x-y|$, hence

$$\sigma(x,y,z)\mathbb{1}_{I_{|x|,|y|}}(|z|) = \frac{z+x+y}{z}\,\frac{(x-y)^2-z^2}{-2xy}\mathbb{1}_{I_{|x|,|y|}}(|z|)$$

and $\gamma_{x,y}$ are positive. Thus

$$\int_{\mathbb{R}} |d\gamma_{x,y}(z)| = \int_{\mathbb{R}} d\gamma_{x,y}(z) = 1.$$

Case 2: $xy > 0$. By symmetry, we may reduce to $0 < x \le y$. Then

$$\int_{\mathbb{R}} |d\gamma_{x,y}(z)| = \int_{-\infty}^{0} |d\gamma_{x,y}(z)| + \int_{0}^{+\infty} |d\gamma_{x,y}(z)|$$
$$= 2d \int_{y-x}^{y+x} \frac{x+y}{2xyz} \left(\frac{z^2 - x^2 - y^2 + 2xy}{2xyz} \right)^k \left(\frac{x^2 + y^2 + 2xy - z^2}{2xyz} \right)^{k-1} z^{2k}\, dz.$$

After performing the change of variables $z = \sqrt{x^2 + y^2 - 2xy\cos\theta}$ and setting $y = sx$, we get

$$\int_{\mathbb{R}} |d\gamma_{x,y}(z)| = \frac{\Gamma(k+1/2)}{\sqrt{\pi}\Gamma(k)} (1+s) \int_0^\pi \frac{(1-\cos\theta)\sin^{2k-1}\theta}{\sqrt{1+s^2-2s\cos\theta}}\, d\theta. \tag{10}$$

Denote by $F(s)$ the right hand side of (10). Since

$$F'(s) = \frac{\Gamma(k+1/2)}{\sqrt{\pi}\Gamma(k)} (1-s) \int_0^\pi \frac{\sin^{2k+1}\theta}{(1+s^2-2s\cos\theta)^{3/2}}\, d\theta$$

is nonpositive, $F(s)$ is a decreasing function on $[1, +\infty)$, which reaches its maximum at $s = 1$. Let us compute it:

$$\begin{aligned}
A_k = F(1) &= \frac{\sqrt{2}\,\Gamma(k+1/2)}{\sqrt{\pi}\,\Gamma(k)} \int_0^\pi (1-\cos\theta)^{k-1/2}(1+\cos\theta)^{k-1} \sin\theta\, d\theta \\
&= 2^{2k} \frac{\Gamma(k+1/2)}{\sqrt{\pi}\Gamma(k)} \int_0^1 t^{k-1/2}(1-t)^{k-1}\, dt \\
&= 2^{2k} \frac{\Gamma(k+1/2)}{\sqrt{\pi}\Gamma(k)} B(k+1/2, k) = 2^{2k} \frac{\{\Gamma(k+1/2)\}^2}{\sqrt{\pi}\Gamma(2k+1/2)} \\
&= \sqrt{2}\, \frac{\Gamma(k+1/2)}{\Gamma(k+1/4)} \frac{\Gamma(k+1/2)}{\Gamma(k+3/4)},
\end{aligned}$$

after performing the change of variables $t = (1-\cos\theta)/2$ and using standard properties of the beta and gamma functions.

Finally, let us show that $A_k \overset{\le}{\to} \sqrt{2}$ as $k \to +\infty$. Write

$$A_k = \sqrt{2}\, \frac{G(k+1/4)}{G(k+1/2)}, \quad \text{where} \quad G(u) = \frac{\Gamma(u+1/4)}{\Gamma(u)} \quad \forall u > 0.$$

Since the logarithmic derivative Γ'/Γ is a strictly increasing analytic function on $(0, +\infty)$, the logarithmic derivative

$$\frac{G'(u)}{G(u)} = \frac{\Gamma'(u+1/4)}{\Gamma(u+1/4)} - \frac{\Gamma'(u)}{\Gamma(u)}$$

is positive. Hence G is strictly increasing and $A_k < \sqrt{2}$. On the other hand, using Stirling's formula

$$\Gamma(u) \sim \sqrt{2\pi}\, u^{u-1/2} e^{-u} \quad \text{as } u \to +\infty,$$

we get $G(k+1/4) \sim G(k+1/2)$, hence $A_k \to \sqrt{2}$ as $k \to +\infty$. ∎

As a first consequence, we obtain the $L^1 \to L^1$ operator norm of Dunkl translations in dimension $N = 1$.

