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CYCLIC SUBSPACES FOR UNITARY REPRESENTATIONS OF LCA GROUPS; GENERALIZED ZAK TRANSFORM

BY

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To commemorate Andrzej Hulanicki

Abstract. We just published a paper showing that the properties of the shift invariant spaces, $\langle f\rangle$, generated by the translates by $\mathbb{Z}^n$ of an f in $L^2(\mathbb{R}^n)$ correspond to the properties of the spaces $L^2(\mathbb{T}^n,p)$, where the weight p equals $[\hat{f},\hat{f}]$. This correspondence helps us produce many new properties of the spaces $\langle f\rangle$. In this paper we extend this method to the case where the role of $\mathbb{Z}^n$ is taken over by locally compact abelian groups G, $L^2(\mathbb{R}^n)$ is replaced by a separable Hilbert space on which a unitary representation of G acts, and the role of $L^2(\mathbb{T}^n,p)$ is assumed by a weighted space $L^2(\widehat{G},w)$, where $\widehat{G}$ is the dual group of G. This provides many different extensions of the theory of wavelets and related methods for carrying out signal analysis.

Professor Hulanicki was a very dear friend. He was also a very good mathematician who helped me by sending several young people to work with me either as PhD students or post doctoral fellows. His work and all he has done to Polish mathematics has left a very special place in my heart for him and for the mathematical school he created. I miss him very much and am very happy to be a part of this volume dedicated to him.

Guido L. Weiss

1. Introduction. Suppose $\mathbb{H}$ is a Hilbert space and $T : g \mapsto T_g$ is a strongly continuous representation of a topological group G acting on $\mathbb{H}$ by bounded linear operators on $\mathbb{H}$. For short, we say that T is a representation of G on $\mathbb{H}$. If $\psi \in \mathbb{H}$, the *T-cyclic subspace* $\langle\psi\rangle_T$ is the closure in $\mathbb{H}$ of the linear subspace spanned by $\{T_g\psi : g \in G\}$. For the special case $\mathbb{H} = L^2(\mathbb{R}^n), G = \mathbb{Z}^n$ and $(T_k\psi)(x) = \psi(x+k)$, $k \in \mathbb{Z}^n$, $\langle\psi\rangle_T$ is often called the *principal shift invariant subspace generated by* ψ. These spaces play an

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important role in wavelet theory. In this case let

$$p_\psi(\xi) = \sum_{j\in\mathbb{Z}^n} |\hat{\psi}(\xi+j)|^2, \quad \text{where} \quad \hat{\psi}(\xi) = \int_{\mathbb{R}^n} \psi(x)e^{-2\pi i\xi\cdot x}\,dx$$

is the Fourier transform of ψ. It is well known ([9]) that $\varphi \in \langle\psi\rangle_T$ if and only if $\hat{\varphi} = m\hat{\psi}$, where m is a $\mathbb{Z}^n$-periodic function satisfying

$$\|\varphi\|^2_{L^2(\mathbb{R}^n)} = \int_{[0,1)^n} |m(\xi)|^2 p_\psi(\xi)\,d\xi < \infty. \tag{1.1}$$

It is not hard to see that the map $J_\psi : L^2([0,1)^n, p_\psi) \to L^2(\mathbb{R}^n)$, where $J_\psi m = (m\hat{\psi})^\vee$, is an isometry onto $\langle\psi\rangle_T$. Let $e_k(\xi) = e^{2\pi ik\cdot\xi}$, $k \in \mathbb{Z}^n$, $\xi \in \mathbb{T}^n = [0,1)^n$. Then $J_\psi e_k = T_k\psi$ since the Fourier transform of translation by k is multiplication by e_k. The isometry J_ψ that gives us the correspodence between the exponential system $\{e_k : k \in \mathbb{Z}^n\}$ and the generating system $\mathcal{B} = \{T_k\psi : k \in \mathbb{Z}^n\}$ of $\langle\psi\rangle_T$ allows us to show that each property of $\langle\psi\rangle_T$ (or of $\mathcal{B}$) corresponds to properties of the weight p_ψ (or the space $L^2(\mathbb{T}^n, p_\psi)$). In many cases this is easily seen to be true; for example, it is clear that $\mathcal{B}$ is an orthonormal basis of $\langle\psi\rangle_T$ if and only if $p_\psi(\xi) = 1$ a.e. There are, however, surprising results that are definitely not obvious. The result of Nielsen and Šikić ([15], [16]) is a good example: The system $\mathcal{B}$ is a Schauder basis if and only if p_ψ is an A_2 weight (in the sense of Hunt, Muckenhoupt and Wheeden).

The goal of this article is to extend these results to considerably more general situations. For a better understanding of this we briefly describe a more immediate extension of the results we just mentioned involving $\langle\psi\rangle_T$ and $L^2(\mathbb{T}^n, p_\psi)$. We replace $\mathbb{Z}^n$ by $G = \mathbb{Z}^n \times \mathbb{Z}^n$ and consider the representation $(k,l) \mapsto (T_k\mathcal{M}_l\psi)(x) \equiv e^{2\pi il\cdot x}\psi(x+k)$ of G acting on $L^2(\mathbb{R}^n)$. The role of the Fourier transform is assumed by the *Zak transform* Z. Let $(x,\xi) \in \mathbb{T}^n \times \mathbb{T}^n$ and

$$(Z\psi)(x,\xi) = \sum_{l\in\mathbb{Z}^n} \psi(x+l)e^{2\pi il\cdot\xi} \equiv \varphi(x,\xi) \tag{1.2}$$

for $\psi \in L^2(\mathbb{R}^n)$. It is not hard to see that Z is an isometry from $L^2(\mathbb{R}^n)$ onto $L^2(\mathbb{T}^n \times \mathbb{T}^n)$:

$$\int_{\mathbb{T}^n\times\mathbb{T}^n} |\varphi(t,\xi)|^2\,dt\,d\xi = \int_{\mathbb{T}^n\times\mathbb{T}^n} |(Z\psi)(t,\xi)|^2\,dt\,d\xi = \int_{\mathbb{R}^n} |\psi(t)|^2\,dt$$

(see [4]). We replace the system of translates $\mathcal{B} = \{T_k\psi : k \in \mathbb{Z}^n\}$ by the system $\widetilde{\mathcal{B}} = \{T_k\mathcal{M}_l\psi : (k,l) \in G\}$. This is an example of a *Gabor system* in $L^2(\mathbb{R}^n)$. The weight p_ψ on $\mathbb{T}^n$ is replaced by $q_\psi(t,\xi) \equiv |(Z\psi)(t,\xi)|^2$, $(t,\xi) \in \mathbb{T}^n \times \mathbb{T}^n$. The T-cyclic space $\langle\psi\rangle_T$ is replaced by $\langle\psi\rangle_{T,\mathcal{M}}$, the closure in $L^2(\mathbb{R}^n)$ of the span of $\widetilde{\mathcal{B}}$. One can again show that the properties of

$\widetilde{\mathcal{B}}$ correspond to properties of the weight q_ψ. Again this follows from an isometry between $\langle\psi\rangle_{T,\mathcal{M}}$ and $L^2(\mathbb{T}^n \times \mathbb{T}^n, q_\psi)$ and the correspondence, via this isometry, between the system $\widetilde{\mathcal{B}}$ and the system $\{e^{2\pi ikx}e^{2\pi il\xi} : (k,l)\in G\}$. Some of these results can be found in the work of Heil and Powell ([7]) (we will be more explicit about this later). We will also discuss further the relation between these two examples as well as derive the following relationship between the two weights p_ψ and q_ψ,

$$p_\psi(\xi) = \int_{\mathbb{T}^n} q_\psi(x,\xi)\,dx. \tag{1.3}$$

As we indicated above, this paper is devoted to the fact that these results are particular cases of a more general theory involving representations of LCA groups. In order to explain this we need to establish appropriate notation, some definitions and other examples. We begin doing this in the second section.

