COLLOQUIUM MATHEMATICUM

VOL. 118 2010 NO. 1

A.E. CONVERGENCE OF ANISOTROPIC PARTIAL FOURIER INTEGRALS ON EUCLIDEAN SPACES AND HEISENBERG GROUPS

BY

D. MÜLLER (Kiel) and E. PRESTINI (Roma)

Dedicated, in admiration and affection, to Andrzej Hulanicki. We shall always remember his enthusiasm for mathematics, his warmth, generosity, and sense of humor.

Abstract. We define partial spectral integrals S_R on the Heisenberg group by means of localizations to isotropic or anisotropic dilates of suitable star-shaped subsets V containing the joint spectrum of the partial sub-Laplacians and the central derivative. Under the assumption that an L^2-function f lies in the logarithmic Sobolev space given by $\log(2 + L_\alpha)f \in L^2$, where L_α is a suitable "generalized" sub-Laplacian associated to the dilation structure, we show that $S_R f(x)$ converges a.e. to $f(x)$ as $R \to \infty$.

1. Introduction. Under the assumption that f belongs to the logarithmic Sobolev space given by $\log(2 - \Delta)f \in L^2(\mathbb{R}^d)$, where Δ denotes the Euclidean Laplacian, a short and simple proof of the almost everywhere convergence as $R \to \infty$ of the partial spectral integrals $S_R f$, associated to the dilates RV of any bounded measurable region V star-shaped with respect to the origin and containing the origin in its interior, has been given in [CMP]. The proof was based on Rademacher–Men'shov's theorem.

By choosing V equal to the unit ball, the spherical partial integrals studied in [CS] are obtained. The fact that the proof in [CMP] makes use of only very basic properties allows for wide generalizations. In [MMP] e.g. the above result has been extended to arbitrary connected Lie groups, the partial spectral integrals being defined in terms of a sub-Laplacian as well as the corresponding logarithmic Sobolev space.

More general partial integrals S_R can be defined by means of the group Fourier transform, for specific groups. We shall demonstrate this in the present article for the case of the Heisenberg group $\mathbb{H}_n$, where such spectral integrals can also be defined by means of the joint spectral resolution of

2010 *Mathematics Subject Classification*: 22E30, 43A50.

Key words and phrases: Rademacher–Men'shov theorem, Heisenberg group, sub-Laplacian, spectral integral.

DOI: 10.4064/cm118-1-18

the partial sub-Laplacians $L_1, \ldots, L_n$ and the central derivative $-iU$. For V we choose suitable star-shaped subsets containing, as an interior point, the origin of the ambient space $\mathbb{R}^{n+1}$ which contains the joint spectrum (the "Heisenberg fan" F_n) of these operators. We shall work with arbitrary isotropic or anisotropic dilations on $\mathbb{R}^{n+1}$.

As a simpler model case we shall first consider the Euclidean space $\mathbb{R}^d$, whose dual is endowed with arbitrary isotropic or anisotropic dilations. We remark that our results in this setting, being independent of the geometry of V, do not fall under the scope of C. Fefferman's method in [F] which requires V to be a rectangular box.

As in [MMP], our approach to this problem on the Heisenberg group makes use of an asymptotic estimate for $R \to \infty$ of $W(R) = \|K_R\|_2^2$, where K_R denotes the convolution kernel associated to our partial integrals S_R, together with the right-continuity of $W(R)$. To show this latter property of $W(R)$, in the case of anisotropic dilations, we furthermore assume V to be closed. A counter-example is given to show that without some extra assumption the right-continuity of $W(R)$ may fail to be true. This is related to the fact that anisotropic dilations will not leave the Heisenberg fan invariant.

Throughout the article, C and c will denote constants which may change from line to line.

2. The Euclidean case. Before studying the Heisenberg group, we shall consider the simpler case of $\mathbb{R}^d$ endowed with an anisotropic dilation structure. As in [CMP] a basic tool will be the classical

THEOREM 2.1 (Rademacher–Men'shov). *Suppose that (X, μ) is a positive measure space. Then there is a positive constant c with the following property: For each orthogonal subset $\{f_n : n \in \mathbb{N}\}$ in $L^2(X, \mu)$ satisfying*

$$\sum_{n=0}^{\infty} (\log(n+2))^2 \|f_n\|_2^2 < \infty, \tag{2.1}$$

the maximal function $F^*(x) := \sup_{N\in\mathbb{N}} |\sum_{n=0}^{N} f_n(x)|$ is in $L^2(X, \mu)$, and

$$\|F^*\|_2 \le c\Big(\sum_{n=0}^{\infty} (\log(n+2))^2 \|f_n\|_2^2\Big)^{1/2}. \tag{2.2}$$

In particular, (2.1) *implies that the series $\sum_{n=0}^{\infty} f_n(x)$ converges almost everywhere on X.*

See [S] or Theorem XIII.10.21 from [Z] for a proof. Here log means the logarithm to the base 2.

Let $\{\delta_r\}_{r>0}$ be a fixed family of (usually) anisotropic dilations in $\mathbb{R}^d$ given by $\delta_r x = (r^{\alpha_1} x_1, \ldots, r^{\alpha_d} x_d)$ with $\alpha_j > 0$ for $j = 1, \ldots, d$. A set $V \subset \mathbb{R}^d$ is said to be *star-shaped* with respect to these dilations if for every $x \in V$,

$$\delta_r x \in V \quad \text{for all } 0 \le r < 1.$$

Let $D = \sum_{j=1}^d \alpha_j$ denote the *homogeneous dimension* of $\mathbb{R}^d$ with respect to the above dilations. For any measurable subset $W \subset \mathbb{R}^d$ we denote by S_W the Fourier multiplier operator given by $\widehat{S_W f}(\xi) = \chi_W(\xi)\hat{f}(\xi)$. By the method developed in [CMP] we can easily prove the following

THEOREM 2.2. *Let $V \subset \mathbb{R}^d$ be a bounded, measurable, star-shaped subset containing the origin as an interior point, and set $S_R = S_{\delta_R V}$. If*

$$\int_{\mathbb{R}^d} |\hat{f}(\xi) \log(2 + |\xi|)|^2 \, d\xi < \infty$$

then $S_R f(x) \to f(x)$ a.e. as $R \to \infty$. Moreover, if B is any set with finite measure in $\mathbb{R}^d$ then

$$\|Mf\|^2_{L^2(B)} \le C_{B,d} \int |\hat{f}(\xi) \log(2 + |\xi|)|^2 \, d\xi, \tag{2.3}$$

where M denotes the maximal function defined by $Mf(x) = \sup_{R>1} |S_R f(x)|$.