COROLLARY 4.3. *Let $x \in \mathbb{R}^*$. Then τ_x is a bounded operator on $L^1(\mathbb{R}, |x|^{2k}dx)$, with $\|\tau_x\|_{L^1\to L^1} = A_k$.*

Proof. The inequality $\|\tau_x\|_{L^1\to L^1} \le A_k$ follows from (9), together with (8), and it remains to prove the reverse inequality. By symmetry, we may assume that $x > 0$. Since

$$A_k = \lim_{y\to x} \int_{\mathbb{R}} |\gamma(x,y,z)|\, |z|^{2k}\, dz,$$

for every $0 < \varepsilon < A_k$ there exists $0 < \eta < x$ such that, for every $y \in [x-\eta, x+\eta]$,

$$\int_{\mathbb{R}} |\gamma(x,y,z)|\, |z|^{2k}\, dz > A_k - \varepsilon. \tag{11}$$

Let f be a nonnegative measurable function on $\mathbb{R}$ such that

$$\operatorname{supp} f \subset [-x-\eta, -x+\eta] \quad \text{and} \quad \|f\|_{L^1} = \int_{\mathbb{R}} f(z)|z|^{2k}\, dz = 1.$$

Since

$$\begin{cases} \gamma(x,y,z) \ge 0 & \forall y < 0,\, \forall z < 0, \\ \gamma(x,y,z) \le 0 & \forall y > 0,\, \forall z < 0, \end{cases}$$

we have

$$|(\tau_x f)(y)| = \int_{-x-\eta}^{-x+\eta} f(z)|\gamma(x,y,z)|\, |z|^{2k}\, dz.$$

Hence, using (8) and (11), we find that

$$\begin{aligned} \|\tau_x f\|_{L^1} &= \int_{\mathbb{R}} |(\tau_x f)(y)|\, |y|^{2k}\, dy \\ &= \int_{-x-\eta}^{-x+\eta} \Big\{ \int_{\mathbb{R}} |\gamma(x,-z,-y)|\, |y|^{2k}\, dy \Big\} f(z)|z|^{2k}\, dz \end{aligned}$$

is bounded from below by $A_k - \varepsilon$. Consequently, $\|\tau_x\|_{L^1\to L^1} \ge A_k - \varepsilon$ and we conclude by letting $\varepsilon \to 0$. ∎

Let us next compute the $L^2 \to L^2$ operator norm of Dunkl translations.

LEMMA 4.4. *Let $x \in \mathbb{R}$. Then τ_x is a bounded operator on $L^2(\mathbb{R}, |x|^{2k}dx)$, with $\|\tau_x\|_{L^2\to L^2} = 1$.*

Proof. The proof is straightforward, via the Plancherel formula, and generalizes to higher dimensions. On one hand, the inequality $\|\tau_x\|_{L^2\to L^2} \le 1$ follows from the estimate $|E(i\xi,x)| \le 1$. On the other hand, let

$$f_\varepsilon(x) = \varepsilon^{k+1/2} f(\varepsilon x)$$

be a rescaled normalized function in $L^2(\mathbb{R}, |x|^{2k}dx)$. Then

$$\|f_\varepsilon\|_{L^2} = \|f\|_{L^2} = 1$$

while

$$\begin{aligned}\|\tau_x f_\varepsilon\|^2_{L^2} &= \int_{\mathbb{R}} |E(i\xi,x)|^2 \varepsilon^{-2k-1} |\mathcal{D}f(\varepsilon^{-1}\xi)|^2\, |\xi|^{2k}\, d\xi \\ &= \int_{\mathbb{R}} |E(i\varepsilon\xi,x)|^2\, |\mathcal{D}f(\xi)|^2 |\xi|^{2k}\, d\xi\end{aligned}$$

tends to

$$\int_{\mathbb{R}} |\mathcal{D}f(\xi)|^2 |\xi|^{2k}\, d\xi = \|f\|^2_{L^2} = 1$$

as $\varepsilon \to 0$. This concludes the proof of the lemma. ∎

Eventually, Corollary 4.3 and Lemma 4.4 imply the following result, by interpolation and duality.

