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ABSOLUTELY S-DOMAINS AND PSEUDO-POLYNOMIAL RINGS

BY

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Abstract. A domain R is called an absolutely S-domain (for short, AS-domain) if each domain T such that $R \subseteq T \subseteq \mathrm{qf}(R)$ is an S-domain. We show that R is an AS-domain if and only if for each valuation overring V of R and each height one prime ideal q of V, the extension $R/(q \cap R) \subseteq V/q$ is algebraic. A Noetherian domain R is an AS-domain if and only if $\dim(R) \leq 1$. In Section 2, we study a class of R-subalgebras of $R[X]$ which share many spectral properties with the polynomial ring $R[X]$ and which we call pseudo-polynomial rings. Section 3 is devoted to an affirmative answer to D. E. Dobbs's question of whether a survival pair must be a lying-over pair in the case of transcendental extension.

0. Introduction. In this paper, all rings considered are commutative with identity. An inclusion of rings signifies that the smaller ring is a subring of the larger and has the same identity. Let R be a ring and n a positive integer. We denote by $R[n]$ the ring of polynomials in n indeterminates over R and by $R[X]$ the ring of polynomials in one indeterminate. We denote by $\dim(R)$ the *Krull dimension* of R and by $\dim_v(R)$ its *valuative dimension*, that is, the limit of the sequence $(\dim(R[n]) - n,\ n \in \mathbb{N})$. If p is a prime ideal of R, we denote by $\mathrm{ht}(p)$ the height of p, and by $\mathrm{ht}_v(p)$ the limit of the sequence $(\mathrm{ht}(p[n]),\ n \in \mathbb{N})$.

Given a finite-dimensional ring R, we say that R is a *Jaffard ring* if $\dim(R) = \dim_v(R)$ [2]. This property is not local; we say that R is a *locally Jaffard ring* if R_p is a Jaffard ring for each prime ideal p of R. A domain R is said to be an *S-domain* if for each height 1 prime ideal p of R, we have $\mathrm{ht}(p[X]) = 1$. A *strong S-ring* is a ring R such that for each prime ideal p of R, R/p is an S-domain; equivalently for any consecutive primes $p \subset q$ in R, $p[X] \subset q[X]$ are consecutive in $R[X]$. An *overring* of a domain R is a ring contained between R and its quotient field $\mathrm{qf}(R)$.

For an extension of domains $R \subseteq T$, we denote by $\mathrm{tr.deg}[T : R]$ the transcendence degree of $\mathrm{qf}(T)$ over $\mathrm{qf}(R)$. Recall that an extension $R \subseteq T$

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is said to satisfy the *altitude inequality* (resp., the *altitude formula*) if for any prime ideal q of T over a prime ideal p of R, we have respectively

$$\mathrm{ht}(q) + \mathrm{tr.deg}[T/q : R/p] \leq \mathrm{ht}(p) + \mathrm{tr.deg}[T : R],$$
$$\mathrm{ht}(q) + \mathrm{tr.deg}[T/q : R/p] = \mathrm{ht}(p) + \mathrm{tr.deg}[T : R].$$

A. Ayache and P.-J. Cahen in [4] studied the domains each of whose overrings is Jaffard; these are domains R whose integral closure R' is a Prüfer domain ([4, Théorème 2.6]). They are called domains *satisfying absolutely the altitude inequality* or *absolutely Jaffard domains*.

Our purpose in Section 1 is to complete this circle of ideas by dealing with *absolutely S-domains* (for short *AS-domains*); that is, domains each of whose overrings is an S-domain. The main result of this section is Theorem 1.5 which states that R is an AS-domain if and only if for each valuation overring V of R and each height 1 prime ideal q of V, the extension $R/(q \cap R) \subseteq V/q$ is algebraic. It is clear that absolutely Jaffard domains are AS-domains, but the converse does not hold (see Corollary 1.18). On the other hand, Proposition 1.11 points out a relationship between these two classes of domains: R is an absolutely Jaffard domain if and only if R/p is an AS-domain for each prime ideal p of R. Among the several interesting consequences of Theorem 1.5, we just point out that $R[X]$ is an AS-domain if and only if R is a field (Proposition 1.14) and that a Noetherian domain R is absolutely S if and only if $\dim(R) \leq 1$ (Proposition 1.15). The last part of Section 1 is devoted to the transfer of this notion to some pullback constructions.

In [5], it is proved that if R is a Jaffard domain, then each domain contained between R and $R[n]$ is Jaffard. It is an open question whether each domain between R and $R[X]$ is a locally Jaffard (resp., an S-) domain when R is a locally Jaffard (resp., an S-) domain. The difficulty in the study of intermediate rings between R and $R[X]$ is that the nature of R as well as that of $R[X]$ has a very subtle influence on intermediate rings. Because of this, it appears to be too difficult to understand all domains between R and $R[X]$ except in case R has very simple structure.

In Section 2 we begin with a description of the valuative height of $p[X] \cap T$, for any ring T between R and $R[X]$ and any prime ideal p of R. Moreover, we note that if T is an intermediate ring between R and $R[X]$ and P is a prime ideal of T, then the inclusion relation $p[X] \cap T \subseteq P$ with $p = P \cap R$ may not hold (Propositions 2.6 and 3.5). Our concern in that section is with rings T between R and $R[X]$ satisfying the previous inclusion for each prime P of T; we call them *R-pseudo-polynomial rings*, or briefly *pseudo-polynomial* rings. The name is justified by the fact that these rings share many ideal-theoretic properties with polynomial rings in one variable to which they are closely related. The motivation for studying this generalization is diverse. To have an example of a pseudo-polynomial ring, one can consider an $(R[X], I[X], D)$ construction ring T, where R is a ring, I is an ideal of R such that $\dim(R/I) = 0$ and D is a ring contained between R/I and $(R/I)[X]$ (Proposition 2.3). Note that a pseudo-polynomial ring need not be a polynomial ring. For instance, if K is a field, then the domain $K[X^2, X^3]$ is not a polynomial ring since it is not integrally closed, while it is obviously a K-pseudo-polynomial ring.

Pseudo-polynomial rings are characterized in terms of lying-over extensions (Proposition 2.6). We study the structure of the prime spectrum of T, clarifying the relation between the spectrum of T and those of R and $R[X]$. We generalize some well-known results previously established for polynomial rings [9]. The final aim of this section is to prove our promised results which state that if R is a locally Jaffard (resp., an S-) domain, then each pseudo-polynomial ring is locally Jaffard (resp., S). Section 3 is concluded with a study of the relationship between pseudo-polynomial rings and lying-over pairs. We give an affirmative answer to D. E. Dobbs' question [13] of whether a survival pair must be a lying-over pair in the case of transcendental extension.

1. Absolutely S-domains. A domain R is said to be an *S-domain* if for each prime ideal p of R such that $\mathrm{ht}(p) = 1$, we have $\mathrm{ht}(p[X]) = 1$. Now, we recall some definitions given in the introduction.

DEFINITION 1.1. A domain R is said to be an *absolutely S-domain* (for short, *AS-domain*) if each overring of R is an S-domain.