2. Locally compact abelian (LCA) groups and their duals. For G an LCA group, a *character* of G is a continuous homomorphism from G into the multiplicative abelian group $\{z \in \mathbb{C} : |z| = 1\}$. For simplicity we restrict our attention to separable LCA groups G and we write the group operation additively. For example, if $G = \mathbb{R}^n$, each character of G has the form $x \mapsto e_\xi(x) = e^{2\pi i\xi\cdot x}$ for a unique $\xi \in \mathbb{R}^n$; if $G = \mathbb{Z}^n$ each character has the form $e_\xi(k) = e^{2\pi i\xi\cdot k}$ for a unique $\xi \in \mathbb{R}^n/\mathbb{Z}^n = \mathbb{T}^n$. In each of these two cases there is a natural duality: in the first case, the element ξ is also a member of an LCA group which we denote by $\widehat{\mathbb{R}}^n$ (here $\widehat{\mathbb{R}}^n$ is also $\mathbb{R}^n$) and each $x \in \mathbb{R}^n$ corresponds to a homomorphism

$$\xi \mapsto e_x(\xi) = e^{2\pi ix\cdot\xi}.$$

In the second case, ξ is also a member of the LCA group $\mathbb{T}^n$ and each $k \in \mathbb{Z}^n$ corresponds to a homomorphism $\xi \mapsto e_k(\xi) = e^{2\pi ik\cdot\xi}$ of $\mathbb{T}^n$ into $\{z \in \mathbb{C} : |z| = 1\}$.

Motivated by these two examples, we will consider the *dual group* $\widehat{G}$ of an LCA group G to be an LCA group together with a continuous bi-additive map $(\xi,x) \mapsto \langle\xi,x\rangle \in \mathbb{T} = \mathbb{R}/\mathbb{Z}$, where $(\xi,x) \in \widehat{G} \times G$, such that every character of G is of the form $e_\xi(x) = e^{2\pi i\langle\xi,x\rangle}$ for a unique $\xi \in \widehat{G}$; vice versa, every character of $\widehat{G}$ is of the form $e_x(\xi)\,(= e_\xi(x)) = e^{2\pi i\langle\xi,x\rangle}$ for a unique $x \in G$. It is easy to see that choices for $\widehat{G}$ exist and any two choices are canonically isomorphic.

As is well known (see [2] or [17]), any two Haar measures (translation invariant Borel measures) differ by a positive scalar and, for each choice of a Haar measure dg on G, there is a unique Haar measure, $d\xi$, on $\widehat{G}$ for which the *Fourier transform*

$$\mathcal{F}_G : f \mapsto \hat{f}(\xi) = \int_G f(g) e_{-\xi}(g)\, dg \tag{2.1}$$

is a unitary operator from $L^2(G, dg)$ onto $L^2(\widehat{G}, d\xi)$.

Fix an LCA group G and let T be a unitary representation of G on $\mathbb{H}$ with inner product $\langle\, ,\rangle$. Thus, T_g is a unitary operator on $\mathbb{H}$ for each $g \in G$ and the bounded functions $g \mapsto \langle \varphi, T_g\psi\rangle$, $\varphi, \psi \in \mathbb{H}$, are continuous. We say that T is *dual integrable* if and only if there exist a Haar measure $d\xi$ on $\widehat{G}$ and a function $[\cdot\,,\cdot]_T : \mathbb{H} \times \mathbb{H} \to L^1(\widehat{G}, d\xi)$, called the *bracket function* for T, such that

$$\langle \varphi, T_g\psi\rangle = \int_{\widehat{G}} [\varphi, \psi]_T(\xi) e_{-g}(\xi)\, d\xi \quad \text{for all } g \in G \text{ and } \varphi, \psi \in \mathbb{H}. \tag{2.2}$$

Examples. The *modulation representation* $g \mapsto \mathcal{M}_g$ of G, acting on $\mathbb{H} = L^2(\widehat{G}, d\xi)$, is defined by $(\mathcal{M}_g\varphi)(\xi) = e_g(\xi)\varphi(\xi)$. This representation is trivially dual integrable when we use $[\varphi, \psi]_{\mathcal{M}}(\xi) = \varphi(\xi)\overline{\psi(\xi)}$:

$$\int_{\widehat{G}} [\varphi, \psi]_{\mathcal{M}}(\xi) e_{-g}(\xi)\, d\xi = \int_{\widehat{G}} \varphi(\xi)\, \overline{(\mathcal{M}_g\psi)(\xi)}\, d\xi = \langle \varphi, \mathcal{M}_g\psi\rangle. \tag{2.3}$$

The *regular representation* R of G acting on $\mathbb{H} = L^2(G, dg)$ is defined by

$$(R_g f)(x) = f(g + x).$$

Clearly $(R_g f)^\wedge = \mathcal{M}_g \hat{f}$. It follows that R_g is unitarily equivalent to $\mathcal{M}_g$ via the Fourier transform $\mathcal{F}_G$ (defined by (2.1)). Consequently, R is dual integrable with $[f_1, f_2]_R = [\hat{f}_1, \hat{f}_2]_{\mathcal{M}}$.

Remark. Dual integrability is a property of equivalence classes of unitary representations in the sense that, if T is a unitary representation of G on $\mathbb{H}$ and T is equivalent to a unitary representation T' on $\mathbb{H}'$ via a unitary operator $U : \mathbb{H} \to \mathbb{H}'$ (that is, $UT_g = T'_g U$ for all $g \in G$), then T' is dual integrable with $[U\varphi, U\psi]_{T'} = [\varphi, \psi]_T$.

The following well known result and the corollaries that follow it are important to us since they will be used in §3 to establish properties of dual integrable representations. The next result is known as Stone's theorem (see [13, p. 147]).

Theorem (2.4). *Let T be a unitary representation of an LCA group G on a Hilbert space $\mathbb{H}$.*

(i) *There exists a Borel measure P on $\widehat{G}$ with values in $\{E : E$ a self-adjoint projection operator on $\mathbb{H}\}$ such that*

$$T_g = \int_{\widehat{G}} e_g(\xi)\, dP(\xi) \quad \textit{for all } g \in G.$$

(ii) *For each* $\varphi, \psi \in \mathbb{H}$, *part* (i) *allows us to define a* $\mathbb{C}$*-valued Borel measure* $\mu_{\varphi,\psi}$ *by letting*

$$\mu_{\varphi,\psi}(S) = \langle P(S)\varphi, \psi\rangle = \langle \varphi, P(S)\psi\rangle = \langle P(S)\varphi, P(S)\psi\rangle.$$

It follows that

$$\langle T_{-g}\varphi, \psi\rangle = \langle \varphi, T_g\psi\rangle = \int_{\widehat{G}} \overline{e_g(\xi)}\, d\mu_{\varphi,\psi}(\xi) \quad \text{for all } g \in G.$$

COROLLARY (2.5). *Let* T *be a unitary representation of an LCA group* G *acting on a Hilbert space* $\mathbb{H}$. *The following are equivalent:*

(i) T *is dual integrable.*
(ii) *For each* $\varphi, \psi \in \mathbb{H}$ *the measure* $\mu_{\varphi,\psi}$ *(defined in Theorem (2.4)) is absolutely continuous with respect to* $d\xi$.

In this situation, the bracket $[\varphi,\psi]_T$ *is the Radon–Nikodym derivative* $\frac{d\mu_{\varphi,\psi}}{d\xi}$.

Proof. (ii)⇒(i). If $\mu_{\varphi,\psi}$ is absolutely continuous with respect to $d\xi$, we write $d\mu_{\varphi,\psi}(\xi) = [\varphi,\psi]_T(\xi)\, d\xi$. The result now follows from part (ii) of Theorem (2.4).