Proof. We just give a brief sketch. Since $|\delta_R V| = R^D |V|$, by choosing $R_n = n^{1/D}$, we see that

$$|\delta_{R_n} V \setminus \delta_{R_{n-1}} V| = |V| \quad \text{for every } n = 1, 2, \ldots.$$

Then, as in [CMP], by the Rademacher–Men'shov Theorem 2.1 it follows that

$$S_{R_n} f(x) \quad \text{converges a.e. in } \mathbb{R}^d$$

and that the maximal function $\tilde{M}f(x) = \sup_n |S_{R_n} f(x)|$ belongs to $L^2(B)$. Since the origin is an interior point of V it follows that $\bigcup_{n\ge1} \delta_{R_n} V = \mathbb{R}^d$, hence $S_{R_n} f \to f$ in $L^2(\mathbb{R}^d)$ as $n \to \infty$ and so $S_{R_n} f(x) \to f(x)$ a.e.

Finally, for $R_{n-1} \le R < R_n$ the error term $S_R f(x) - S_{R_{n-1}} f(x)$ can be shown to tend to zero for every $x \in \mathbb{R}^d$ as $n \to \infty$ by Cauchy–Schwarz' inequality, as in [CMP]. Then the estimate (2.3) for Mf easily follows. ∎

REMARK 2.3. If Q is the unit cube, say in $\mathbb{R}^2$ for simplicity, then a stronger result is known. For instance, if we use the parabolic dilations $\delta_r(x_1, x_2) = (rx_1, r^2 x_2)$, then

$$S_{\delta_R Q} f(x, y) \to f(x, y) \quad \text{a.e. as } R \to \infty, \text{ for every } f \in L^2(\mathbb{R}^2).$$

Indeed, one can derive this result easily from Carleson's theorem following C. Fefferman's idea in [F] as follows: Notice first that $\delta_R Q$ is the rectangle $\{(\xi, \eta) : |\xi| \le R, |\eta| \le R^2\}$. We therefore decompose $f = f_1 + f_2$, where $\widehat{f_1} = \hat{f}\chi_P$, with $P = \{(\xi, \eta) : |\eta| \le \xi^2\}$. Notice that χ_P is an L^2-bounded Fourier multiplier. Then it is easy to see that

$$\sup_{R\ge1} |S_{\delta_R Q} f_1(x, y)| \lesssim (C_x f_1)(x, y),$$

where C_x denotes Carleson's maximal operator acting in the variable x. Similarly

$$\sup_{R\geq 1} |S_{\delta_R Q} f_2(x,y)| \lesssim (C_y f_2)(x,y),$$

where C_y denotes Carleson's maximal operator acting in the variable y. Then the result follows by standard arguments.

The method just described does not work for dilations (isotropic or anisotropic) of sets V with curved boundary, and not even for anisotropic dilations of general polygonal regions since the slope of the edges of the dilated polygons might change under anisotropic dilations.

The advantage of our method, which however requires a stronger regularity assumption on the function f, lies in the fact that it is independent of the geometry of the set V.

3. The case of the Heisenberg group

3.1. Statement of the main result. Recall that the *Heisenberg group* $\mathbb{H}_n$ can be defined as $\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}$ endowed with the product

$$(x,y,u)\cdot(x',y',u') := \left(x+x', y+y', u+u' + \frac{1}{2}\,(x\cdot y' - y\cdot x')\right).$$

We denote by $\mathfrak{h}_n$ its Lie algebra, which can again be identified with $\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}$. Then the exponential mapping $\exp : \mathfrak{h}_n \to \mathbb{H}_n$ is the identity mapping. Identifying as usual an element $X \in \mathfrak{h}_n$ with its Lie derivative

$$(L_X\varphi)(g) := \frac{d}{dt}\,\varphi(g\exp tX)\Big|_{t=0}, \qquad g\in\mathbb{H}_n,$$

we shall consider the elements of the Lie algebra as left-invariant vector fields. A natural basis of $\mathfrak{h}_n$ is then given by the vector fields

$$X_j = \frac{\partial}{\partial x_j} - \frac{1}{2}\,y_j\,\frac{\partial}{\partial u}, \qquad Y_j := \frac{\partial}{\partial y_j} + \frac{1}{2}\,x_j\,\frac{\partial}{\partial u}, \qquad j=1,\dots,n,$$

and $U = \partial/\partial u$. They satisfy the "Heisenberg commutation relations"

$$[X_j, Y_k] = \delta_{jk} U,$$

all other brackets being zero. In particular, U spans the center of $\mathfrak{h}_n$, and $\mathfrak{h}_n$ is two-step nilpotent. Denote by $\mathfrak{u}(\mathfrak{h}_n)$ its universal enveloping algebra, regarded as the associative algebra of all left-invariant differential operators on $\mathbb{H}_n$.