It follows immediately from this definition that each domain which satisfies absolutely the altitude inequality [4, Lemme 2.1] is an AS-domain. Moreover, this notion is stable under localization. We first give the following straightforward result.

PROPOSITION 1.2. *Let R be a domain. Then the following statements are equivalent*:

(i) *R is an AS-domain*;
(ii) *R_p is an AS-domain for each prime ideal p of R*;
(iii) *R_m is an AS-domain for each maximal ideal m of R*;
(iv) *$N^{-1}R$ is an AS-domain for each multiplicative subset N of R.*

Recall that a domain R is said to be *going-down* if for each overring T of R the extension $R \subseteq T$ has the going-down property ([14]).

PROPOSITION 1.3. *A going-down S-domain is an AS-domain.*

Proof. Let R be a going-down S-domain, T a overring of R and q a height 1 prime ideal in T. Since R is going-down, $\mathrm{ht}(p) = 1$, where $p = q \cap R$.

The domain R_p is a one-dimensional Jaffard domain [8, Corollary 6.3], hence so is T_q. Therefore T is an S-domain. ■

In particular, a one-dimensional S-domain is an AS-domain. However, an S-domain need not be an AS-domain. To see this, it suffices to consider a ring of polynomials with coefficients in an integral domain which is not a field (see Proposition 1.14).

We begin by presenting some terminology. Let B be a ring, I an ideal of B and D a subring of B/I. Consider the pullback construction of commutative rings:

$$\begin{array}{ccc} R & \rightarrow & D \\ \downarrow & & \downarrow \\ B & \rightarrow & B/I \end{array}$$

Following [10], we say that R is the *ring of the* (B, I, D) *construction* and we write $R = (B, I, D)$. Next we consider the case where I is assumed to be maximal. Denote by M the ideal I, by T the domain B, by K the field T/M, and by $\varphi : T \rightarrow K$ the natural epimorphism.

We recall a few well-known properties of pullbacks. First, M is a common ideal of both R and T, $M = (R : T) = \{x \in T \mid xT \subseteq R\}$ (if $D \neq K$), and $R/M \simeq D$. For each $p \in \mathrm{Spec}(R)$ with $M \subseteq p$, there is a (unique) $q \in \mathrm{Spec}(D)$ such that $\varphi^{-1}(q) = p$; and $\varphi^{-1}(D_q) = R_p$. If T is local, then each prime ideal of R compares with M, and thus $\dim(R) = \dim(D) + \dim(T)$. Also R is local if and only if both D and T are local (cf. [16, Theorem 1.4 and Proposition 2.1]).

Before stating Theorem 1.5, we establish a proposition which serves both to motivate this theorem and to dispatch the difficult implication in its proof.

PROPOSITION 1.4. *Let T be a domain, M a maximal ideal of T, D a subring of the residue field $K = T/M$, k the quotient field of D and $R := (T, M, D)$.*

(a) *If* $\mathrm{ht}_T(M) \geq 2$, *then the following statements are equivalent*:

(i) *R is an S-domain*;
(ii) *T is an S-domain.*

(b) *If* $\mathrm{ht}_T(M) = 1$, *then the following statements are equivalent*:

(i) *R is an S-domain*;
(ii) *T is an S-domain and K is algebraic over k.*

Proof. (a) (i)⇒(ii). Let q be a height 1 prime ideal of T. Then q does not contain M. Thus $T_q = R_p$, where $p = q \cap R$. Hence $\mathrm{ht}(p) = 1$ and so $\mathrm{ht}(p[X]) = 1$ since R is an S-domain. Thus $\mathrm{ht}(q[X]) = \mathrm{ht}(p[X]) = 1$.

(ii)⇒(i). Let p be a height 1 prime ideal of R. Then p does not contain M. Thus there exists a unique prime ideal q of T such that $T_q = R_p$.

Hence $\mathrm{ht}(q) = 1$ and as T is an S-domain, we have $\mathrm{ht}(q[X]) = 1$. Thus $\mathrm{ht}(p[X]) = \mathrm{ht}(q[X]) = 1$.

(b) (i)$\Rightarrow$(ii). First notice that $R_M = (T_M, MT_M, k)$. Since R_M is a one-dimensional S-domain, it is a Jaffard domain. Thus K is algebraic over k ([2, Theorem 2.6]). Now our task is to show that T is an S-domain. Let q be a height 1 prime ideal of T. If $q = M$, then since $R_M = (T_M, MT_M, k)$ is a Jaffard domain, it follows readily from [2, Theorem 2.6] that T_M is a Jaffard domain. Thus $\mathrm{ht}_{T[X]}(q[X]) = 1$. Now if $q \neq M$, then $T_q = R_p$, where $p = q \cap R$. Hence $\mathrm{ht}(p) = 1$ and so $\mathrm{ht}(p[X]) = 1$ since R is an S-domain. Thus $\mathrm{ht}(q[X]) = \mathrm{ht}(p[X]) = 1$.

(ii)$\Rightarrow$(i). Let p be a height 1 prime ideal of R. If $p = M$, then since $R_M = (T_M, MT_M, k)$, K is algebraic over k and T_M is a Jaffard domain it follows from [2, Theorem 2.6] that R_M is a Jaffard domain. Thus $\mathrm{ht}_{R[X]}(p[X]) = 1$. Now assume that $p \neq M$. Then there exists a unique prime ideal q of T such that $T_q = R_p$. Hence $\mathrm{ht}(q) = 1$ and as T is an S-domain, we have $\mathrm{ht}(q[X]) = 1$. Thus $\mathrm{ht}(p[X]) = \mathrm{ht}(q[X]) = 1$. ∎

Recall that an extension $R \subseteq T$ of domains is said to be *residually algebraic* if $R/(q \cap R) \subseteq T/q$ is algebraic for each prime ideal q of T ([7] and [17]).

Now, we establish the main result of this section.

THEOREM 1.5. *Let R be a domain. Then the following statements are equivalent*:

(1) *R is an AS-domain*;

(2) *for each overring T of R and primes $q \subseteq q_1$ in T, if $q \cap R = q_1 \cap R$ and* $\mathrm{ht}(q) = 1$, *then $q = q_1$*;

(3) *for each valuation overring V of R and primes $q \subseteq q_1$ in V, if $q \cap R = q_1 \cap R$ and* $\mathrm{ht}(q) = 1$, *then $q = q_1$*;

(4) *for each overring T of R and each height* 1 *prime ideal q of T, we have* $\mathrm{tr.deg}[T/q : R/(q \cap R)] = 0$;

(5) *for each one-dimensional overring T of R, the extension $R \subseteq T$ is residually algebraic*;

(6) *for each valuation overring V of R and each height* 1 *prime ideal q of V, we have* $\mathrm{tr.deg}[V/q : R/(q \cap R)] = 0$;

(7) *for each one-dimensional valuation overring V of R, the extension $R \subseteq V$ is residually algebraic.*

Proof. The equivalences (4)$\Leftrightarrow$(5) and (6)$\Leftrightarrow$(7) as well as the implications (2)$\Rightarrow$(3) and (5)$\Rightarrow$(7) are clear. The implication (3)$\Rightarrow$(2) is due to [18, Corollary 19.7].