(i)⇒(ii). Let $d\nu_{\varphi,\psi}(\xi) = d\mu_{\varphi,\psi}(\xi) - [\varphi,\psi]_T(\xi)\, d\xi$. It follows that $\nu_{\varphi,\psi}$ is a bounded Borel measure whose Fourier transform satisfies

$$\begin{aligned}\mathcal{F}_{\widehat{G}}(\nu_{\varphi,\psi})(g) &= \int_{\widehat{G}} \overline{e_g(\xi)}\, d\mu_{\varphi,\psi}(\xi)\, d\xi - \int_{\widehat{G}} \overline{e_g(\xi)}\, [\varphi,\psi]_T(\xi)\, d\xi \\ &= \langle \varphi, T_g\psi\rangle - \langle \varphi, T_g\psi\rangle = 0\end{aligned}$$

for all $g \in G$. By the uniqueness theorem for the Fourier transform (see p. 103 of [2]) we conclude that $\nu_{\varphi,\psi} = 0$; this shows that $\mu_{\varphi,\psi}$ is absolutely continuous with respect to $d\xi$. ∎

COROLLARY (2.6). *Suppose* T *is a dual integrable unitary representation of an LCA group* G *on a Hilbert space* $\mathbb{H}$. *Then* $[\varphi,\psi]_T : \mathbb{H}\times\mathbb{H} \to L^1(\widehat{G}, d\xi)$ *is a sesquilinear hermitian symmetric map having the following properties:*

(i) (Positivity) $[\varphi,\varphi]_T \geq 0$ *a.e.*
(ii) (Cauchy–Schwarz) $|[\varphi,\psi]_T| \leq [\varphi,\varphi]_T^{1/2}[\psi,\psi]_T^{1/2}$ *a.e.*
(iii) $\|[\varphi,\psi]_T\|_{L^1(\widehat{G})} \leq \|\varphi\|_{\mathbb{H}}\|\psi\|_{\mathbb{H}}$ *for all* $\varphi, \psi \in \mathbb{H}$.

Proof. For each measurable $S \subset \widehat{G}$, $\mu_{\varphi,\psi}(S)$ is linear in φ, conjugate linear in ψ and $\overline{\mu_{\varphi,\psi}(S)} = \mu_{\psi,\varphi}(S)$. It follows that $[\varphi,\psi]_T$ has the same properties.

To prove (i) observe that $\mu_{\varphi,\varphi}(S) = \langle P(S)\varphi, P(S)\varphi\rangle = \|P(S)\varphi\|^2 \geq 0$ by Theorem (2.4)(ii). Thus, $[\varphi,\varphi]_T \geq 0$ a.e.

From Theorem (2.4)(ii) and the Cauchy–Schwarz inequality in $\mathbb{H}$ we have, for a measurable set $S \subset \widehat{G}$,

$$\begin{aligned}|\mu_{\varphi,\psi}(S)| &= |\langle P(S)\varphi, P(S)\psi\rangle| \\ &\le \|P(S)\varphi\|\,\|P(S)\psi\| = (\mu_{\varphi,\varphi}(S))^{1/2}(\mu_{\psi,\psi}(S))^{1/2}.\end{aligned}$$

Part (ii) now follows immediately.

The third inequality follows from (ii), the Cauchy–Schwarz inequality for functions in $L^2(\widehat{G})$ and the fact that

$$\int_{\widehat{G}} [\varphi,\varphi]_T(\xi)\,d\xi = \|\varphi\|^2;$$

see (2.2) with $g = 0$ and $\psi = \varphi$. ∎

Corollary (2.7). *Suppose T is a dual integrable unitary representation of an LCA group G acting on a Hilbert space $\mathbb{H}$.*

(i) *For $g \in G$ and $\varphi, \psi \in \mathbb{H}$, we have*

$$[T_g\varphi,\psi]_T = e_g[\varphi,\psi]_T = [\varphi, T_{-g}\psi]_T \qquad \text{a.e. in } \widehat{G}.$$

(ii) *Let Γ be a finite subset of G and $\varphi,\psi \in \mathbb{H}$. For $p_\Gamma(\xi) = \sum_{g\in\Gamma} a_g e_g(\xi)$ a trigonometric polynomial on $\widehat{G}$ and*

$$p_\Gamma(T) = \sum_{g\in\Gamma} a_g T_g$$

we have

$$[p_\Gamma(T)\varphi,\psi]_T = p_\Gamma[\varphi,\psi]_T = [\varphi, \overline{p_\Gamma(T)}\psi]_T \qquad \text{a.e. in } \widehat{G};$$

also

$$[p_\Gamma(T)\psi, p_\Gamma(T)\psi]_T = |p_\Gamma|^2[\psi,\psi]_T \qquad \text{a.e. in } \widehat{G}.$$

Proof. Let $g, k \in G$ and use (2.2) to obtain

$$\begin{aligned}\langle T_g\varphi, T_k\psi\rangle = \langle \varphi, T_{k-g}\psi\rangle &= \int_{\widehat{G}} [\varphi,\psi]_T(\xi)\,\overline{e_{k-g}(\xi)}\,d\xi \\ &= \int_{\widehat{G}} e_g(\xi)[\varphi,\psi]_T(\xi)\,\overline{e_k(\xi)}\,d\xi.\end{aligned}$$

But (2.2) also implies

$$\langle T_g\varphi, T_k\psi\rangle = \int_{\widehat{G}} [T_g\varphi,\psi]_T(\xi)\,\overline{e_k(\xi)}\,d\xi.$$

By the uniqueness of the Fourier transform on $\widehat{G}$ we thus obtain

$$e_g[\varphi,\psi]_T = [T_g\varphi,\psi]_T \qquad \text{a.e. on } \widehat{G}.$$

The other properties follow from the linearity and sesquilinearity of the bracket. ∎

LEMMA (2.8). *Let $\varphi, \psi \in \mathbb{H}$ and T a dual integrable representation of G acting on $\mathbb{H}$. Let*

$$\langle\psi\rangle_T = \overline{\operatorname{span}\{T_g\psi : g \in G\}}$$

(the closure in $\mathbb{H}$). Then $\varphi \perp \langle\psi\rangle_T$ if and only if $[\varphi, \psi]_T = 0$ a.e. in $\widehat{G}$.

Proof. $\varphi \perp \langle\psi\rangle_T$ if and only if $\langle\varphi, T_g\psi\rangle = 0$ for all $g \in G$. By (2.2) this is equivalent to

$$\int_{\widehat{G}} [\varphi, \psi]_T(\xi) e_{-g}(\xi)\, d\xi = 0 \quad \text{for all } g \in G.$$

By the uniqueness of the Fourier transform this is equivalent to $[\varphi, \psi]_T = 0$ in $L^2(\widehat{G}, d\xi)$. ∎

LEMMA (2.9). *Let $p_\psi = [\psi, \psi]_T$ and $\Omega_\psi = \{\xi \in \widehat{G} : p_\psi(\xi) > 0\}$. If $\varphi, \psi \in \mathbb{H}$, then $[\varphi, \psi]_T(\xi) = 0$ for a.e. $\xi \in \Omega_\psi^c$.*

Proof. By Corollary (2.6)(ii) we have

$$0 \le \int_{\Omega_\psi^c} |[\varphi, \psi]_T(\xi)|\, d\xi \le \int_{\Omega_\psi^c} ([\varphi, \varphi]_T(\xi))^{1/2} ([\psi, \psi]_T(\xi))^{1/2}\, d\xi = 0$$

since $[\psi, \psi]_T(\xi) = 0$ if $\xi \in \Omega_\psi^c$. ∎

In the introduction we presented the example of the principal shift invariant space $\langle\psi\rangle_T$ generated by $\psi \in L^2(\mathbb{R}^n)$ and the translates $T_k\psi$, $k \in \mathbb{Z}^n\,(= G)$. We explained how the properties of $\langle\psi\rangle_T$ (or the system $\mathcal{B} = \{T_k\psi : k \in \mathbb{Z}^n\}$) correspond to the properties of

$$p_\psi(\xi) = [\hat{\psi}, \hat{\psi}]_T(\xi) = \sum_{k \in \mathbb{Z}^n} |\hat{\psi}(\xi + k)|^2.$$

This correspondence is obtained through a basic linear isometry between the spaces $\langle\psi\rangle_T$ and $L^2(\widehat{G}, p_\psi) = L^2(\mathbb{T}, p_\psi)$. We are now prepared, because of the result we have just developed, to establish this isometry between the more general spaces $\langle\psi\rangle_T$ and the weighted space $L^2(\widehat{G}, [\psi, \psi]_T)$.

3. An isometric isomorphism

THEOREM (3.1). *Let $T : g \mapsto T_g$ be a dual integrable unitary representation of an LCA group G acting on a Hilbert space $\mathbb{H}$.*

(i) *If $\psi \neq 0$ belongs to $\mathbb{H}$, then the map $S_\psi : \langle\psi\rangle_T \to L^2(\widehat{G}, p_\psi)$ defined by*

$$S_\psi\varphi = \chi_{\Omega_\psi} \frac{[\varphi, \psi]_T}{[\psi, \psi]_T}$$

is a linear isometry between these two spaces.