COROLLARY 4.5. *Let $x \in \mathbb{R}$ and $1 \le p \le \infty$. Then τ_x is a bounded operator on $L^p(\mathbb{R}, |x|^{2k}dx)$, with $\|\tau_x\|_{L^p\to L^p} \le A_k^{2|1/p-1/2|}$.*

REMARK 4.6. In the product case, where $G = \mathbb{Z}_2^N$ acts on $\mathbb{R}^N$, we have

$$\|\tau_x\|_{L^p\to L^p} \le A_k^{2|1/p-1/2|N}$$

for every $x \in \mathbb{R}^N$ and $1 \le p \le \infty$.

5. A support theorem for generalized translations. As mentioned in the introduction, we lack information about Dunkl translations in general. In this section, we locate more precisely the support of the distribution

$$\langle \gamma_{x,y}, f\rangle = (\tau_x f)(y),$$

which is known [18] to be contained in the closed ball of radius $|x| + |y|$.

THEOREM 5.1.

(i) *The distribution $\gamma_{x,y}$ is supported in the spherical shell*

$$\big\{z \in \mathbb{R}^N \,\big|\, \big||x| - |y|\big| \le |z| \le |x| + |y|\big\}.$$

(ii) *If G is crystallographic, then the support of $\gamma_{x,y}$ is contained in*

$$\{z \in \mathbb{R}^N \mid z_+ \preccurlyeq x_+ + y_+,\ z_+ \succcurlyeq y_+ + g_0.x_+ \text{ and } z_+ \succcurlyeq x_+ + g_0.y_+\}.$$

Here $\preccurlyeq$ denotes the partial order on $\mathbb{R}^N$ associated to the cone $\overline{\Gamma^+}$:

$$a \preccurlyeq b \ \Leftrightarrow\ b - a \in \overline{\Gamma^+},$$

and g_0 is the longest element in G, which interchanges the chambers Γ_+ and $-\Gamma_+$.

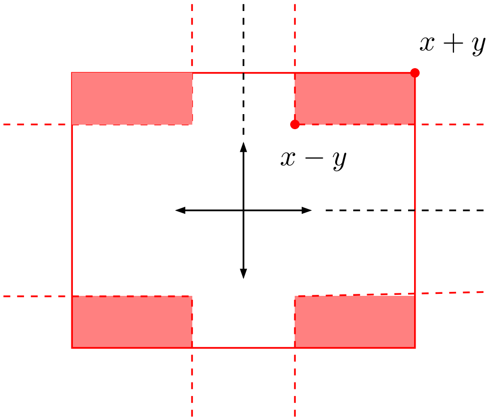

Fig. 3. Support of $\gamma_{x,y}$ for the root system $A_1 \times A_1$

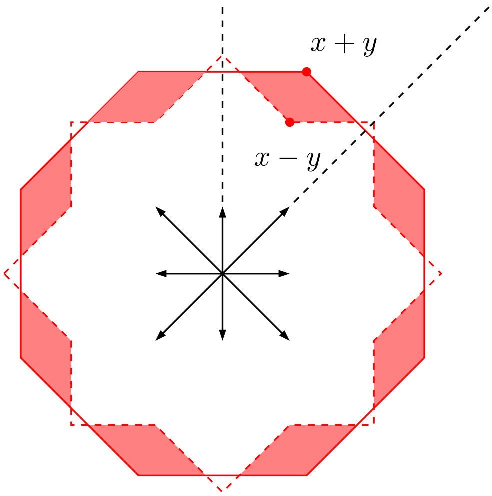

Fig. 4. Support of $\gamma_{x,y}$ for the root system B_2

Proof. Let $h \in \mathcal{C}_c^\infty(\mathbb{R}^N)$ be an auxiliary radial function such that

$$\int\limits_{\mathbb{R}^N} h(x)w(x)\,dx = 1$$

and $\operatorname{supp} h \subset -\operatorname{co}(G.u)$, where $u \in \Gamma_+$ is a unit vector. For every $\varepsilon > 0$ and $x, y, z \in \mathbb{R}^N$, set

$$\gamma_\varepsilon(x,y,z)=\frac{1}{c^2}\int\limits_{\mathbb{R}^N}\mathcal{D}h(\varepsilon\xi)\,E(i\xi,x)E(i\xi,y)E_k(-i\xi,z)w(\xi)\,d\xi.$$