(1)⇒(4). Let T be an overring of R, and q a height 1 prime in T. Since $R + qT_q$ obtained by the $(T_q, qT_q, R/(q \cap R))$ construction is an S-domain, Proposition 1.4 implies that $\text{tr.deg}[T/q : R/(q \cap R)] = 0$.

We prove that if (1) fails, then (6) fails. Assume that T is an overring of R containing a prime ideal q of height 1 such that $\text{ht}(q[X]) = 2$. Then there is a nonzero prime ideal Q of $T[X]$ contained in $q[X]$ such that $Q \cap T = (0)$. Thus T is a subring of $T_1 = T[X]/Q$ which is isomorphic to $T[u]$, where u is an algebraic element over T. By [18, Corollary 19.7], there is a valuation overring W of T_1 containing a prime ideal J of height 1 such that $J \cap T_1 = q[X]/Q$. Set $V = W \cap \text{qf}(R)$. Then V is a valuation overring of R containing a height 1 prime ideal $J \cap \text{qf}(R)$ ([18, Theorem 19.16]) such that $(J \cap \text{qf}(R)) \cap R = q \cap R$. Now, $\text{tr.deg}[W/J : V/(J \cap \text{qf}(R))] = 0$ ([18, Theorem 19.16]). Hence

$$\begin{aligned}\text{tr.deg}[V/(J \cap \text{qf}(R)) : R/(q \cap R)] &= \text{tr.deg}[W/J : R/(q \cap R)] \\ &\geq \text{tr.deg}[T_1/(q[X]/Q) : T/q] = \text{tr.deg}[(T[X]/Q)/(q[X]/Q) : T/q] \\ &= \text{tr.deg}[T[X]/q[X] : T/q] = 1.\end{aligned}$$

We prove that if (3) fails, then (6) fails. Assume that V is a valuation overring of R containing primes $q \subset M$ lying over p and such that $\text{ht}(q) = 1$. If $x \in M \setminus q$, we show that $\overline{x}$ is transcendental over R/p in V/q. Let $\overline{a_k}\,\overline{x^k} + \ldots + \overline{a_1 x} + \overline{a_0} = 0$, where $a_0, a_1, \ldots, a_k$ are elements of R. Then $a_k x^k + \ldots + a_1 x + a_0 \in q$. Now, $a_0 \in M \cap R = p$, $x(a_k x^{k-1} + \ldots + a_1) \in q$ and $a_k x^{k-1} + \ldots + a_1 \in q$. By induction, we conclude that $\overline{a}_k = \overline{a}_{k-1} = \ldots = \overline{a}_1 = \overline{a}_0 = 0$, and consequently $\text{tr.deg}[V/q : R/p] \geq 1$.

We prove that if (1) fails, then (3) fails. Assume that T is an overring of R containing a prime ideal q of height 1 such that $\text{ht}(q[X]) = 2$. Then there is a nonzero prime ideal Q of $T[X]$ contained in $q[X]$ such that $Q \cap T = (0)$. Thus, T is a subring of $T_1 = T[X]/Q$ which is isomorphic to $T[u]$, where u is an algebraic element over T. We can assume that T is local with maximal ideal q. The ideal $p_0 = q[X]/Q$ is prime in T_1, but not maximal (since $T_1/p_0 \cong (T/q)[X]$, which is not a field). Therefore if M is a maximal ideal of T_1 containing p_0, then p_0 and M are prime ideals of T_1 lying over q. By [18, Corollary 19.7], there is a valuation overring W of T_1 containing prime ideals $p' \subset M'$ such that $\text{ht}(p') = 1$, $p' \cap T_1 = p_0$ and $M' \cap T_1 = M$. Set $V = W \cap \text{qf}(R)$. Then V is a valuation overring of R containing distinct prime ideals $p' \cap \text{qf}(R) \subset M' \cap \text{qf}(R)$ such that $\text{ht}(p' \cap \text{qf}(R)) = 1$ and

$$(p' \cap \text{qf}(R)) \cap R = p_0 \cap R = q \cap R = M \cap R = (M' \cap \text{qf}(R)) \cap R. \blacksquare$$

Among the several interesting consequences of Theorem 1.5, we limit ourselves to pointing out the following three propositions.

PROPOSITION 1.6. *Let $R \subset T$ be an algebraic extension. If R is an AS-domain, then so is T.*

Proof. Assume that T is not an AS-domain. Then by the previous theorem, there is a valuation overring W of T containing distinct prime ideals p and M such that $\mathrm{ht}(p) = 1$ and $p \cap T = M \cap T$. By [18, Theorem 19.16], $V = W \cap \mathrm{qf}(R)$ is a valuation overring of R containing distinct prime ideals $p \cap \mathrm{qf}(R)$ and $M \cap \mathrm{qf}(R)$ such that $(p \cap \mathrm{qf}(R)) \cap R = (M \cap \mathrm{qf}(R)) \cap R$ and $\mathrm{ht}(p \cap \mathrm{qf}(R)) = 1$. In view of Theorem 1.5, we conclude that R is not an AS-domain. ∎

REMARK 1.7. The converse of the previous proposition does not hold (see Corollary 1.18).

PROPOSITION 1.8. *Let $R \subset T$ be an integral extension. R is an AS-domain if and only if T is an AS-domain.*

Proof. According to Proposition 1.6, if R is an AS-domain, then so is T. Assume now that T is an AS-domain, and let C be an overring of R. Denote by C' the integral closure of C in $\mathrm{qf}(T)$. Since C' is an overring of T, it is an S-domain. Therefore C is also an S-domain. ∎

PROPOSITION 1.9. *If R is an AS-domain, then it has the following property:*

($*$) *For each prime ideal p of R, if $(0) \subset P \subset p[X]$ is a saturated chain of primes in $R[X]$, then $P \cap R \neq (0)$.*

Proof. We show that if the property ($*$) fails, then so does assertion (3) in Theorem 1.5. For this, consider the ring $T_1 = R[X]/P$ and use the same idea as in the proof of (3)⇒(1) in Theorem 1.5. ∎

Recall that a domain R is said to be *strong* S if R/p is an S-domain for each prime ideal p of R; equivalently, if for each pair of consecutive prime ideals $p \subset q$ of R, the extended primes $p[X] \subset q[X]$ are consecutive. R is said to be *stably strong* S if $R[n]$ is strong S for each integer n ([20, 21]). A domain R is said to be *totally Jaffard* if R/p is locally Jaffard for each prime p of R ([11]). We now recall the following theorem.

THEOREM 1.10 [6, Théorème 5.1]. *Let R be a domain. Then the following statements are equivalent:*

(i) *each overring of R is a strong S-domain;*
(ii) *each overring of R is Jaffard;*
(iii) *each overring of R is totally Jaffard;*
(iv) *each overring of R is a stably strong S-domain;*
(v) *each overring of R satisfies the altitude inequality;*
(vi) *each overring of R is locally Jaffard;*
(vii) *R satisfies absolutely the altitude inequality.*

This raises the question whether the class of rings satisfying absolutely the altitude inequality coincides with that of absolutely S-domains. It is clear that for one-dimensional domains these two notions coincide, but in the general case, the answer is negative. Indeed, if V is a valuation domain with maximal ideal M and k is a subfield of V/M such that $\dim(V) \geq 2$ and $\mathrm{tr.deg}[V/M : k] \geq 1$, then by Corollary 1.18, $R := (V, M, k)$ is an AS-domain and by [2, Theorem 2.6], R is not a Jaffard domain.