The *partial sub-Laplacians* $L_j \in \mathfrak{u}(\mathfrak{h}_n)$ are defined by

$$L_j = -(X_j^2 + Y_j^2), \qquad j=1,\dots,n.$$

These play a basic role within $\mathfrak{u}(\mathfrak{h}_n)$ because of the following well-known facts:

Identify $(x,y) \in \mathbb{R}^n \times \mathbb{R}^n$ with $z = (z_1, \ldots, z_n) = (x_1 + iy_1, \ldots, x_n + iy_n) \in \mathbb{C}^n$, and call a function f on $\mathbb{H}_n$ *polyradial* if $f(z,u) = \tilde{f}(|z_1|, \ldots, |z_n|, u)$ for some function $\tilde{f}$ on $\mathbb{R}_+^n \times \mathbb{R}$. Under this identification of the underlying manifold of $\mathbb{H}_n$ with $\mathbb{C}^n \times \mathbb{R}$, the n-torus $\mathbb{T}^n = \{(e^{i\varphi_1}, \ldots, e^{i\varphi_n}) : \varphi_i \in [0, 2\pi[\}$ acts by (symplectic) automorphisms $(z_1, \ldots, z_n, u) \mapsto (e^{i\varphi_1} z_1, \ldots, e^{i\varphi_n} z_n, u)$ on $\mathbb{H}_n$, and f is polyradial if and only if $f \circ \tau = f$ for every $\tau \in \mathbb{T}^n$. The pair $(\mathbb{H}_n, \mathbb{T}^n)$ is then known to be a Gelfand pair in the sense that the algebra

$$L^1_{\mathrm{pr}}(\mathbb{H}_n) := \{f \in L^1(\mathbb{H}_n) : f \text{ is polyradial}\}$$

is a commutative subalgebra of $L^1(\mathbb{H}_n)$, whose Gelfand spectrum has been identified by A. Hulanicki and F. Ricci in [HR].

The counterpart of this algebra within $\mathfrak{u}(\mathfrak{h}_n)$ is the subalgebra $\mathfrak{u}_{\mathrm{pr}}(\mathfrak{h}_n)$ of all $\mathbb{T}^n$-invariant elements (notice here that the subgroup $\mathbb{T}^n$ of the automorphism group of $\mathbb{H}_n$ acts in a natural way by automorphisms on $\mathfrak{u}(\mathfrak{h}_n)$). This subalgebra is then generated by the partial sub-Laplacians $L_1, \ldots, L_n$ and U, so that the harmonic analysis for polyradial functions can be viewed as the joint spectral theory of these operators.

A bi-invariant Haar measure on $\mathbb{H}_n$ is given by the Lebesgue measure $dg = dx\, dy\, du$, and we shall denote by $L^2(\mathbb{H}_n)$ the L^2-space with respect to this measure. The operators L_j and iU, initially defined on $C_0^\infty(\mathbb{H}_n)$, are known to be essentially self-adjoint on $L^2(\mathbb{H}_n)$, and their closures will again be denoted by the same symbols.

By our previous remarks these operators form a commutative set of self-adjoint operators on $L^2(\mathbb{H}_n)$ so that, for every Borel measurable function ψ on $\mathbb{R}^{n+1}$, the *joint spectral multiplier operator* $T_\psi = \psi(L_1, \ldots, L_n, -iU)$ can be defined as a (possibly unbounded) operator on $L^2(\mathbb{H}_n)$ by means of the spectral theorem. This functional calculus will be made explicit later by means of the representation theory of the Heisenberg group. In particular, for any Borel measurable subset $W \subset \mathbb{R}^{n+1}$ we denote by S_W the joint spectral multiplier operator corresponding to the characteristic function of W, i.e.

$$S_W = T_{\chi_W} = \chi_W(L_1, \ldots, L_n, -iU).$$

Notice that $\mathbb{R}^{n+1}$ contains the joint spectrum of the operators $L_1, \ldots, L_n$, $-iU$, the so-called Heisenberg fan (see Subsection 3.2). In analogy with the dilations considered in the Euclidean setting, which were acting on the dual space, let us fix a one-parameter family of dilations on the space $\mathbb{R}^{n+1} = \mathbb{R}^n \times \mathbb{R}$, with coordinates $\xi = (\lambda, \mu) = (\lambda_1, \ldots, \lambda_n, \mu)$, of the form

$$\delta_r \xi = (r^{\alpha_1} \lambda_1, \ldots, r^{\alpha_n} \lambda_n, r^\beta \mu),$$

with $\alpha_j > 0$ for $j = 1, \ldots, n$ and $\beta > 0$. Notice that these dilations leave the subspace $\mathbb{R}^n \setminus \{0\}$ invariant, which we identify with $\mathbb{R}^n$. We shall denote the corresponding dilations of $\mathbb{R}^n$ again by δ_r. We set

$$\alpha_{\min} = \min_{j=1,\dots,n} \alpha_j, \qquad D = \sum_{j=1}^{n} \alpha_j.$$

Notice that D is the homogeneous dimension of $\mathbb{R}^n$ with respect to these dilations.

Next assume that $V \subset \mathbb{R}^{n+1}$ is a measurable, star-shaped subset containing the origin as an interior point. Because of the special role played by the operator $-iU$ compared to the L_j, we now distinguish two cases:

Case 1. If $\beta < \alpha_{\min}$ we assume that V is a bounded set and define

$$\mathcal{L} = L_1^{1/\alpha_1} + \cdots + L_n^{1/\alpha_n} + |U|^{1/\beta}. \tag{3.1}$$

Case 2. If $\beta \geq \alpha_{\min}$ we only assume that the projection of V onto the space $\mathbb{R}^n$ of λ-variables is a bounded set and define

$$\mathcal{L} = L_1^{1/\alpha_1} + \cdots + L_n^{1/\alpha_n}. \tag{3.2}$$

Notice that the corresponding joint spectral multipliers $\psi_\alpha(\xi) = \lambda_1^{1/\alpha_1} + \cdots + \lambda_n^{1/\alpha_n}$ respectively $\psi_\alpha(\xi) = \lambda_1^{1/\alpha_1} + \cdots + \lambda_n^{1/\alpha_n} + |\mu|^{1/\beta}$ are homogeneous of degree one with respect to our dilations. A particular case of such an operator $\mathcal{L}$ is the sub-Laplacian $L = L_1 + \cdots + L_n$ (Case 2) and the full Laplacian $L - U^2$. We can now state our main result.