(ii) *If $g \in G$ and $\varphi, \psi \in \mathbb{H}$, then $S_\psi(T_g\varphi)(\xi) = e_g(\xi) S_\psi\varphi(\xi)$ for a.e. $\xi \in \widehat{G}$.*

We remind our readers that p_ψ and Ω_ψ are defined in Lemma (2.9).

Proof. (i) If $\varphi \in \mathbb{H}$, then, from the definition of S_ψ and Corollary (2.6)(ii), we have

$$\begin{aligned}\int_{\widehat{G}} |S_\psi\varphi(\xi)|^2 p_\psi(\xi)\,d\xi &= \int_{\Omega_\psi} \left|\frac{[\varphi,\psi]_T(\xi)}{[\psi,\psi]_T(\xi)}\right|^2 [\psi,\psi]_T(\xi)\,d\xi \\ &\le \int_{\Omega_\psi} \frac{[\varphi,\varphi]_T(\xi)[\psi,\psi]_T(\xi)}{([\psi,\psi]_T(\xi))^2}\,[\psi,\psi]_T(\xi)\,d\xi \\ &= \int_{\Omega_\psi} [\varphi,\varphi]_T(\xi)\,d\xi = \langle\varphi,\varphi\rangle = \|\varphi\|^2,\end{aligned}$$

where we used the definition of the bracket in (2.2) for the penultimate equality. This shows that S_ψ maps $\mathbb{H}$ into $L^2(\widehat{G}, p_\psi)$. Recall that the linearity of S_ψ is a consequence of the fact that the bracket $[\varphi,\psi]_T$ is the Radon–Nikodym derivative $d\mu_{\varphi,\psi}/d\xi$ (see Corollary (2.5)).

Let us now show the isometry property of S_ψ (and, thus, the one-to-one property). Let φ be a finite sum $\sum a_h T_h\psi$ $(h \in G)$. By Corollary (2.7)(ii), we see that

$$\int_{\widehat{G}} |(S_\psi\varphi)(\xi)|^2 p_\psi(\xi)\,d\xi = \int_{\widehat{G}} \Big|\sum a_h e_h(\xi)\Big|^2 p_\psi(\xi)\,d\xi = \int_{\widehat{G}} [\varphi,\varphi]_T(\xi)\,d\xi = \|\varphi\|^2$$

(see (2.2)). Since these finite sums are dense in $\langle\psi\rangle_T$ we have this isometry for all $\varphi \in \langle\psi\rangle_T$.

We now show that S_ψ is onto $L^2(\widehat{G}, p_\psi)$. Suppose $S_\psi(\langle\psi\rangle_T)$ does not contain a non-zero $m \in L^2(\widehat{G}, p_\psi)$; we can, in fact, assume

$$m \perp S_\psi(\langle\psi\rangle_T).$$

Thus,

$$0 = \int_{\widehat{G}} m(\xi) S_\psi(T_h\psi)(\xi) p_\psi(\xi)\,d\xi \qquad \text{for all } h \in G.$$

By Corollary (2.7)(i), $S_\psi(T_h\psi) = \chi_{\Omega_\psi} e_h$. Thus,

$$0 = \int_{\widehat{G}} m(\xi) e_h(\xi) p_\psi(\xi)\,d\xi \qquad \text{for all } h \in G.$$

By the uniqueness of the Fourier transform it follows that $m(\xi)p_\psi(\xi) = 0$ for a.e. $\xi \in \widehat{G}$. But this means that m, as a function in $L^2(\widehat{G}, p_\psi)$, is the zero function, contrary to assumption.

Part (ii) is an immediate consequence of Corollary (2.7)(i). ∎

Remark. In the special case described at the beginning of the Introduction, $G = \mathbb{Z}^n$, $\mathbb{H} = L^2(\mathbb{R}^n)$, $(T_k\psi)(x) = \psi(x+k)$, $\widehat{G} = \mathbb{T}^n$, the map we denoted by J_ψ corresponds to the inverse S_ψ^{-1} of the isometry S_ψ introduced in this section.

Observe that $m \mapsto m p_\psi^{1/2}$ is an isometry from $L^2(\widehat{G}, p_\psi)$ into $L^2(\Omega_\psi) \subset L^2(G)$. Using this, we obtain the following alternative version of Theorem (3.1):

Corollary (3.2). *Under the same hypotheses of Theorem* (3.1), *the map* $V_\psi : \langle\psi\rangle_T \to L^2(\widehat{G})$ *defined by*

$$V_\psi \varphi = \chi_{\Omega_\psi} \frac{[\varphi, \psi]_T}{p_\psi^{1/2}}, \qquad \varphi \in \mathbb{H},$$

is a unitary map from $\langle\psi\rangle_T$ *onto* $L^2(\Omega_\psi, d\xi)$. *Moreover, for* $g \in G$ *and* $\varphi, \psi \in \mathbb{H}$, $V_\psi T_g \varphi = e_g V_\psi \varphi$ *a.e. in* $\widehat{G}$.

We originally defined S_ψ and V_ψ on the cyclic space $\langle\psi\rangle_T$ generated by ψ. The reader can observe that their definitions make sense for all $\varphi \in \mathbb{H}$ and these operators are 0 for $\varphi \perp \langle\psi\rangle_T$.

Remark (3.3). Suppose T is a dual integrable unitary representation of an LCA group acting on a Hilbert space $\mathbb{H}$, and T' is another such representation acting on $\mathbb{H}'$. We shall denote the maps S_ψ and V_ψ corresponding to T and T' by $S_{\psi,T}$, $V_{\psi,T}$ and $S_{\psi,T'}$, $V_{\psi,T'}$. Suppose T is unitarily equivalent to T' via $U : \mathbb{H} \to \mathbb{H}'$; we then have the following commutative diagram:

$$\begin{tikzcd} \langle\psi\rangle_T \arrow[rr, "U"] \arrow[dr, "V_{\psi,T}"'] & & \langle U(\psi)\rangle_{T'} \arrow[dl, "V_{U(\psi),T'}"] \\ & L^2(\widehat{G}) & \end{tikzcd}$$

When $\varphi, \psi \in \mathbb{H}$, $\psi \neq 0$, we have $\varphi' = U\varphi$, $\psi' = U\psi$ and $V_{\psi',T'}\varphi' = V_{\psi,T}\varphi$.

Consider the example of the modulation representation $g \mapsto \mathcal{M}_g$ of G introduced after (2.2). The regular representation R of G, acting on $\mathbb{H} = L^2(G)$, is unitarily equivalent, via the Fourier transform $\mathcal{F}_G$, to the modulation representation $\mathcal{M}$ of G acting on $L^2(\widehat{G})$. For $\psi \in L^2(G)$ with $\hat{\psi}(\xi) \neq 0$ for a.e. ξ, we have $\langle\psi\rangle_R = L^2(G)$, $\langle\hat{\psi}\rangle_{\mathcal{M}} = L^2(\widehat{G})$, $p_\psi = p_{\hat{\psi}} = |\hat{\psi}|^2$ and, when $\varphi \in L^2(G)$, $V_{\psi,R}\,\varphi = V_{\hat{\psi},\mathcal{M}}\hat{\varphi} = \hat{\varphi}\overline{\hat{\psi}}/|\hat{\psi}|$ a.e.

Corollary (3.4). *Let T be a unitary representation of G on a separable Hilbert space $\mathbb{H}$. Then the following are equivalent:*

(i) *T is dual integrable.*
(ii) *T is unitarily equivalent to a subrepresentation of the direct sum of countably many copies of the modulation representation $\mathcal{M}$.*
(iii) *T is unitarily equivalent to a subrepresentation of the direct sum of countably many copies of the regular representation R.*

(iv) *T is square integrable in the sense that there is a dense subspace $K \subset \mathbb{H}$ such that, for each $\psi \in K$,*

$$(W_\psi \varphi)(g) = \langle \varphi, T_g \psi \rangle$$

defines a bounded linear map $W_\psi : \mathbb{H} \to L^2(G)$.