The following result establishes a relationship between absolutely S-domains and domains satisfying the altitude inequality.

PROPOSITION 1.11. *Let R be a domain. The following statements are equivalent*:

(i) *R satisfies absolutely the altitude inequality*;
(ii) *R/p is an AS-domain for each prime ideal p of R.*

Proof. (i)⇒(ii). For each prime p of R, R/p satisfies absolutely the altitude inequality ([4, Proposition 2.2]), and therefore is an AS-domain.

(ii)⇒(i). Let $P \subset q[X]$ be consecutive primes in $R[X]$ and $p = P \cap R$. Then $P/p[X] \subset (q/p)[X]$ are consecutive in $(R/p)[X]$. Since R/p is an AS-domain, it follows that $P = p[X]$ (Proposition 1.9). Thus R satisfies absolutely the altitude inequality [6, Proposition 1.4]. ∎

As an immediate consequence of Proposition 1.11, we have:

COROLLARY 1.12. *Let R be a domain such that* $\dim(R) \leq 2$ *and* $\dim(R[X]) = 1 + \dim(R)$. *Then the following statements are equivalent*:

(i) *R satisfies absolutely the altitude inequality*;
(ii) *R is an AS-domain.*

In the course of proof of Proposition 1.14, we need the following lemma.

LEMMA 1.13. *If $R[X]$ is an AS-domain, then so is R.*

Proof. Let V be a one-dimensional valuation overring of R with maximal ideal M. By [18, Proposition 20.11], $V[X]_{M[X]}$ is a one-dimensional valuation overring of $R[X]$. Therefore, by Theorem 1.5, we have

$$\begin{aligned}
&\mathrm{tr.deg}[V[X]_{M[X]}/MV[X]_{M[X]} : R[X]/(M \cap R)[X]] = 0\\
&\qquad = \mathrm{tr.deg}[V[X]/M[X] : R[X]/(M \cap R)[X]]\\
&\qquad = \mathrm{tr.deg}[(V/M)[X] : (R/(M \cap R))[X]] = \mathrm{tr.deg}[V/M : R/(M \cap R)].
\end{aligned}$$

Hence by Theorem 1.5, R is an AS-domain. ∎

PROPOSITION 1.14. *$R[X]$ is an AS-domain if and only if R is a field.*

Proof. By way of contradiction, suppose that R has a height 1 prime ideal p. From the previous lemma, it follows that R is an S-domain. Thus R_p is a Jaffard domain. Since $R_p[X]$ is a localization of $R[X]$, it is an AS-domain

(Proposition 1.2). Moreover, R_p is a one-dimensional Jaffard domain, so by Corollary 1.12, $R_p[X]$ satisfies absolutely the altitude inequality and therefore R_p is a field [4], contrary to the hypothesis. ∎

According to the previous proposition, a Noetherian domain need not be an AS-domain. Indeed, a polynomial ring with two indeterminates over a field is not an AS-domain. Nevertheless, we have the following

Proposition 1.15. *A Noetherian domain R is an AS-domain if and only if* $\dim(R) \leq 1$.

Proof. Assume that R is a Noetherian domain such that $\dim(R) \geq 2$. Let p be a prime ideal of R of height 2. Since R_p is a two-dimensional Noetherian AS-domain (Proposition 1.2), Corollary 1.12 shows that R_p satisfies absolutely the altitude inequality, contrary to [4, Remarques §3].

Conversely, if $\dim(R) \leq 1$, then R satisfies absolutely the altitude inequality and therefore R is an AS-domain. ∎

We now turn to transferring the "AS-domain" property to pullback constructions. We first record the following lemma, which is an immediate consequence of Proposition 1.4.

Lemma 1.16. *Let T be a domain, M a maximal ideal of T, D a subring of the field $K = T/M$ and $R := (T, M, D)$. If R is an S-domain, then each intermediate ring between R and T is an S-domain.*

Now, we establish the following useful result.

Proposition 1.17. *Let T be a Prüfer domain, M a maximal ideal of T, D a subring of the residue field $K = T/M$ and $R := (T, M, D)$.*

(a) *If* $\mathrm{ht}_T(M) \geq 2$, *then R is an AS-domain.*
(b) *If* $\mathrm{ht}_T(M) = 1$, *then the following statements are equivalent*:

(i) *R is an S-domain*;
(ii) *K is algebraic over* $\mathrm{qf}(D)$;
(iii) *R is an AS-domain.*

Proof. Assertion (a) is straightforward and we omit its proof.

(b) (i) and (ii) are equivalent by virtue of Proposition 1.4. Now it remains to show that (i)⇒(iii). We discuss two cases:

Case 1: If T is local, then it is a valuation domain. Let R_1 be an overring of R. Then by [3, Lemme 4.9], either R_1 is an overring of T and therefore is an S-domain, or R_1 is a domain between R and T and therefore it is an S-domain by the previous lemma. It follows that each overring of R is an S-domain.

Case 2: If T is not local, we show that for each prime ideal p of R, R_p is an AS-domain. Then we conclude using Proposition 1.2.

If M is not contained in p, then there is a unique prime ideal q of T such that $R_p = T_q$. Thus R_p is an AS-domain (since T_q is a Prüfer domain).

If $M \subseteq p$, then there is a unique prime ideal q of D such that $R_p = (T_M, MT_M, D_q)$ and therefore R_p is an AS-domain by Case 1. ∎

The following corollary is in the same vein.

COROLLARY 1.18. *Let $R := (V, M, k)$, where V is a valuation domain with maximal ideal M and k a subfield of V/M.*

(a) *If* $\dim(R) \geq 2$, *then R is an AS-domain.*

(b) *If* $\dim(R) = 1$, *then R is an AS-domain if and only if the residue field V/M is algebraic over R/M.*

PROPOSITION 1.19. *Let T be a domain, M a maximal ideal of T, k a subfield of the residue field $K = T/M$ and $R := (T, M, k)$. If T is an AS-domain and K is algebraic over k, then R is an AS-domain.*

Proof. Since K is algebraic over k, we see that T is integral over R ([2, Lemma 2.1]). By Proposition 1.8, R is an AS-domain. ∎

It is clear that under the same hypotheses of the previous proposition, if $\mathrm{ht}_T(M) = 1$, then R is an AS-domain if and only T is an AS-domain and K is algebraic over k. In the general case, if $\mathrm{ht}_T(M) \geq 2$, then R may be an AS-domain even if K is not algebraic over k (see Corollary 1.18).