Theorem 3.1. *Let $V \subset \mathbb{R}^{n+1}$ be a measurable, star-shaped subset containing the origin as an interior point, which furthermore has the properties as described in Case* 1, *respectively Case* 2. *In case the dilations δ_r are anisotropic, assume in addition that V is closed. Choose $\mathcal{L}$ as in* (3.1) *respectively* (3.2) *and let $S_R = S_{\delta_R V}$.*

If $\log(2+\mathcal{L})f \in L^2(\mathbb{H}_n)$ then $S_R f(x) \to f(x)$ a.e. as $R \to \infty$. Moreover, if B is any subset with finite measure of $\mathbb{H}_n$, then

$$\|Mf\|^2_{L^2(B)} \leq C_{B,n} \|\log(2+\mathcal{L})f\|_{L^2}, \tag{3.3}$$

where M denotes the maximal function defined by

$$Mf(x) = \sup_{R \geq 1} |S_R f(x)|.$$

3.2. Concrete realization of joint spectral multiplier operators. Let us first recall some well-known facts about the Heisenberg group and its representation theory (see e.g. [Fo], [St], [T], and the original papers by D. Geller [G1], [G2]).

The group Fourier transform on the Heisenberg group $\mathbb{H}_n$ is defined in terms of the Schrödinger representations, i.e. the irreducible unitary representations of infinite dimension: For every $\mu \in \mathbb{R}^\times := \mathbb{R}\setminus\{0\}$ the *Schrödinger* *representation* π_μ, acting on $L^2(\mathbb{R}^n)$, is given by

$$[\pi_\mu(x,y,u)\phi](t) := e^{i\mu(u+y\cdot t+\frac{1}{2}x\cdot y)}\phi(t+x), \qquad \phi \in L^2(\mathbb{R}^n).$$

One checks that $\pi_\mu : \mathbb{H}_n \to \mathcal{U}(L^2(\mathbb{R}^n))$ is a strongly continuous homomorphism from $\mathbb{H}_n$ into the group $\mathcal{U}(L^2(\mathbb{R}^n))$ of unitary operators on the representation space $L^2(\mathbb{R}^n)$.

The *Fourier transform* of a function $f \in L^1(\mathbb{H}_n)$ is the operator-valued mapping $\hat{f} : \mathbb{R}^\times \to \mathcal{B}(L^2(\mathbb{R}^n))$ given (in the strong operator sense) by

$$\hat{f}(\mu) := \int_{\mathbb{H}_n} f(z,u)\pi_\mu(z,u)\,dz\,du, \qquad \mu \in \mathbb{R}^\times.$$

One also writes $\pi_\mu(f)$ instead of $\hat{f}(\mu)$. Then

$$\widehat{f_1 * f_2}(\mu) = \widehat{f_1}(\mu)\widehat{f_2}(\mu) \qquad \forall f_1, f_2 \in L^1(\mathbb{H}_n),$$

where the convolution of f_1 and f_2 on $\mathbb{H}_n$ is defined by

$$f_1 * f_2(g) := \int_{\mathbb{H}_n} f_1(h)f_2(h^{-1}g)\,dh.$$

For sufficiently "nice" functions, such as Schwartz functions, one then has the following *Fourier inversion formula*:

$$f(z,u) = (2\pi)^{-n-1}\int_{\mathbb{R}^\times} \operatorname{tr}(\pi_\mu(z,u)^*\hat{f}(\mu))|\mu|^n\,d\mu. \tag{3.4}$$

Here $\operatorname{tr}(T)$ denotes the trace of the operator T. Equivalently, one has *Plancherel's formula*: If $f \in L^1 \cap L^2(\mathbb{H}_n)$, then

$$\|f^2\|_2 = (2\pi)^{-n-1}\int_{\mathbb{R}^\times} \|\hat{f}(\mu)\|_{\mathrm{HS}}^2|\mu|^n\,d\mu, \tag{3.5}$$

where $\|\cdot\|_{\mathrm{HS}}$ denotes the Hilbert–Schmidt norm. Denote by $d\pi_\mu$ the derived representation of the Lie algebra $\mathfrak{h}_n$. Then

$$d\pi_\mu(X_j) = \partial_{t_j}, \qquad d\pi_\mu(Y_j) = i\mu t_j, \qquad d\pi_\mu(U) = i\mu,$$

so that

$$d\pi_\mu(L_j) = -\partial_{t_j}^2 + \mu^2 t_j^2, \qquad j = 1,\dots,n.$$

These are rescaled *Hermite operators*, acting on the jth coordinate only. The joint eigenfunctions of $d\pi_\mu(L_1),\dots,d\pi_\mu(L_n)$ (and $d\pi_\mu(-iU)$) are therefore given by

$$h_k^\mu(t) := |\mu|^{n/4}\prod_{j=1}^n h_{k_j}(|\mu|^{1/2}t_j),$$

where $k = (k_1,\dots,k_n) \in \mathbb{N}^n$ and $h_m(s)$ is the L^2-normalized Hermite function given by

$$h_m(s) := (2^m\sqrt{\pi}\,m!)^{-1/2}H_m(s)e^{-s^2/2}.$$

Here $H_m(s)$ denotes the Hermite polynomial of degree m, i.e.

$$H_m(s) := (-1)^m e^{s^2} \frac{d^m}{ds^m}(e^{-s^2}).$$

Then

$$d\pi_\mu(L_j)h_k^\mu = |\mu|(2k_j+1)h_k^\mu, \quad j = 1, \dots, n, \tag{3.6}$$

$$d\pi_\mu(-iU)h_k^\mu = \mu h_k^\mu. \tag{3.7}$$

Therefore if ψ is any Borel measurable joint spectral multiplier on $\mathbb{R}^{n+1}$, say of moderate growth, we can explicitly define the (possibly unbounded) operator $T_\psi = \psi(L_1, \dots, L_n, -iU)$ by means of its Fourier transform given by

$$\pi_\mu(T_\psi f)h_k^\mu := \psi(|\mu|(2k_1+1), \dots, |\mu|(2k_n+1), \mu)\pi_\mu(f)h_k^\mu, \tag{3.8}$$

and the Fourier inversion formula (3.4). This makes sense for instance for Schwartz functions $f \in \mathcal{S}(\mathbb{H}_n)$.