Firstly, according to (3) and (2),

$$\xi\mapsto\mathcal{D}h(\varepsilon\xi)\,E(i\xi,x)E(i\xi,y)$$

is an entire function on $\mathbb{C}^N$ satisfying

$$(12)\qquad |\mathcal{D}h(\varepsilon\xi)\,E(i\xi,x)E(i\xi,y)|\le C_M(1+|\xi|)^{-M}e^{-\langle g_0.(x_++y_++\varepsilon u),(\operatorname{Im}\xi)_+\rangle}.$$

Secondly,

$$\begin{aligned}\langle\gamma_{x,y},f\rangle&=\frac{1}{c}\int\limits_{\mathbb{R}^N}\mathcal{D}f(\xi)\,E(i\xi,x)E(i\xi,y)w(\xi)\,d\xi\\&=\lim_{\varepsilon\to0}\frac{1}{c}\int\limits_{\mathbb{R}^N}\mathcal{D}h(\varepsilon\xi)\,\mathcal{D}f(\xi)\,E(i\xi,x)E(i\xi,y)w(\xi)\,d\xi\\&=\lim_{\varepsilon\to0}\int\limits_{\mathbb{R}^N}f(z)\gamma_\varepsilon(x,y,z)w(z)\,dz,\end{aligned}$$

i.e. the distribution $\gamma_{x,y}$ is the weak limit of the measures $\gamma_\varepsilon(x,y,z)w(z)\,dz$. Thirdly, notice the symmetries

$$(13)\qquad \gamma_\varepsilon(x,y,z)=\begin{cases}\gamma_\varepsilon(y,x,z),\\\gamma_\varepsilon(g.x,g.y,g.z)\quad\forall g\in G\cup\{-\mathrm{Id}\},\\\gamma_\varepsilon(-z,y,-x)=\gamma_\varepsilon(x,-z,-y).\end{cases}$$

If G is crystallogaphic, we use Corollary 3.2 (actually the third version of the Paley–Wiener theorem in [8]), and deduce from (12) that the function $z\mapsto\gamma_\varepsilon(x,y,z)$ is supported in

$$\operatorname{co}\{G.(x_++y_++\varepsilon u)\}=\operatorname{co}(G.x)+\operatorname{co}(G.y)+\varepsilon\operatorname{co}(G.u).$$

Equivalently,

$$\gamma_\varepsilon(x,y,z)\neq0\;\Rightarrow\;z_+\prec x_++y_++\varepsilon u.$$

Using the symmetries (13), we see that $\gamma_\varepsilon(x,y,z)\neq0$ implies also

$$\begin{cases}-g_0.x_+\prec-g_0.z_++y_++\varepsilon u,\quad\text{i.e.}\quad z_+\succ x_++g_0.y_++\varepsilon g_0.u,\\-g_0.y_+\prec-g_0.z_++x_++\varepsilon u,\quad\text{i.e.}\quad z_+\succ g_0.x_++y_++\varepsilon g_0.u.\end{cases}$$

The conclusion of Theorem 5.1 in the crystallographic case is obtained by letting $\varepsilon\to0$.

If G is not crystallographic, we can only use the spherical Paley–Wiener theorem and we deduce this way that $\gamma_\varepsilon(x,y,z)\neq0$ implies

$$\begin{cases}|z|\le|x|+|y|+\varepsilon,\\|x|\le|z|+|y|+\varepsilon,\\|y|\le|x|+|z|+\varepsilon,\end{cases}$$

hence

$$\big||x|-|y|\big|-\varepsilon \le |z| \le |x|+|y|+\varepsilon.$$

We conclude again by letting $\varepsilon \to 0$. ∎

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