2. Pseudo-polynomial rings. Let R be a ring and $R[X]$ the polynomial ring in one indeterminate X over R. The purpose of this section is to study pseudo-polynomial rings. We define an *R-pseudo-polynomial ring* to be a ring T contained between R and $R[X]$ in which every prime ideal P contains the prime ideal $p[X] \cap T$, where $p = P \cap R$. Clearly R, $R[X^2]$, $R[X]$ are R-pseudo-polynomial rings. In this section we collect more information on rings between R and $R[X]$ in order to clear up their relationship with the polynomial ring $R[X]$. We start with a proposition which generalizes the fact that if p is a prime ideal of a ring R and X is an indeterminate over R, then $\mathrm{ht}(p[X]) \geq \mathrm{ht}(p)$, $\dim(R[X]) \geq \dim(R) + 1$, and $\mathrm{ht}_v(p[X]) = \mathrm{ht}_v(p)$ ([9, 15]).

PROPOSITION 2.1. *Let R be a domain, p a prime ideal of R, X an indeterminate over R, and T a domain such that $R \subset T \subseteq R[X]$. Then:*

(i) $\mathrm{ht}(p[X] \cap T) \geq \mathrm{ht}(p)$;

(ii) *if $p[X] \cap T$ is not maximal among the primes of T lying over p, then* $\mathrm{ht}_v(p[X] \cap T) = \mathrm{ht}_v(p)$;

(iii) *if $p[X] \cap T$ is maximal among the primes of T lying over p, then* $\mathrm{ht}_v(p[X] \cap T) = \mathrm{ht}_v(p) + 1$.

Proof. The case where $\mathrm{ht}(p) = \infty$ is clear. Hence we assume that $\mathrm{ht}(p)$ is finite. By localization of R at p, we can suppose that R is local with maximal ideal p.

(i) It is clear that a chain $p_0 \subset p_1 \subset \ldots \subset p_k = p$ of prime ideals in R gives rise to a chain $p_0[X] \cap T \subset p_1[X] \cap T \subset \ldots \subset p_k[X] \cap T$ of prime ideals in T. Thus $\mathrm{ht}(p[X] \cap T) \geq \mathrm{ht}(p)$.

(ii) Since $T_{p[X]\cap T} \subseteq R[X]_{p[X]}$ is an algebraic extension, [18, Theorem 30.8] yields $\mathrm{ht}_v(p[X] \cap T) \geq \mathrm{ht}_v(p[X]) = \mathrm{ht}_v(p)$. On the other hand, using the fact that $p[X] \cap T$ is not maximal among the primes of T lying over p and [4, Lemme 1.1], we have $\mathrm{ht}_v(p[X] \cap T) \leq \dim_v(T_p) - 1 \leq \dim_v(R) = \mathrm{ht}_v(p)$. Thus $\mathrm{ht}_v(p[X] \cap T) = \mathrm{ht}_v(p)$.

(iii) It is clear that $\mathrm{ht}_v(p[X] \cap T) \leq \dim_v(T_p) \leq \dim_v(R) + 1 = \mathrm{ht}_v(p) + 1$ [4, Lemme 1.1]. Since $T_p/(p[X] \cap T)_p$ is a field contained between R/p and $(R/p)[X]$, necessarily $T_p/(p[X] \cap T)_p = R/p$. Thus denoting by A the ring of the $(R[X], p[X], R/p)$ construction (in fact $A = R + p[X]$), we have the algebraic extension $T_p = R + (p[X] \cap T)_p \subseteq A$ and, by [18, Theorem 30.8], we get $\mathrm{ht}_v(p[X] \cap T) \geq \mathrm{ht}_{v_A}(p[X])$. On the other hand, for any positive integers m and r, we have, according to [11, Lemme 3],

$$\mathrm{ht}_{A[m+r]}(p[X][m+r]) \geq \mathrm{ht}_{R[X][m]}(p[X][m]) + \inf(\mathrm{tr.deg}[(R/p)[X] : R/p], r).$$

Then by passage to the limit $m, r \to \infty$, we obtain

$$\mathrm{ht}_{v_A}(p[X]) \geq 1 + \mathrm{ht}_{v_{R[X]}}(p[X]) = 1 + \mathrm{ht}_v(p). \ \blacksquare$$

For instance, let R be a domain, p a maximal ideal of R, X an indeterminate over R, and set $T := (R[X], p[X], D)$, where D is a ring contained between R/p and $(R/p)[X]$. Then $\mathrm{ht}_{v_T}(p[X]) = 1 + \mathrm{ht}_v(p)$ if D is a field, and $\mathrm{ht}_{v_T}(p[X]) = \mathrm{ht}_v(p)$ if not.

Recall that according to [15], a prime ideal p of a ring R is said to be a J*affard prime ideal* if $\mathrm{ht}(p) = \mathrm{ht}_v(p) < \infty$. It is an open question whether if R is locally Jaffard, then each ring between R and $R[X]$ is locally Jaffard. Nevertheless, we have the following

Corollary 2.2. *Let R be a domain, X an indeterminate over R and T a domain such that $R \subseteq T \subseteq R[X]$. Then for each Jaffard prime ideal p of R, $p[X] \cap T$ is a Jaffard prime ideal of T.*

Proof. We ignore the case $T = R$ and by localization of R at p, we can suppose that R is local with maximal ideal p. If $\mathrm{ht}(p[X] \cap T) = \mathrm{ht}(p) + 1$, then Proposition 2.1 gives $\mathrm{ht}(p[X] \cap T) = \mathrm{ht}_v(p[X] \cap T) = \mathrm{ht}_v(p) + 1 = \mathrm{ht}(p) + 1$. If $\mathrm{ht}(p[X] \cap T) = \mathrm{ht}(p) = \mathrm{ht}_v(p)$, then using Proposition 2.1, to prove that $\mathrm{ht}(p[X] \cap T) = \mathrm{ht}_v(p[X] \cap T) = \mathrm{ht}_v(p) = \mathrm{ht}(p)$, it suffices to prove that $p[X] \cap T$ is not maximal in T.

Indeed, assume that $p[X]\cap T$ is maximal in T and $\mathrm{ht}(p[X]\cap T)=\mathrm{ht}(p)$. Let q be a prime ideal of R such that $\mathrm{ht}(q)=\mathrm{ht}(p)-1$. By passage to the quotients $R/q\subseteq T/(q[X]\cap T)\subseteq (R/q)[X]$, we can suppose that $\mathrm{ht}(p)=1$ and that $p[X]\cap T$ is a height 1 maximal ideal of T. By [5, Proposition 1.7], we have necessarily $R/p=T/(p[X]\cap T)$ and therefore $R\subset T\subseteq R+p[X]$ and $T=R+p[X]\cap T$. But in this case $\mathrm{ht}(p[X]\cap T)=2>1$, since $(0)\subset (XR[X])\cap T\subset p[X]\cap T$, contradicting the hypothesis $\mathrm{ht}(p[X]\cap T)$ $=\mathrm{ht}(p)=1$. ∎

Now, we turn our attention to pseudo-polynomial rings. We start with a simple result showing how we can construct a pseudo-polynomial ring.