Denote by F_n the *Heisenberg fan,* i.e. the closure of the set

$$\{(|\mu|(2k_1+1), \dots, |\mu|(2k_n+1), \mu) : \mu \in \mathbb{R}^\times,\ k = (k_1, \dots, k_n) \in \mathbb{N}^n\} \subset \mathbb{R}^{n+1}.$$

Then clearly the operator T_ψ depends only on the restriction of ψ to F_n. Moreover, if ψ is bounded on the Heisenberg fan then, by Plancherel's theorem, T_ψ is bounded on $L^2(\mathbb{H}_n)$ with norm

$$\|T_\psi\| = \|\psi|_{F_n}\|_\infty. \tag{3.9}$$

Also by the Schwartz kernel theorem and left-invariance there exists a unique convolution kernel $K_\psi \in \mathcal{S}'(\mathbb{H}_n)$ such that

$$T_\psi f = f * K_\psi \quad \text{for every } f \in \mathcal{S}(\mathbb{H}_n). \tag{3.10}$$

We shall use the abbreviation

$$\tilde{k} = (2k_1+1, \dots, 2k_n+1) \quad \text{for } k \in \mathbb{N}^n.$$

We also define a *spectral measure* σ supported in F_n by

$$\int h(\xi)\, d\sigma(\xi) = c_n \int_{\mathbb{R}^\times} \sum_{k \in \mathbb{N}^n} h(|\mu|\tilde{k}, \mu)|\mu|^n \, d\mu, \tag{3.11}$$

for suitable Borel measurable functions h on $\mathbb{R}^{n+1}$, where $c_n = (2\pi)^{-n-1}$. The following identity follows then easily from (3.5) and (3.8):

$$\|K_\psi\|_2^2 = \int |\psi(\xi)|^2 \, d\sigma(\xi). \tag{3.12}$$

3.3. Proof of Theorem 3.1. Let $V \subset \mathbb{R}^{n+1}$ be as in the theorem, and let again $S_R = S_{\delta_R V}$. We set $K_R = K_{\chi_{S_R}}$ so that by (3.10),

$$S_R f = f * K_R \quad \text{for every } f \in \mathcal{S}(\mathbb{H}_n).$$

We distinguish the cases: $\beta < \alpha_{\min} = \min_{j=1,\dots,n} \alpha_j$ (Case 1), and $\beta \geq \alpha_{\min}$ (Case 2). In Case 1 we introduce a homogeneous norm $|\cdot|$ on $\mathbb{R}^{n+1}$, in the sense of [FS], as follows:

$$|(\lambda, \mu)| = \max(|\lambda_1|^{1/\alpha_1}, \dots, |\lambda_n|^{1/\alpha_n}, |\mu|^{1/\beta}).$$

In Case 2 we work with the homogenous seminorm

$$|(\lambda, \mu)| = \max(|\lambda_1|^{1/\alpha_1}, \dots, |\lambda_n|^{1/\alpha_n})$$

instead. Then in particular $|\delta_r \xi| = r|\xi|$. Notice that an equivalent homogeneous norm, respectively seminorm, is given by

$$\|(\lambda, \mu)\| = |\lambda_1|^{1/\alpha_1} + \dots + |\lambda_n|^{1/\alpha_n} + |\mu|^{1/\beta},$$

and

$$\|(\lambda, \mu)\| = |\lambda_1|^{1/\alpha_1} + \dots + |\lambda_n|^{1/\alpha_n}.$$

Hence in both Case 1 and Case 2 we have, in the sense of functional calculus,

$$\mathcal{L} = \|(L_1, \dots, L_n, -iU)\|. \tag{3.13}$$

Then our assumptions on V imply that there exists a constant $M \geq 1$ such that, in both cases, for every $R \geq M$ we have

$$|\xi| \leq MR \quad \text{for every } \xi \in \delta_R V; \tag{3.14}$$

$$\xi \in \delta_R V \quad \text{for every } \xi \in F_n \text{ with } |\xi| \leq R/M. \tag{3.15}$$

Indeed, in Case 1, V is bounded so that there is a constant M such that $|\xi| \leq M$ for every $\xi \in V$. Similarly, in Case 2, the projection of V onto $\mathbb{R}^n$ is bounded so that the same conclusion holds. Thus (3.14) follows by scaling.

As for (3.15), in Case 1, we can use a similar scaling argument making use of the fact that V contains 0 as an interior point. Statement (3.15) then even holds for any $\xi \in \mathbb{R}^{n+1}$. The reasoning in Case 2 is a bit more subtle:

Assume $\xi = (|\mu|\tilde{k}, \mu) \in F_n$ satisfies $|\xi| \leq R/M$. Then $(2k_j + 1)|\mu| \leq (R/M)^{\alpha_j}$ for $j = 1, \dots, n$ and in particular $|\mu| \leq (R/M)^{\alpha}_{\min}$. Since in Case 2, $\beta \geq \alpha_{\min}$, we get $|\mu| \leq (R/M)^{\beta}$. Therefore $\delta_{R^{-1}}\xi = \eta$, where η lies in the set U given by $|\eta_j| \leq (1/M)^{\alpha_j}$, $j = 1, \dots, n$, and $|\eta_{n+1}| \leq (1/M)^{\beta}$. By choosing M sufficiently large we may assume that $U \subset V$ and then $\xi \in \delta_R V$.

Our strategy to prove Theorem 3.1 will be to adapt the method of [MMP] by means of the following two lemmas.

Lemma 3.2. *$K_R \in L^2(\mathbb{H}_n)$ for every $R > 0$. Moreover, if we set $W(R) = \|K_R\|_2^2$ then W is an increasing function and there is a constant $C \geq 1$ such that for $R \gg 1$,*

$$\frac{1}{C} R^{D+\nu} \leq W(R) \leq C R^{D+\nu},$$

where $\nu = \min\{\alpha_{\min}, \beta\} = \min\{\alpha_1, \dots, \alpha_n, \beta\}$.