Proof. (i)⇒(ii) Since T is unitary, we can choose a countable $\Psi = \{\psi_i : i \in I\}$ such that $\mathbb{H}$ is the orthogonal direct sum of the cyclic subspaces $\langle \psi_i \rangle_T$, $i \in I$. Then $V_\Psi \varphi \equiv \{V_{\psi_i}\varphi\}_{i \in I}$ defines a linear isometry from $\mathbb{H}$ into $\ell^2(I, L^2(\widehat{G}))$ and, via V_Ψ, T is unitarily equivalent to a subrepresentation of the representation on $\ell^2(I, L^2(G))$ of $|I|$ copies of $\mathcal{M}$.

(ii)⇔(iii) is immediate from the unitary equivalence of $\mathcal{M}$ and R.

(ii)⇒(i)&(iv) $\mathcal{M}$ is dual integrable (see the example following (2.2)) and it is also square integrable with $K = L^2(\widehat{G}) \cap L^\infty(\widehat{G})$. It follows easily that any subrepresentation of the direct sum of countably many copies of $\mathcal{M}$ is both dual integrable and square integrable.

(iv)⇒(iii) When T is square integrable, for each $\psi \in K$, W_ψ has the polar decomposition $W_\psi = U_\psi |W_\psi|$ with U_ψ a linear isometry from $\langle \psi \rangle_T$ into $L^2(G)$ and zero on the orthogonal complement of $\langle \psi \rangle_T$. Moreover, $U_\psi T_g = R_g U_\psi$ for all $g \in G$. Using the Gram–Schmidt process, we can construct a countable subset $\Psi = \{\psi_i : i \in I\} \subset K$ for which $\mathbb{H}$ is the orthogonal direct sum of the subspaces $\langle \psi_i \rangle_T$, $i \in I$, and deduce that T is unitarily equivalent to a subrepresentation of the direct sum of $|I|$ copies of R via the map $U_\Psi : \mathbb{H} \to \ell^2(I, L^2(G))$ defined by $U_\Psi \varphi = \{U_{\psi_i}\varphi\}_{i \in I}$. ∎

We note that, in practice, direct verification that a representation T is square integrable is often difficult. In contrast, as we will illustrate below in Sections 4 and 6, determination that T is dual integrable is often elementary. Also, a variety of calculations for T are most easily carried out using the properties of $[\cdot,\cdot]_T$ discussed above.

4. Integer translations on $L^2(\mathbb{R}^n)$ and Gabor systems. We can extend the regular representation R of the LCA group $\mathbb{Z}^n$ to a representation T of $\mathbb{Z}^n$ acting on $L^2(\mathbb{R}^n)$ by $(T_k f)(x) = f(x+k)$. As in the example after (2.3), the Fourier transform from $L^2(\mathbb{R}^n)$ onto $L^2(\widehat{\mathbb{R}}^n)$ converts T to the multiplier representation $(\mathcal{M}_k \hat{f}) = e_k \hat{f}$, where $e_k(\xi) = e^{2\pi i \xi \cdot k}$. Since $\widehat{\mathbb{R}}^n$ is the disjoint union of $\mathbb{Z}^n$ translates of $[0,1)^n \sim \widehat{\mathbb{R}}^n / \mathbb{T}^n = \widehat{\mathbb{Z}}^n$, we see that T is equivalent to countably many copies of $\mathbb{R}$ and, hence, by Corollary (3.4), is dual integrable. Explicitly, for $\varphi, \psi \in L^2(\mathbb{R}^n)$ the bracket function

$$[\varphi, \psi]_T(\xi) = [\hat{\varphi}, \hat{\psi}]_{\mathcal{M}}(\xi) = \sum_{k \in \mathbb{Z}^n} (\hat{\varphi}\overline{\hat{\psi}})(\xi + k)$$

satisfies (2.2).

The Gabor representation, $(T,\mathcal{M})$, of the product group $\mathbb{Z}^n\times\mathbb{Z}^n$ acts on $L^2(\mathbb{R}^n)$ by

$$(T_k\mathcal{M}_l f)(x)=f(x+k)e_l(x)f(x).$$

To show that $(T,\mathcal{M})$ is dual integrable, we make use of the *Zak transform* Z defined by

$$(Zf)(x,\xi)=\sum_{j\in\mathbb{Z}^n} f(x+j)e_j(\xi). \tag{4.1}$$

Since

$$\|f\|^2_{L^2(\mathbb{R}^n)}=\int\limits_{[0,1)^n}\sum_k |f(x+k)|^2\,dx,$$

$(Zf)(x,\cdot)$ is a square integrable $\mathbb{Z}^n$-periodic function for a.e. x whose Fourier coefficients are $f(x+k)$, $k\in\mathbb{Z}^n$. Using this, for f, $g\in L^2(\mathbb{R}^n)$ we obtain

$$\begin{aligned}\langle f,g\rangle_{L^2(\mathbb{R}^n)}&=\int\limits_{[0,1)^n}\sum_k f(x+k)\,\overline{g(x+k)}\,dx\\&=\int\limits_{[0,1)^n}\int\limits_{[0,1)^n}(Zf)(x,\xi)\,\overline{(Zg)(x,\xi)}\,d\xi\,dx.\end{aligned} \tag{4.2}$$

Now by a simple change of indices

$$Z(T_k\mathcal{M}_l f)(x,\xi)=e_{-k}(x)\,e_l(\xi)(Zf)(x,\xi). \tag{4.3}$$

It follows from (4.2) and (4.3) that $(Zf)\,\overline{(Zg)}$ is a $\mathbb{Z}^n\times\mathbb{Z}^n$-periodic function, which is integrable on the dual group $\mathbb{T}^n\times\mathbb{T}^n$ of $\mathbb{Z}^n\times\mathbb{Z}^n$, and

$$\int\limits_{[0,1)^n}\int\limits_{[0,1)^n}(Zf)(x,\xi)\overline{(Zg)(x,\xi)}\,e_k(x)\,e_{-l}(\xi)\,dx\,d\xi=\langle f,T_k\mathcal{M}_{-l}g\rangle. \tag{4.4}$$

As a consequence, $(T,\mathcal{M})$ is dual integrable with bracket function

$$[f,g]_{T,\mathcal{M}}=Zf\,\overline{Zg}.$$

We can now show, as we claimed in §1, that $p_\psi(\xi)=[\psi,\psi]_T(\xi)=[\hat\psi,\hat\psi]_{\mathcal{M}}(\xi)$ coincides a.e. with

$$\int\limits_{[0,1)^n} q_\psi(x,\xi)\,dx=\int\limits_{[0,1)^n}|(Z\psi)(x,\xi)|^2\,dx.$$

One easy way to see this is to observe, from (4.4), that both $[\varphi,\psi]_T(\xi)$ and $\xi\mapsto\int_{[0,1)^n}(Z\varphi)(x,\xi)\,dx$ satisfy the dual integrable criterion (2.2) for T, and hence must coincide a.e. There are also a variety of direct ways to establish this result using connections between the Zak and the Fourier transforms.

Some recent results about the Zak transform can be found in [11], [12] and Chapter 8 of [4].

5. Properties of the set $\mathcal{B}_\psi = \{T_k\psi : k \in G\}$ and its span $\langle\psi\rangle_T$. In the introduction we asserted that if $G = \mathbb{Z}^n$, $\psi \in L^2(\mathbb{R}^n)$, and $\mathcal{B}_\psi = \{T_k\psi : k \in G\}$ is the collection of translates of ψ, then the T-cyclic subspace $\langle\psi\rangle_T$ (also known as the principal shift invariant subspace generated by $\mathcal{B}_\psi$) has properties that correspond to those of the weight p_ψ. These properties also correspond to properties of the generating system $\mathcal{B}_\psi$. In this section we show how this extends to the more general cases we have been considering that involve LCA groups.

We consider a countable abelian group G with the discrete topology and a dual integrable representation T of G, $T : k \mapsto T_k$, acting on a separable Hilbert space $\mathbb{H}$. For $\psi \in \mathbb{H} \setminus \{0\}$ the cyclic T-invariant subspace $\langle\psi\rangle_T$ is, by definition, the closure (in $\mathbb{H}$) of the span of the set $\mathcal{B}_\psi = \{T_k\psi : k \in G\}$. We shall show how properties of $\mathcal{B}_\psi$ correspond to properties of the weight function $p_\psi = [\psi,\psi]_T$ introduced in §2.