PROPOSITION 2.3. *Let R be a ring, I an ideal of R such that* $\dim(R/I)=0$, *and D a ring contained between R/I and $(R/I)[X]$. Then the ring T of the $(R[X],I[X],D)$ construction is a pseudo-polynomial ring.*

Proof. Let Q be a prime ideal of T and $q=Q\cap R$. If Q does not contain $I[X]$, then by [10, Proposition 0] there is a prime ideal Q' of $R[X]$ lying over Q. Since $q[X]\subseteq Q'$, we find that $q[X]\cap T\subseteq Q$. If Q contains $I[X]$, the result is a consequence of Proposition 3.5 since $R/I\subseteq T/I[X]\subseteq (R/I)[X]$. ∎

REMARK 2.4. (a) The "$\dim(R/I)=0$" condition is essential in the previous proposition. Indeed, if $\dim(R/I)\geq 1$, then by Proposition 3.5, there exists a ring D contained between R/I and $(R/I)[X]$ which is not pseudo-polynomial. Therefore, the ring $T:=(R[X],I[X],D)$ is not pseudo-polynomial.

(b) If J is not an extended ideal of $R[X]$ (i.e. is not of the form $I[X]$), then an $(R[X],J,D)$ construction ring need not be pseudo-polynomial as will be illustrated by the following example.

EXAMPLE 2.5. Let t_1,t_2 be two independent indeterminates over a field K. Setting $B=K[t_1,t_2]$, $M=(t_1-1)$ and $N=(t_1,t_2)$, we have $B/M\cong K[t_2]$ and $B/N\cong K$. Let $I=M\cap N$, $R=K+I$, $q=(t_1+t_2)B$ and $p=q\cap R$. Pick an element α of q whose class $\overline{\alpha}$ in B/M is transcendental over $R/I=K$ (for instance $\alpha=t_1+t_2$). By [11, Exemple 8], for $P=(\alpha X-1)\cap R[X]$, $(0)\subset P\subset I[X]$ is a chain of prime ideals in $R[X]$. Consider the ring T obtained by the $(R[X],P,R)$ construction. It is clear that $R\subset T\subset R[X]$ and that $p+P$ is a prime ideal of T lying over p in R. Select two elements x and y in I such that $x+y\in p$, $x\notin p$ and $y\notin p$ (for instance $x=(t_1-1)t_1$ and $y=(t_1-1)t_2$). Then the polynomial $f=x(\alpha X-1)-y=\alpha xX-x-y$ belongs to $p[X]\cap T$, whereas f does not belong to $p+P$. Thus $p+P$ does not contain $p[X]\cap T$ and T is not pseudo-polynomial.

Recall that a ring extension $R\subseteq T$ is *lying-over* if each prime ideal p of R lifts to T.

Proposition 2.6. *Let T be a ring contained between R and $R[X]$. Then T is pseudo-polynomial if and only if $T \subseteq R[X]$ is a lying-over extension.*

Proof. Of course the "if" part is immediate since $R[X]$ is pseudo-polynomial.

For the "only if" part, let Q be a prime ideal of T and set $q = Q \cap R$. Since T is pseudo-polynomial, we have $q[X] \cap T \subseteq Q$. By localization of R at q, we can assume that R is local with maximal ideal q. If $q[X] \cap T \subset Q$, then $Q/(q[X] \cap T)$ is a nonzero prime ideal of $T/(q[X] \cap T)$ which is contained between R/q and $(R/q)[X]$. Since $(R/q, (R/q)[X])$ is a lying-over pair [13, Proposition 2.9], there is a prime ideal $Q'/q[X]$ of $(R/q)[X]$ such that $Q/(q[X] \cap T) = (Q'/q[X]) \cap (T/(q[X] \cap T))$. Thus $Q' \cap T = Q$. ∎

Now, we give some results concerning the spectrum of pseudo-polynomial rings which generalize well-known results about the polynomial ring in one variable.

Proposition 2.7. *Let R be a domain, p a prime ideal of R and T a pseudo-polynomial ring distinct from R. Then*:

(i) $\mathrm{ht}(p) \leq \mathrm{ht}(p[X] \cap T) \leq 2\mathrm{ht}(p)$;
(ii) $\dim(R) + 1 \leq \dim(T) \leq 2\dim(R) + 1$.

To prove this result we need the following lemma.

Lemma 2.8. *Let R be a domain and T a pseudo-polynomial ring. If $P_1 \subseteq P_2 \subseteq P_3$ is a chain of primes in T such that $P_1 \cap R = P_2 \cap R = P_3 \cap R$, then necessarily $P_1 = P_2$ or $P_2 = P_3$.*

Proof. By localization of R at $p = P_1 \cap R$, we can suppose that R is local with maximal ideal p. Since T is pseudo-polynomial, $p[X] \cap T \subseteq P_1$. On the other hand, we have the inclusions $R/p \subseteq T/(p[X] \cap T) \subseteq (R/p)[X]$ and $\dim(T/(p[X] \cap T)) \leq 1$ [4, Lemme 1.1], hence necessarily $P_1/(p[X] \cap T) = P_2/(p[X] \cap T)$ or $P_2/(p[X] \cap T) = P_3/(p[X] \cap T)$. Thus $P_1 = P_2$ or $P_2 = P_3$. ∎

Proof of Proposition 2.7. The prime ideal $(X) \cap T$ is nonzero and such that $T/(X) \cap T \simeq R$. Thus $\dim(R) + 1 \leq \dim(T)$. The inequality $\mathrm{ht}(p) \leq \mathrm{ht}(p[X] \cap T)$ is established in Proposition 2.1(i). It is straightforward to check that, with the use of Lemma 2.8, the proof of [18, Corollary 30.3] may be adapted to establish the remaining inequalities. ∎

Remark 2.9. Let R be a one-dimensional domain and M a maximal ideal of R, and set $T := (R[X], M[X], R/M) = R + M[X]$. Then, by Proposition 2.3, T is a pseudo-polynomial ring. Moreover, we have $\mathrm{ht}_T(M[X]) = 2$ and $\dim(T) = 2$. Thus the upper bound in statement (i) of Proposition 2.7 is attained.

Notice that if R is a Jaffard domain, then for each T such that $R \subset T \subseteq R[X]$ we have $\dim(T) = \dim_v(T) = \dim_v(R) + 1 = \dim(R) + 1$. On the other hand, consider a local domain R with maximal ideal M such that $\dim(R[X]) = 2\dim(R) + 1$. Let $T := (R[X], M[X], (R/M)[X^2])$. By Proposition 2.3, T is a pseudo-polynomial ring and $\dim(T) = 2\dim(R) + 1$ since $R[X]$ is integral over T. Hence the upper bound in statement (ii) of Proposition 2.7 is attained.

LEMMA 2.10. *Let R be a domain and T a pseudo-polynomial ring. If P is a prime ideal of T such that $p[X] \cap T \subset P$, where $p = P \cap R$, then* $\mathrm{ht}(P) = \mathrm{ht}(p[X] \cap T) + 1$ *and* $\mathrm{ht}_v(P) = \mathrm{ht}_v(p[X] \cap T) + 1 = \mathrm{ht}_v(p) + 1$.