Proof. By Plancherel's formula (3.12) for spectral multipliers and (3.11) we have

$$W(R) = \|K_R\|_2^2 = \sigma(\delta_R V) = c_n \int\limits_{\mathbb{R}^\times} \Big(\sum_{k\in\mathbb{N}^n,\, (|\mu|\tilde{k},\mu)\in\delta_R V} 1\Big) |\mu|^n \, d\mu.$$

This function is clearly increasing in R. We shall prove that the right-hand side in this display is finite and of order $O(R^{D+\nu})$. Indeed, $(|\mu|\tilde{k},\mu) \in \delta_R V$ implies by (3.14) that

$$\begin{aligned} (2k_j+1)|\mu| &\le (MR)^{\alpha_j}, \qquad j=1,\dots,n,\\ |\mu| &\le (MR)^\nu, \end{aligned}$$

since $\nu = \beta$ in Case 1, and $\nu = \alpha_{\min}$ in Case 2. Notice that, in Case 2, the last inequality is a consequence of the first n inequalities. Therefore

$$\begin{aligned} W(R) &\le C \int\limits_{|\mu|\le(MR)^\nu} \prod_{j=1}^n \Big(\sum_{2k_j+1\le (MR)^{\alpha_j}/|\mu|} 1\Big)|\mu|^n \, d\mu \\ &\le C(MR)^{\alpha_1+\dots+\alpha_n} \int\limits_{|\mu|\le(MR)^\nu} d\mu \le CR^{D+\nu}. \end{aligned}$$

The lower bound is derived in a similar way by using (3.15) in place of (3.14). Indeed, by (3.15), in Case 2, $(|\mu|\tilde{k},\mu) \in \delta_R V$ whenever

$$(2k_j+1)|\mu| \le (R/M)^{\alpha_j} \quad \text{for every } j = 1\dots,n, \tag{3.16}$$

and, in Case 1, if in addition $|\mu| \le (R/M)^\beta$. In particular, in both cases, we see that $(|\mu|\tilde{k},\mu) \in \delta_R V$ whenever $|\mu| \le (R/M)^\nu$ and (3.16) holds. Notice also that for such μ and $R \ge M$ we have $(R/M)^{\alpha_j}/|\mu| \ge 1$ for every j. Therefore

$$\begin{aligned} W(R) &\ge c \int\limits_{|\mu|\le(R/M)^\nu} \prod_{j=1}^n \Big(\sum_{2k_j+1\le (R/M)^{\alpha_j}/|\mu|} 1\Big)|\mu|^n \, d\mu \\ &\ge c(R/M)^{\alpha_1+\dots+\alpha_n} \int\limits_{|\mu|\le(R/M)^\nu} d\mu \ge cR^{D+\nu}, \end{aligned}$$

for positive constants $c > 0$ which may change from line to line. ∎

LEMMA 3.3. *Under the assumptions on V of Theorem* 3.1 *the function $W(R)$ is right-continuous.*

Proof. We shall prove that W is continuous from the right at $R = 1$. For general values of R the proof is similar.

We have seen that

$$W(R) = \sigma(\delta_R V).$$

Let $\{R_j\}_j$ be a decreasing sequence such that $\lim_{j\to\infty} R_j = 1$. We have to prove that $W(R_j) \to W(1)$ as $j \to \infty$. To this end we first observe that

$$V \subseteq \bigcap_{j=1}^{\infty} \delta_{R_j} V \subseteq \overline{V}. \tag{3.17}$$

Indeed, if $x \in V$ then $\delta_r x \in V$ for all $r \le 1$ since V is star-shaped. Write $x = \delta_{R_j}(\delta_{1/R_j} x)$ to see that $x \in \delta_{R_j} V$. Hence the first inclusion is clear. To prove the second inclusion let $y \in \bigcap_{j=1}^{\infty} \delta_{R_j} V$. Then there are $z_j \in V$ such that $y = \delta_{R_j} z_j$. Trivially $y = \lim \delta_{1/R_j} y$, hence $y = \lim z_j \in \overline{V}$.

First case: anisotropic dilations. In this case we assume that V is closed so that $V = \bigcap_{j=1}^{\infty} \delta_{R_j} V$ by (3.17). Since $\delta_{R_1} V \supset \delta_{R_2} V \supset \cdots \supset V$ and $\sigma(\delta_{R_1} V) < \infty$ the dominated convergence theorem implies that $\sigma(V) = \lim_{j\to\infty} \sigma(\delta_{R_j} V)$. Hence $W(1) = \lim_{j\to\infty} W(R_j)$.

Second case: isotropic dilations $\delta_R x = Rx$. Set $\tilde{V} = \bigcap_{j=1}^{\infty} \delta_{R_j} V$. Then our reasoning above shows that $\sigma(\delta_{R_j} V) \to \sigma(\tilde{V})$ as $j \to \infty$. Therefore it will suffice to prove that $\sigma(\tilde{V} \setminus V) = 0$.

Fix any ray

$$\mathit{\Gamma}_k^{\pm} = \{((2k_1+1)|\mu|, \ldots, (2k_n+1)|\mu|, \mu) : \pm\mu > 0\}$$

in the Heisenberg fan F_n and take $x \in \tilde{V}$. Then we may write $x = R_j x_j$ with $x_j = (1/R_j)x \in V$. Clearly $x_j \to x$.

Since V is star-shaped, $Rx \in V$ for $0 \le R < 1$. Therefore $\mathit{\Gamma}_k^{\pm} \cap (\tilde{V} \setminus V)$ contains at most one point. So $\sigma(\mathit{\Gamma}_k^{\pm} \cap (\tilde{V} \setminus V)) = 0$ since the measure σ is absolutely continuous with respect to the Lebesgue measure along such a ray. Our claim now follows since there are only a countable number of rays within F_n. ■

Remark 3.4. If V is not closed and if the dilations are anisotropic then the set $\tilde{V}$ defined in the previous proof may satisfy $\sigma(\tilde{V} \setminus V) > 0$, so that $W(R)$ is not right-continuous.