We begin with the property that $\mathcal{B}_\psi$ is an orthonormal set:

$$\langle T_k\psi, T_l\psi\rangle = \delta_{k,l} \quad \text{for } k,l \in G.$$

Since each T_k is unitary this is equivalent to $\langle\psi, T_k\psi\rangle = \langle T_0\psi, T_k\psi\rangle = \delta_{0k} \equiv \delta_k$ for all $k \in G$ (0 is the zero element of G). Property (2.2) tells us that we must have all the Fourier coefficients of the bracket $[\psi,\psi]_T$, as a function in $L^1(\widehat{G})$, equal to 0, except the one corresponding to $k = 0$. Hence, $p_\psi = [\psi,\psi]_T$ equals 1 a.e. in $\widehat{G}$. Conversely, if $p_\psi(\xi) = 1$ for a.e. $\xi \in \widehat{G}$, then

$$\langle\psi, T_k\psi\rangle = \int_{\widehat{G}} e_{-k}(\xi)\, d\xi = \delta_{k0} \quad \text{for all } k \in G;$$

thus, $\mathcal{B}_\psi$ is an o.n. system.

We have shown:

Proposition (5.1). *Let G be a countable abelian group and T a dual integrable unitary representation of G on a Hilbert space $\mathbb{H}$. Then $\mathcal{B}_\psi$ is an orthonormal basis of $\langle\psi\rangle_T$ if and only if $[\psi,\psi]_T = p_\psi$ equals 1 a.e. in $\widehat{G}$.*

From Theorem (3.1) we see that

$$S_\psi(T_k\psi) = e_k S_\psi(\psi) = \chi_{\Omega_\psi} e_k \tag{5.2}$$

for all $k \in G$. This makes it clear that properties of $\mathcal{B}_\psi$ should be readable from properties of the spanning set $\{\chi_{\Omega_\psi} e_k : k \in G\}$ in $L^2(\widehat{G}, p_\psi)$. The following propositions make use of this last observation.

Proposition (5.3). *Under the same assumptions of Proposition* (5.1), *$\mathcal{B}_\psi$ is a Riesz basis for $\langle\psi\rangle_T$ with constants $0 < A \le B < \infty$ if and only if for a.e. $\xi \in \widehat{G}$,*

$$A \le p_\psi(\xi) \le B.$$

REMARK. By definition, $\mathcal{B}_\psi$ is a Riesz basis with constants A and B provided

$$A\sum_{k\in G}|a_k|^2 \le \Big\|\sum_{k\in G} a_k T_k\psi\Big\|_{\mathbb{H}}^2 \le B\sum_{k\in G}|a_k|^2 \tag{5.4}$$

for all $\{a_k : k\in G\}\in \ell^2(G)$.

It is well known that Riesz bases are precisely those that are images, under invertible bounded linear operators (on $\mathbb{H}$), of orthonormal bases. In the particular case we are considering,

$$S_\psi(T_k\psi) = \chi_{\Omega_\psi} e_k, \qquad k\in G. \tag{5.5}$$

When $A \le p_\psi(\xi) \le B$ a.e., the spaces $L^2(\widehat{G}, p_\psi)$ and $L^2(\widehat{G}, d\xi)$ are equal (and their norms are equivalent). Proposition (5.3) follows easily from this remark.

Following Chapter 2, §11, of [19] we say that $\mathcal{B}_\psi$ has the *Besselian property* if and only if the convergence of $\sum_{k\in G} a_k T_k\psi$ in $\langle\psi\rangle_T$ implies $\sum_{k\in G}|a_k|^2 < \infty$. Theorem 11.1 in [19] tells us that this last property is equivalent to the existence of a constant $A>0$ such that

$$A\sum_{k\in G}|a_k|^2 \le \Big\|\sum_{k\in G} a_k T_k\psi\Big\|^2$$

for all finite sequences $\{a_k : k\in G\}$. It follows from the observations made after Proposition (5.3) that $\mathcal{B}_\psi$ has the Besselian property in $\langle\psi\rangle_T$ if and only if $p_\psi \ge A$ a.e. in $\widehat{G}$.

In Chapter 2, §11 of [19] the system $\mathcal{B}_\psi$ has the *Hilbertian property* if and only if $\sum_{k\in G}|a_k|^2<\infty$ implies that $\sum_{k\in G} a_k T_k\psi$ converges in $\langle\psi\rangle_T$. Theorem 11.1 in [19] asserts that this is equivalent to the existence of $B<\infty$ such that

$$\Big\|\sum_{k\in G} a_k T_k\psi\Big\|^2 \le B\sum_{k\in G}|a_k|^2$$

for all finite sequences $\{a_k : k\in G\}$. Again, the above observations show that $\mathcal{B}_\psi$ has the Hilbertian property in $\langle\psi\rangle_T$ if and only if $p_\psi(\xi)\le B$ a.e. in $\widehat{G}$.

$\mathcal{B}_\psi$ is a *frame* with constants $0<A\le B<\infty$ for $\langle\psi\rangle_T$ if and only if

$$A\|\varphi\|^2 \le \sum_{k\in G}|\langle\varphi, T_k\psi\rangle|^2 \le B\|\varphi\|^2 \tag{5.6}$$

for all $\varphi\in\langle\psi\rangle_T$. Notice the difference between (5.4) and (5.6). In fact, frames are different from Riesz bases; however, if $p_\psi(\xi)>0$ for a.e. ξ, then $\mathcal{B}_\psi$ is a frame if and only if $\mathcal{B}_\psi$ is a Riesz basis. This will be a consequence of the theorem we prove next that characterizes when $\mathcal{B}_\psi$ is a frame.

THEOREM (5.7). *Under the same hypothesis of Proposition* (5.3), $\mathcal{B}_\psi$ *is a frame for* $\langle\psi\rangle_T$ *with constants* $0 < A \le B < \infty$ *if and only if*

$$A \le p_\psi(\xi) \le B \quad \textit{for a.e. } \xi \in \Omega_\psi. \tag{5.8}$$

Proof. Suppose (5.8) is true, then, by Theorem (3.1) and equality (5.2), if $\varphi \in \langle\psi\rangle_T$,

$$\begin{aligned}\sum_{k\in G} |\langle \varphi, T_k\psi\rangle|^2 &= \sum_{k\in G} |\langle S_\psi \varphi, S_\psi(T_k\psi)\rangle_{L^2(\widehat{G},p_\psi)}|^2 \\ &= \sum_{k\in G} |\langle S_\psi(\varphi), \chi_{\Omega_\psi} e_k\rangle_{L^2(\widehat{G},p_\psi)}|^2 = \sum_{k\in G} \Big| \int_{\widehat{G}} S_\psi(\varphi)\chi_{\Omega_\psi} e_k p_\psi \Big|^2.\end{aligned}$$

Since $\{e_k : k \in G\}$ is an o.n. basis of $L^2(\widehat{G}, d\xi)$, Plancherel's theorem and the above equalities give us

$$\sum_{k\in G} |\langle \varphi, T_k\psi\rangle|^2 = \int_{\widehat{G}} |S_\psi(\varphi) p_\psi \chi_{\Omega_\psi}|^2 \, d\xi. \tag{5.9}$$

This equality and (5.8) imply

$$A \int_{\widehat{G}} |S_\psi(\varphi)|^2 p_\psi \le \sum_{k\in G} |\langle \varphi, T_k\psi\rangle|^2 \le B \int_{\widehat{G}} |S_\psi(\varphi)|^2 p_\psi.$$

These inequalities and the isometry result of Theorem (3.1)(i) are equivalent to those in (5.6). Thus, (5.8) implies that $\mathcal{B}_\psi$ is a frame for $\langle\psi\rangle_T$.