Proof. Both assertions of the lemma are clear if $\mathrm{ht}(P) = \infty$. Hence assume that $\mathrm{ht}(P)$ is finite. To prove the first part, we proceed as in [18, Lemma 30.17]. By localization of R at p, we can suppose that R is local with maximal ideal p. If $p = (0)$, then $\mathrm{qf}(R) \subseteq S^{-1}T \subseteq \mathrm{qf}(R)[X]$, where $S = R \setminus \{0\}$. By [4, Lemme 1.1], we have $\dim(S^{-1}T) \leq 1$. Thus $\mathrm{ht}(P) = 1 = 1 + \mathrm{ht}(p[X] \cap T)$. We assume the result for all $k < m$, where $m \geq 0$ and $\mathrm{ht}(p) = m$. To prove that $\mathrm{ht}(P) = \mathrm{ht}(p[X] \cap T) + 1$, it suffices to show that $\mathrm{ht}(Q) \leq \mathrm{ht}(p[X] \cap T)$ for each prime ideal $Q \subset P$. Let $q = Q \cap R$. If $q = p$, then $(p[X] \cap T) \subseteq Q \subset P$. By the previous lemma, we have necessarily $Q = p[X] \cap T$ and $\mathrm{ht}(Q) \leq \mathrm{ht}(p[X] \cap T)$. If $q \subset p$, then the induction hypothesis implies that $\mathrm{ht}(Q) = \mathrm{ht}(q[X] \cap T) + 1 \leq \mathrm{ht}(p[X] \cap T)$. This completes the proof of $\mathrm{ht}(P) = \mathrm{ht}(p[X] \cap T) + 1$.

Let $Y_1, \ldots, Y_n$ be n indeterminates over $R[X]$. We have the inclusions $R[Y_1, \ldots, Y_n] \subseteq T[Y_1, \ldots, Y_n] \subseteq R[Y_1, \ldots, Y_n][X]$. From the first part of the lemma, $\mathrm{ht}(P[n]) = \mathrm{ht}(P[Y_1, \ldots, Y_n]) = \mathrm{ht}((p[X] \cap T)[Y_1, \ldots, Y_n]) + 1$. By Proposition 2.1 and letting $n \to \infty$, we have $\mathrm{ht}_v(P) = \mathrm{ht}_v(p[X] \cap T) + 1$ $= \mathrm{ht}_v(p) + 1$. ∎

By combining Proposition 2.1 and Lemma 2.10, we have the following theorem which generalizes the special chain theorem ([9], [19]) and the valuative special chain theorem [15] for a given pseudo-polynomial ring.

THEOREM 2.11. *Let R be a domain, X an indeterminate over R and T an R-pseudo-polynomial ring. Let P be a prime ideal of T and let $p = P \cap R$. Then* $\mathrm{ht}(P) = \mathrm{ht}(p[X] \cap T) + \mathrm{ht}(P/(p[X] \cap T)) \leq \mathrm{ht}(p[X] \cap T) + 1$ *and* $\mathrm{ht}_v(P) = \mathrm{ht}_v(p) + \mathrm{ht}(P/(p[X] \cap T)) \leq \mathrm{ht}_v(p) + 1$.

DEFINITION 2.12. Let R be a ring, X an indeterminate over R and T a ring such that $R \subseteq T \subseteq R[X]$. We say that a chain $P_0 \subset P_1 \subset \ldots \subset P_k$ of primes in T is a *generalized special chain* if for each $0 \leq i \leq k$, $(p_i[X] \cap T)$ is a member of this chain, where $p_i = P_i \cap R$.

It is clear that if $T = R[X]$, then this definition coincides with the definition of a special chain set by P. Jaffard [19, Chapitre II, Section 4]. The next theorem is a generalization of [19, chapitre II, Théorème 3] for pseudo-polynomial rings R.

THEOREM 2.13. *Let R be a ring and T a pseudo-polynomial ring. If P is a prime ideal of T of finite height, then* $\mathrm{ht}(P)$ *can be realized as the length of a generalized special chain of primes in T with terminal element P.*

Proof. We proceed as in [18]. If $\mathrm{ht}(P) = 0$, then $P = (P \cap R)[X] \cap T$ and the desired conclusion holds. If $\mathrm{ht}(P) = m > 0$, and if the theorem is true for primes of height less than m, then we take a chain of prime ideals $P_0 \subset P_1 \subset \ldots \subset P_m = P$ of length m. It is clear that $\mathrm{ht}(P_j) = j$ for each j. By Theorem 2.11, we can assume that $P_{m-1} = (P \cap R)[X] \cap T$ or $P_m = (P \cap R)[X] \cap T$. If $P_{m-1} = (P \cap R)[X] \cap T$, then the induction hypothesis implies that there is a generalized special chain $P'_0 \subset P'_1 \subset \ldots \subset P'_{m-1} = P_{m-1}$ of length $m-1$, and $P'_0 \subset \ldots \subset P'_{m-1} = P_{m-1} \subset P$ is a generalized special chain of length m with terminal element P. If $P = (P \cap R)[X] \cap T$, then we can obtain a generalized special chain $P'_0 \subset \ldots \subset P'_{m-1}$ of length $m-1$ with terminal element P_{m-1}, and $P'_0 \subset \ldots \subset P'_{m-1} \subset P_m = P$ is a generalized special chain of length m with terminal element P. ∎

It is known that if R is a Jaffard domain, then each domain between R and $R[X]$ is Jaffard [5]. We give here an analogous result in the case where R is locally Jaffard for pseudo-polynomial rings. Note that it is an open question whether each ring between R and $R[X]$ is locally Jaffard whenever R is locally Jaffard.

THEOREM 2.14. *If R is a locally Jaffard domain, then each pseudo-polynomial ring is locally Jaffard.*

Proof. Let T be a pseudo-polynomial ring distinct from R, P a prime ideal of T, and $p = P \cap R$. By localization of R at p, we can suppose that R is local with maximal ideal p. Two cases may occur:

Case 1: $p[X] \cap T \subset P$. Then by Lemma 2.10, $\mathrm{ht}_v(P) = \mathrm{ht}_v(p) + 1 \leq \mathrm{ht}(P)$ and therefore $\mathrm{ht}_v(P) = \mathrm{ht}(P) = \mathrm{ht}(p) + 1$.

Case 2: $p[X] \cap T = P$. By Corollary 2.2, we have $\mathrm{ht}(P) = \mathrm{ht}_v(P) = \mathrm{ht}(p)$ or $\mathrm{ht}(p) + 1$. ∎

COROLLARY 2.15. *If R is a one-dimensional S-domain, then each pseudo-polynomial ring is totally Jaffard.*

Proof. From [4, Lemme 1.1], it follows that if T is a domain such that $R \subseteq T \subseteq R[X]$, then $\dim(T) \leq \dim(R) + 1 = 2$. Thus, each ring between R and $R[X]$ is catenarian. Since a locally Jaffard catenarian domain is totally Jaffard [11, Corollaire 1], the previous theorem yields the desired conclusion. ∎

COROLLARY 2.16. *If R is an S-domain, then each pseudo-polynomial ring is an S-domain.*

Proof. Let T be a pseudo-polynomial ring and let P be a height 1 prime ideal in T. We have $p[X] \cap T \subseteq P$, where $p = P \cap R$. If $p = (0)$, then $\mathrm{qf}(R) \subseteq S^{-1}T \subseteq \mathrm{qf}(R)[X]$, where $S = R \setminus \{0\}$. By [5, Théorème 1.9], we conclude that $S^{-1}T$ is locally Jaffard and therefore $\mathrm{ht}(P) = \mathrm{ht}_v(P) = 1$. If $p \neq (0)$, then necessarily $P = p[X] \cap T$ and $\mathrm{ht}(p) = 1$. By localization of R at p and using the previous corollary, we deduce that $\mathrm{ht}(P) = \mathrm{ht}_v(P) = 1$. Thus T is an S-domain. ■

3. Pseudo-polynomial rings and lying-over pairs. Recall that a ring extension $R \subseteq T$ is *lying-over* (resp., *survival*) if each prime ideal p of R lifts to T (resp., $pT \neq T$). Considering an extension property $(\mathcal{P})$, we say that (R, T) is a $(\mathcal{P})$*-pair* if, for each ring A between R and T, that is, $R \subseteq A \subseteq T$, both extensions $R \subseteq A$ and $A \subseteq T$ are $(\mathcal{P})$ extensions.