Example. For the Heisenberg group $\mathbb{H}_1$ we choose on $\mathbb{R}^2$ parabolic dilations $\delta_r(x_1, x_2) = (rx_1, r^2 x_2)$ and consider the star-shaped set V given by

$$V = A \cup B = \{(x_1, x_2) \in \mathbb{R}^2 : x_1^2 + x_2^2 < 1\} \cup \{\delta_r(t,t) : 0 < t < 2,\ 0 < r < 1\}.$$

Observe that $B = \{(x_1, x_2) : x_1^2/2 < x_2 < x_1,\ 0 < x_1 < 2\}$. Next we consider $\tilde{V}$. Notice that the boundary of V contains the segment $\gamma = \{(t,t) : 1 \le t \le 2\}$. Then, for $R > 1$, the boundary of $\delta_R V$ contains the line segment $\{(Rt, R^2 t) : 1 \le t \le 2\}$ which has slope $R > 1$. We thus see that $\gamma \subset \delta_R V$ for every $R > 1$. Hence $\tilde{V} = \bigcap_{j=1}^{\infty} \delta_{R_j} V \supset \gamma$, whereas γ and V are disjoint. Therefore $\gamma \subset \tilde{V} \setminus V$. Now $\sigma(\gamma) = \int_\gamma \mu \, d\mu > 0$. So we proved that $\sigma(\tilde{V} \setminus V) > 0$ since γ lies on the ray $\varGamma_0^+$ of the Heisenberg fan F_1.

To complete the proof of Theorem 3.1 we define a sequence $0 = R_0 < R_1 < R_2 < \cdots$ recursively as in [MMP] by setting

$$R_{m+1} = \sup\{R \geq R_m : W(R) < W(R_m) + 1\}, \qquad m \geq 0.$$

Then by Lemma 3.2, Lemma 3.3 and arguing as in [MMP], the above recursion leads to an infinite sequence $\{R_m\}_m$ tending to infinity and we have (compare [MMP, (4.1), (4.2)])

$$m \leq W(R_m) \leq C R_m^{D+\nu}, \qquad m \geq 0, \tag{3.18}$$

$$\log(3+m) \leq C \log(2+R_m), \qquad m \geq 1. \tag{3.19}$$

Next we define pairwise orthogonal projections P_m on $L^2(\mathbb{H}_n)$ by setting

$$\begin{aligned} P_0 &= S_0 = \chi_{\{0\}}(L_1, \ldots, L_n, -iU) = 0, \\ P_m &= S_{R_m} - S_{R_{m-1}} = \chi_m(L_1, \ldots, L_n, -iU), \end{aligned}$$

with $\chi_m = \chi_{(\delta_{R_m} V) \setminus (\delta_{R_{m-1}} V)}$, $m \geq 1$. Then

$$S_{R_J} = \sum_{m=0}^{J} P_m.$$

By Plancherel's theorem, (3.19) and since $\chi_m^2 = \chi_m$, we have

$$\begin{aligned} &[\log(2+m)]^2 \|P_m f\|_2^2 \\ &\quad = c_n \int_{\mathbb{R}^\times} \sum_{k \in \mathbb{N}^n} [\log(3+m-1)]^2 \chi_m(|\mu|\tilde{k}, \mu) \|\pi_\mu(f) h_k^\mu\|_2^2 \, |\mu|^n \, d\mu \\ &\quad \leq C \int_{\mathbb{R}^\times} \sum_{k \in \mathbb{N}^n} [\log(2+R_{m-1})]^2 \chi_m(|\mu|\tilde{k}, \mu) \|\pi_\mu(f) h_k^\mu\|_2^2 \, |\mu|^n \, d\mu. \end{aligned}$$

By (3.15) we know that $|\xi| \geq R_{m-1}/M$ if $\xi = (|\mu|\tilde{k}, \mu) \in F_n \cap (\delta_{R_m} V) \setminus (\delta_{R_{m-1}} V)$, hence

$$\log(2+R_{m-1}) \leq \log(2 + M|(|\mu|\tilde{k}, \mu)|) \leq C \log(2 + \|(|\mu|\tilde{k}, \mu)\|).$$

Therefore, by (3.13),

$$\begin{aligned} &[\log(2+m)]^2 \|P_m f\|_2^2 \\ &\quad \leq C \int_{\mathbb{R}^\times} \sum_{k \in \mathbb{N}^n} [\log(2 + \|(|\mu|\tilde{k}, \mu)\|)]^2 \chi_m(|\mu|\tilde{k}, \mu) \|\pi_\mu(f) h_k^\mu\|_2^2 \, |\mu|^n \, d\mu \\ &\quad = C \int_{\mathbb{R}^\times} \sum_{k \in \mathbb{N}^n} \|\pi_\mu(\log(2+\mathcal{L}) f) h_k^\mu\|_2^2 \chi_m(|\mu|\tilde{k}, \mu) |\mu|^n \, d\mu. \end{aligned}$$

Summing over all m we then obtain, by Plancherel's formula,

$$\sum_{m\in\mathbb{N}}[\log(2+m)]^2\|P_m f\|_2^2 \le C\int_{\mathbb{R}^\times}\sum_{k\in\mathbb{N}^n}\|\pi_\mu(\log(2+\mathcal{L})f)h_k^\mu\|_2^2\,|\mu|^n\,d\mu$$
$$= C\|\log(2+\mathcal{L})f\|_2^2.$$

We can thus apply Rademacher–Men'shov's theorem to conclude that

$$\|\tilde{M}f\|_2 \le C\|\log(2+\mathcal{L})f\|_2, \tag{3.20}$$

where $\tilde{M}$ denotes the discrete maximal operator given by

$$\tilde{M}f = \sup_{J\ge 0}\Big|\sum_{m=0}^{J}P_m f(x)\Big| = \sup_{J\ge 0}|S_{R_J}f(x)|.$$

We finally dominate the maximal function over arbitrary $R\ge 1$ as follows:

$$Mf(x) = \sup_{R\ge 1}|S_R f(x)| \le \tilde{M}f(x) + \sup_{m\ge 0}\sup_{R_m\le r<R_{m+1}}|S_r f(x) - S_{R_m}f(x)|.$$

To control the remainder term, observe that by our definition of the sequence R_m we have $W(r) < W(R_m)+1$. Moreover, $S_r f - S_{R_m}f = T_\eta f$, where $\eta = \chi_{(\delta_r V)\setminus(\delta_{R_m}V)}$. Then $S_r f - S_{R_m}f = f * K_\eta$, and since $\eta^2=\eta$ we have

$$\|K_\eta\|_2^2 = c_n\int_{\mathbb{R}^\times}\sum_{k\in\mathbb{N}^n}\eta(|\mu|\tilde{k},\mu)\,|\mu|^n\,d\mu$$
$$= c_n\int_{\mathbb{R}^\times}\sum_{k\in\mathbb{N}^n}(\chi_{\delta_r V}-\chi_{\delta_{R_m}V})(|\mu|\tilde{k},\mu)\,|\mu|^n\,d\mu$$
$$= \|K_r\|_2^2 - \|K_{R_m}\|_2^2 = W(r) - W(R_m) \le 1.$$