Now suppose $\mathcal{B}_\psi$ is a frame for $\langle\psi\rangle_T$. We establish (5.8) by contradiction. Suppose $p_\psi(\xi) < A$ on a set $E \subset \Omega_\psi \subset \widehat{G}$ of positive measure. Since $p_\psi \in L^1(\widehat{G}, d\xi)$ we have $\chi_E \in L^2(\widehat{G}, p_\psi)$. Since S_ψ is onto, there exists $\varphi_E \in \langle\psi\rangle_T$ such that $S_\psi(\varphi_E) = \chi_E$ and

$$\|\chi_E\|_{L^2(\widehat{G},p_\psi)} = \|\varphi_E\|_{\mathbb{H}}. \tag{5.10}$$

An argument similar to the one we used to establish (5.9), with φ_E playing the role of φ, yields

$$\sum_{k\in G} |\langle \varphi_E, T_k\psi\rangle|^2 = \int_{\widehat{G}} \chi_E p_\psi^2 \, d\xi. \tag{5.11}$$

Since $p_\psi(\xi) < A$ for $\xi \in E$, the right side of (5.11) is strictly smaller than

$$A \int_E p_\psi(\xi)\, d\xi = A\|\varphi_E\|^2_{\mathbb{H}}$$

(by (5.10)). Consequently, the left inequality of (5.6) does not hold for $\varphi = \varphi_E$. This shows that $A \le p_\psi(\xi)$ a.e. in Ω_ψ. A completely similar argument shows that $p_\psi(\xi) \le B$ a.e. in Ω_ψ. Hence, (5.8) must be true if $\mathcal{B}_\psi$ is a frame for Ω_ψ. ∎

REMARK. In [7, Theorem 2.8] and [8, Theorem 4.3.3], results that we derived in this section can be found that are associated with the Gabor unitary representation acting on $\mathbb{R}^n$ (see §1, §4). In this situation the bracket is obtained from the Zak transform. In particular, the weight whose properties correspond to $\mathcal{B}_\psi$ is denoted by q_ψ (see §1 where we introduced the weight with the equality $q_\psi(t,\xi) = |(Z\psi)(t,\xi)|^2$). We also remind the reader of equality (1.3) and the proof we gave at the end of §4.

6. Biorthogonality and minimality. In this section we continue to consider a countable abelian group G endowed with the discrete topology and a dual integrable unitary representation T of G acting on a Hilbert space $\mathbb{H}$.

Suppose ψ and $\tilde{\psi}$ belong to $\mathbb{H}\setminus\{0\}$. The collections $\mathcal{B}_\psi = \{T_k\psi : k \in G\}$ and $\mathcal{B}_{\tilde{\psi}} = \{T_k\tilde{\psi} : k \in G\}$ are said to be *biorthogonal* if and only if $\langle T_k\psi, T_l\tilde{\psi}\rangle = \delta_{kl}$ for all $k, l \in G$. If $\psi \in \mathbb{H}\setminus\{0\}$, $\mathcal{B}_\psi = \{T_k\psi : k \in G\}$ is said to be *minimal for* $\langle\psi\rangle_T$ if and only if there does not exist a $k_0 \in G$ such that

$$T_{k_0}\psi \in \overline{\operatorname{span}\{T_k\psi : k \in G,\ k \neq k_0\}}.$$

It can be shown (see [9]) that $\mathcal{B}_\psi$ is minimal for $\langle\psi\rangle_T$ if and only if $\psi \notin \overline{\operatorname{span}\{T_k\psi : k \in G,\ k \neq 0\}}$ (the closure is in $\mathbb{H}$).

THEOREM (6.1). *Under the hypothesis stated at the beginning of this section and* $\psi \in \mathbb{H}\setminus\{0\}$:

(i) *If there exists* $\tilde{\psi} \in \langle\psi\rangle_T$ *such that* $\mathcal{B}_\psi$ *and* $\mathcal{B}_{\tilde{\psi}}$ *are biorthogonal, then* $\mathcal{B}_\psi$ *is minimal.*

(ii) *Conversely, if* $\mathcal{B}_\psi$ *is minimal, then there exists* $\tilde{\psi} \in \langle\psi\rangle_T$ *such that* $\mathcal{B}_\psi$ *and* $\mathcal{B}_{\tilde{\psi}}$ *are biorthogonal.*

Proof. If $\mathcal{B}_\psi$ and $\mathcal{B}_{\tilde{\psi}}$ are biorthogonal then $0 = \langle T_k\psi, \tilde{\psi}\rangle$ for all $k \in G$, $k \neq 0$. Thus,

$$\tilde{\psi} \perp \overline{\operatorname{span}\{T_k\psi : k \in G,\ k \neq 0\}}.$$

Since $\langle\psi, \tilde{\psi}\rangle = 1$, $\psi \notin \overline{\operatorname{span}\{T_k\psi : k \in G,\ k \neq 0\}}$. Thus, $\mathcal{B}_\psi$ is minimal and (i) is established.

Now assume $\mathcal{B}_\psi$ is minimal. Then

$$\overline{\operatorname{span}\{T_k\psi : k \in G,\ k \neq 0\}} \subsetneq \langle\psi\rangle_T.$$

Hence, there exists $\tilde{\psi} \in \langle\psi\rangle_T$, $\tilde{\psi} \neq 0$, such that $\tilde{\psi} \perp \{T_k\psi : k \in G,\ k \neq 0\}$. It follows that $\langle\psi, \tilde{\psi}\rangle \neq 0$ and we can clearly assume $\langle\psi, \tilde{\psi}\rangle = 1$. Consequently, $\langle T_k\psi, T_k\tilde{\psi}\rangle = \langle\psi, \tilde{\psi}\rangle = 1$; while, if $k \neq l$, $\langle T_k\psi, T_l\tilde{\psi}\rangle = \langle T_{k-l}\psi, \tilde{\psi}\rangle = 0$. ∎

THEOREM (6.2). *Under the same hypotheses stated at the beginning of §6 suppose* $\psi \in \mathbb{H}\setminus\{0\}$. *Then there exists* $\tilde{\psi} \in \langle\psi\rangle_T$ *such that* $\mathcal{B}_\psi$ *and* $\mathcal{B}_{\tilde{\psi}}$ *are biorthogonal if and only if* $1/p_\psi \in L^1(\widehat{G}, d\xi)$.

Proof. Suppose $\mathcal{B}_\psi$ and $\mathcal{B}_{\tilde\psi}$ are biorthogonal with $\tilde\psi \in \langle\psi\rangle_T$. By Theorem (3.1), $S_\psi(\tilde\psi) \in L^2(\widehat{G}, p_\psi)$ and $\|\tilde\psi\|_{\mathbb{H}} = \|S_\psi(\tilde\psi)\|_{L^2(\widehat{G},p_\psi)}$. Also

$$\int_{\widehat{G}}(|S_\psi(\tilde\psi)|p_\psi^{1/2})p_\psi^{1/2} \le \Big(\int_{\widehat{G}}|S_\psi(\tilde\psi)|^2 p_\psi\Big)^{1/2}\Big(\int_{\widehat{G}} p_\psi\Big)^{1/2} = \|\tilde\psi\|_{\mathbb{H}}\|\psi\|_{\mathbb{H}} < \infty$$

since S_ψ is an isometry and, from the definition of dual integrability,

$$\Big(\int_{\widehat{G}} p_\psi\Big)^{1/2} = \|\psi\|_{\mathbb{H}} \quad \text{(see (2.2))}.$$

We also have, by (5.2), Theorem (3.1), and the biorthogonality,

$$\begin{aligned}\int_{\widehat{G}} S_\psi(\tilde\psi)e_{-k}p_\psi &= \int_{\widehat{G}} S_\psi(\tilde\psi)\,\overline{S_\psi(T_k\psi)}p_\psi = \langle S_\psi(\tilde\psi), S_\psi(T_k\psi)\rangle_{L^2(\widehat{G},p_\psi)}\\ &= \langle\tilde\psi, T_k\psi\rangle = \delta_{k_0}.\end{aligned}$$

Thus, the Fourier coefficients of $S_\psi(\tilde\psi)(\xi)p_\psi(\xi)$ are zero except the one corresponding to $k = 0$. Hence, $S_\psi(\tilde\psi)(\xi)p_\psi(\xi) = 1$ a.e. in $\widehat{G}$. This means that $p_\psi(\xi) > 0$ a.e. and we can write $|S_\psi(\tilde\psi)|^2 p_\psi(\xi) = 1/p_\psi(\xi)$. We therefore have

$$\int_{\widehat{G}} \frac{1}{p_\psi(\xi)}\,d\xi = \int_{\widehat{G}} |S_\psi(\tilde\psi)|^2 p_\psi(\xi)\,d\xi = \|\tilde\psi\|_{\mathbb{H}}$$

(since S_ψ is an isometry) and we see that $1/p_\psi \in L^1(\widehat{G}, d\xi)$.