Notice that $f*K_\eta = (P_{m+1}f)*K_\eta$. Then, as in [MMP], we may conclude by Cauchy–Schwarz' inequality that for $x\in\mathbb{H}_n$,

$$|S_r f(x) - S_{R_m}f(x)| \le \int|(P_{m+1}f)(y)K_\eta(y^{-1}x)|\,dy$$
$$\le \|P_{m+1}f\|_2 A_\eta(x) \le \|P_{m+1}f\|_2,$$

where

$$A_\eta(x)^2 = \int|K_\eta(y^{-1}x)|^2\,dy = \int|K_\eta(y)|^2\,dy = \|K_\eta\|_2^2 \le 1$$

since $\mathbb{H}_n$ is unimodular. As in [MMP] this implies

$$\sup_{m\ge 0}\big(\sup_{R_m\le r<R_{m+1}}|S_r f(x) - S_{R_m}f(x)|\big) \le C\|f\|_2,$$

hence

$$Mf(x) \le \tilde{M}f(x) + C\|f\|_2.$$

Therefore (3.3) is proved. The remaining statement about a.e. convergence in Theorem 3.1 follows now by standard arguments.

REMARKS 3.5. (a) Our theorem applies for instance to pseudo-differential operators on $\mathbb{H}_n$ of the form $\mathcal{L} = L_1^{a_1} + \cdots + L_n^{a_n} + |U|^b$ and their "subelliptic" variants $\mathcal{L} = L_1^{a_1} + \cdots + L_n^{a_n}$, with $a_1, \ldots, a_n, b > 0$, and the spectrally defined partial sum operators $S_R = \int_0^R dE_\tau$, where $\mathcal{L} = \int_0^\infty \tau\, dE_\tau$ denotes the spectral resolution of $\mathcal{L}$. The associated sets V are here given by $V = \{(\lambda_1, \ldots, \lambda_n, \mu) : |\lambda_1|^{a_1} + \cdots + |\lambda_n|^{a_n} + |\mu|^b \leq 1\}$, respectively by $V = \{(\lambda_1, \ldots, \lambda_n, \mu) : |\lambda_1|^{a_1} + \cdots + |\lambda_n|^{a_n} \leq 1\}$, and the dilations have weights $\alpha_j = 1/a_j$ and $\beta = 1/b$ for the first case (in the second case, β must satisfy $\beta \geq \min_j 1/a_j$). Notice that V is unbounded with respect to the variable μ in the second case.

(b) Extensions of Theorem 3.1 to more general two-step nilpotent Lie groups seem possible.

(c) We take the opportunity to correct a minor error in [MMP] which however has no effect on the proofs in that paper: the exponent α in display (1.2) of [MMP] is not the local homogeneous dimension, but half of it.

Acknowledgements. The authors acknowledge the joint support by the Deutsche Forschungsgemeinschaft (DFG) and the Italian Consiglio Nazionale delle Ricerche (CNR).

REFERENCES

[CS] A. Carbery and F. Soria, *Almost-everywhere convergence of Fourier integrals for functions in Sobolev spaces, and an L^2-localisation principle*, Rev. Mat. Iberoamer. 4 (1988), 319–337.

[CMP] L. Colzani, C. Meaney, and E. Prestini, *Almost everywhere convergence of inverse Fourier transforms*, Proc. Amer. Math. Soc. 134 (2006), 1651–1660.

[F] C. Fefferman, *On the convergence of multiple Fourier series*, Bull. Amer. Math. Soc. 77 (1971), 744–745.

[Fo] G. B. Folland, *Harmonic Analysis in Phase Space*, Ann. of Math. Stud. 122, Princeton Univ. Press, Princeton, NJ, 1989.

[FS] G. B. Folland and E. M. Stein, *Hardy Spaces on Homogeneous Groups*, Math. Notes 28, Princeton Univ. Press, Princeton, NJ, 1982.

[G1] D. Geller, *Fourier analysis on the Heisenberg group*, Proc. Nat. Acad. Sci. U.S.A. 74 (1977), 1328–1331.

[G2] —, *Spherical harmonics, the Weyl transform and the Fourier transform on the Heisenberg group*, Canad. J. Math. 36 (1984), 615–684.

[HR] A. Hulanicki and F. Ricci, *A Tauberian theorem and tangential convergence for bounded harmonic functions on balls in $\mathbb{C}^n$*, Invent. Math. 62 (1980), 325–331.

[MMP] C. Meaney, D. Müller, and E. Prestini, *A.e. convergence of spectral sums on Lie groups*, Ann. Inst. Fourier (Grenoble) 57 (2007), 1509–1520.

[S] R. Salem, *A new proof of a theorem of Menchoff*, Duke Math. J. 8 (1941), 269–272.

[St] E. M. Stein, *Harmonic Analysis: Real-Variable Methods, Orthogonality, and Oscillatory Integrals*, Princeton Math. Ser. 43, Princeton Univ. Press, Princeton, NJ, 1993.

[T] S. Thangavelu, *Harmonic Analysis on the Heisenberg Group*, Progr. Math. 159, Birkhäuser Boston, Boston, MA, 1998.

[Z] A. Zygmund, *Trigonometric Series. Vol. I, II*, 3rd ed., Cambridge Univ. Press, Cambridge, 2002.

D. Müller
Mathematisches Seminar
C.A.-Universität Kiel
Ludewig-Meyn-Str. 4
D-24098 Kiel, Germany
E-mail: mueller@math.uni-kiel.de

E. Prestini
Dipartimento di Matematica
Università di Roma "Tor Vergata"
Via della Ricerca Scientifica
00133 Roma, Italy
E-mail: prestini@mat.uniroma2.it

Received 11 June 2009;
revised 14 July 2009 (5236)