Now suppose $1/p_\psi \in L^1(\widehat{G}, d\xi)$. Then, clearly, $1/p_\psi \in L^2(\widehat{G}, p_\psi)$ and, by Theorem (3.1), $S_\psi^{-1}(1/p_\psi)$ is a well defined element in $\langle\psi\rangle_T$; let us denote it by $\tilde\psi$. We have

$$\begin{aligned}\langle T_k\psi, \tilde\psi\rangle_{\mathbb{H}} &= \langle T_k\psi, S_\psi^{-1}(p_\psi^{-1})\rangle_{\mathbb{H}} = \langle S_\psi(T_k\psi), p_\psi^{-1}\rangle_{L^2(\widehat{G},p_\psi)}\\ &= \langle e_k, 1/p_\psi\rangle_{L^2(\widehat{G},p_\psi)} = \int_{\widehat{G}} e_k\,\frac{1}{p_\psi}\,p_\psi = \int_{\widehat{G}} e_k(\xi)\,d\xi = \delta_{k_0}.\end{aligned}$$

We have used Theorem (3.1), (5.2) and the orthogonality of $\{e_k : k \in G\}$ in $L^2(\widehat{G}, d\xi)$. This shows that $\mathcal{B}_{\tilde\psi}$ with $\tilde\psi = S_\psi^{-1}(1/p_\psi)$ is biorthogonal to $\mathcal{B}_\psi$. ∎

As a corollary of the two theorems, (6.1) and (6.2), we have

THEOREM (6.3). *Under the same hypotheses and $\psi \in \mathbb{H}\setminus\{0\}$, $\mathcal{B}_\psi$ is minimal if and only if $p_\psi^{-1} \in L^1(\widehat{G}, d\xi)$.*

7. A general framework for dual integrable representations. As discussed in §2 and §4, the translation representation T of $\mathbb{Z}^n$ on $L^2(\mathbb{R}^n)$ is dual integrable and can be combined with the modulation representation, $\mathcal{M}$, of a second copy of $\mathbb{Z}^n$ to obtain the dual integrable Gabor representation

$(T, \mathcal{M})$ of $\mathbb{Z}^n \times \mathbb{Z}^n$ on $L^2(\mathbb{R}^n)$. In the theory of wavelets, we consider unitary dilation representations

$$(D_a f)(x) = |\det a|^{1/2} f(ax),$$

where a is a member of a countable abelian subgroup, A, of $GL(n, \mathbb{R})$. In this section, we will give sufficient conditions on A so that $a \mapsto D_a$ is dual integrable. More generally, we will consider a σ-finite measure space $(\mathcal{X}, \mu)$ and an action $(k, x) \mapsto k \bullet x$ of a countable abelian group G on $\mathcal{X}$. Thus,

(1) $(k, x) \mapsto k \bullet x$ is a μ-measurable mapping from $G \times \mathcal{X}$ into $\mathcal{X}$ satisfying $k \bullet (l \bullet x) = (k + l) \bullet x$ for all $k, l \in G$ and $x \in \mathcal{X}$.
(2) $0 \bullet x = x$ for each $x \in \mathcal{X}$.

We will, in addition, assume the following regularity properties:

(3) μ is *quasi-G-invariant* in the sense that for each $k \in G$, the Radon–Nikodym derivative
$$J_k(x) = \frac{d\mu(k \bullet x)}{d\mu(x)}$$
exists and is positive a.e.
(4) There exists a μ-measurable subset $C \subset \mathcal{X}$ which is a *tiling domain for the action of* G in the sense that the tiles $k \bullet C$, $k \in G$, are mutually disjoint and, modulo a μ-null set, $\mathcal{X}$ is their union.

While (3) is a natural assumption, (4) is restrictive and excludes, for example, the action of $G = \mathbb{Z}$ on $\mathbb{R}^2$ defined by $k \bullet x$ equal to the rotation of x by $2\pi k\alpha$, $\alpha \notin \mathbb{Q}$. On the other hand, (4) is satisfied by the dyadic action $j \bullet x = 2^j x$ of $\mathbb{Z}$ on $\mathbb{R}^n$; also, by the shearing action $j \bullet (x, y) = (x, y + jx)$ of $\mathbb{Z}$ on $\mathbb{R}^2$.

From the chain rule for Radon–Nikodym derivatives we have

$$J_{k+l}(x) = J_k(l \bullet x) J_l(x) \quad \text{a.e. for } k, l \in G. \tag{7.1}$$

Moreover, using (1), (7.1) and a simple change of variable arguments we see that

$$k \mapsto D_k, \quad \text{where} \quad (D_k f)(x) = J_k(x)^{1/2} f(k \bullet x), \tag{7.2}$$

is a unitary representation of G on $L^2(\mathcal{X}, \mu)$.

Theorem (7.3). *Given an action of G on $(\mathcal{X}, \mu)$ satisfying the regularity conditions* (3) *and* (4)*, the representation D of G on $L^2(\mathcal{X}, \mu)$, defined by* (7.2)*, is dual integrable.*

Proof. Consistent with our treatment in §4 of dual integrability for translation representations, we define the generalized Zak transform $Z\psi$,

$\psi \in L^2(\mathcal{X},\mu)$, by

$$(Z\psi)(x,\xi) = \sum_{k\in G} (D_k\psi)(x) e_k(\xi). \tag{7.4}$$

Using the fact that $\{e_k : k \in G\}$ is an orthonormal basis, and also the tiling condition (4), we obtain

$$\begin{aligned} \int_C \int_{\widehat{G}} |(Z\psi)(x,\xi)|^2 \, d\xi \, d\mu(x) &= \int_C \sum_{k\in G} |(D_k\psi)(x)|^2 \, d\mu(x) \\ &= \sum_{k\in G} \int_C J_k(x) |\psi(k\bullet x)|^2 \, d\mu(x) \\ &= \sum_{k\in G} \int_{k\bullet C} |\psi(y)|^2 \, d\mu(y) \\ &= \int_{\mathcal{X}} |\psi(y)|^2 \, d\mu(y) = \|\psi\|^2_{L^2(\mathcal{X},\mu)}. \end{aligned} \tag{7.5}$$

Using (7.2) and a change of summation index, for each $l \in G$ we obtain

$$\begin{aligned} (ZD_l\psi)(x,\xi) &= \sum_{k\in G} (D_k D_l \psi)(x,\xi) e_k(\xi) \\ &= \sum_{j\in G} (D_j\psi)(x) e_j(\xi) \overline{e_{-l}(\xi)} \\ &= e_{-l}(\xi)(Z\psi)(x,\xi). \end{aligned} \tag{7.6}$$

By (7.5), $\psi \mapsto Z\psi$ is an isometry from $L^2(\mathcal{X},\mu)$ onto $L^2(C \times \widehat{G}, d\mu(x)\, d\xi)$. Define the bracket function $[\cdot,\cdot]_D : L^2(\mathcal{X},\mu) \times L^2(\mathcal{X},\mu) \to L^1(\widehat{G})$ by

$$[\varphi,\psi]_D(\xi) = \int_C (Z\varphi)(x,\xi) \, \overline{(Z\psi)(x,\xi)} \, d\mu(x). \tag{7.7}$$

Combining (7.6) and polarization of (7.5), we have

$$\begin{aligned} \int_{\widehat{G}} [\varphi,\psi]_D(\xi) e_{-l}(\xi) \, d\xi &= \int_{\widehat{G}} \int_C (ZD_l\varphi)(x,\xi) \, \overline{(Z\psi)(x,\xi)} \, d\mu(x) \, d\xi \\ &= \langle D_l\varphi, \psi\rangle_{L^2(\mathcal{X},\mu)} = \langle \varphi, D_{-l}\psi\rangle_{L^2(\mathcal{X},\mu)}. \end{aligned} \tag{7.8}$$

Thus, $[\cdot,\cdot]_D$ satisfies (2.2) in the definition of dual integrable representations. ∎

The object defined in (7.4) is a generalization of the Zak transform adapted to our situation. It coincides with the Zak transform when $\mathbb{Z}^n$ acts on $\mathbb{R}^n$ by translations. When the action is dilation by 2 in the real line, the object defined in (7.4) is called the multiplicative Zak transform in [3]. It also appears in the work [18] and more generally in [20] and [5]. We thank Professor Wojciech Czaja for pointing out some of these references to us.

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