If $R \subseteq T$ is a lying-over extension, then for each A in between, $R \subseteq A$ is also a lying-over extension. Thus to say that (R, T) is a lying-over pair, it is enough to say that, for each A such that $R \subseteq A \subseteq T$, $A \subseteq T$ is a lying-over extension. On the other hand, in the case of survival pairs it is enough to say that for each A such that $R \subseteq A \subseteq T$, $R \subseteq A$ is a survival extension. A pair (R, T), where $R \subseteq T$, is said to be a *lying-over pair* (resp., *survival pair*) if $A \subseteq T$ (resp., $R \subseteq A$) is a lying-over extension (resp., survival extension) whenever A is an intermediate ring between R and T [21]. It is immediate that (R, T) is a survival pair if and only if each maximal ideal of an intermediate ring between R and T lifts to T.

In [13], D. E. Dobbs has shown that for rings $R \subseteq T$ such that T is semi-local, T is integral over R if and only if (R, T) is a lying-over pair, or equivalently (R, T) is a survival pair. He asked if a survival pair is always a lying-over pair. In this section we answer this question in the affirmative in case T is transcendental over R.

In [13], D. E. Dobbs noted that a survival extension need not be lying-over. He gave two examples; the first derives from Chevalley [12, Lemma 2] and the second is due to W. J. Lewis [14, Example 4.4]. A sharpening of this remark is that a survival extension of the form $R \subseteq R[u]$ (for some u in the quotient field of R) need not be lying-over (see Example 3.6).

We start with the following proposition.

PROPOSITION 3.1. *Let $R \subseteq T$ be a ring extension. Then (R, T) is a survival pair if and only if $(R/(q \cap R), T/q)$ is a survival pair for each prime ideal q of T.*

Proof. For the "only if" part, our task is to show that for each ring C contained between $R/(q \cap R)$ and T/q, C "survives" in T/q. Note that C may be identified as $C = A/q$ where A is a suitable ring contained between $R + q$ and T, and that maximal ideals of C correspond to those of A containing q.

Let M be a maximal ideal of A containing q. Since $(R+q,T)$ is a survival pair and $A \subseteq T$, there exists a prime ideal q_1 of T such that $M = q_1 \cap A$ and so $M/q = (q_1/q) \cap (A/q)$.

For the "if" part, consider a maximal ideal M of a ring C contained between R and T. Let q be a minimal prime ideal contained in M. Then q lifts to T as a minimal prime q'. Since $(R/(q' \cap R), T/q')$ is a survival pair, there exists a prime ideal q_1/q' of T/q' lying over M/q in C/q and hence $M = q_1 \cap C$. ∎

Recall that D. E. Dobbs has established that for any ring R, $(R, R[X])$ is a lying-over pair if and only if $\dim(R) = 0$ [13, Proposition 2.13]. This characterization is sharpened in the next lemma.

LEMMA 3.2. *Let R be a ring. Then $(R, R[X])$ is a survival pair if and only if* $\dim(R) = 0$.

Proof. For the "if" part, assume that $\dim(R) = 0$; then by [13, Proposition 2.13], $(R, R[X])$ is a lying-over pair and a fortiori a survival pair.

For the "only if" part, if p is a minimal prime ideal of R, then by considering the survival pair $(R/p, (R/p)[X])$ (Proposition 3.1), we may assume that R is a domain. If $\dim(R) > 0$, then R contains a maximal ideal M whose height is greater than 1. Select a nonzero element b in M, and consider $T = R[1+bX]$. By hypothesis, the extension $T \subseteq R[X]$ is survival and so there exists a prime ideal Q of $R[X]$ such that $Q \cap T = (M, 1+bX)T$ (note that $(M, 1+bX)T$ is a maximal ideal of T). As $bX \in (MR[X] \cap T) \subseteq Q \cap T$, it follows that $1 = (1+bX) - bX \in Q \cap T$, contradicting $Q \subset T$. ∎

The next result as well as Proposition 3.4 are partial answers to a question left open in [13] of whether a survival pair must be lying-over.

THEOREM 3.3. *Let $R \subseteq T$ be domains such that T is transcendental over R. The following statements are equivalent*:

(i) *(R, T) is a lying-over pair*;
(ii) *(R, T) is a survival pair*;
(iii) *R is a field and for each element x in T which is transcendental over R, T is integral over $R[x]$*;
(iv) *R is a field and T is integral over $R[x]$ for some element x in T which is transcendental over R.*

Proof. Assertions (i) and (iii) are equivalent by [13, Theorem 4.1]. It is clear that (i) implies (ii).

(ii)⇒(iii). Let x be an element of T which is transcendental over R. Since $(R, R[x])$ is a survival pair, the previous lemma shows that R is a field. If there were an element y of T which is transcendental over $R[x]$, then $(R[x], R[x][y])$ would be a survival pair, contradicting Lemma 3.2. Hence, T

is algebraic over $R[x]$. Since $R[x]$ is a one-dimensional Prüfer domain, each ring contained between $R[x]$ and T has dimension less than 1 and $(R[x], T)$ is an INC pair. Using [13, Theorem 2.1 and Corollary 2.4 (bis)], we conclude that T is integral over $R[x]$.

Finally, the equivalence between (iii) and (iv) is clear. ∎

However, in the algebraic case we obtain the following result in the case of (T, M, D) constructions.

PROPOSITION 3.4. *Let T be a domain, M a maximal ideal of T, D a subring of the field $K = T/M$ and $R := (T, M, D)$. Then the following statements are equivalent.*

(i) (R, T) *is a lying-over pair*;
(ii) (R, T) *is a survival pair*;
(iii) D *is a field and* $D \subseteq K$ *is an algebraic extension.*

Proof. The implication (i)⇒(ii) is clear.

(ii)⇒(iii). Since (R, T) is a survival pair, Proposition 3.1 shows that (D, K) is also a survival pair. Hence by [13, Theorem 2.7], K is integral over D. Thus D is a field and $D \subseteq K$ is an algebraic extension.

(iii)⇒(i). Since D is a field and $D \subseteq K$ is an algebraic extension, K is integral over D. Thus T is integral over R (see [10]). Therefore (R, T) is a lying-over pair. ∎

By combining Theorem 3.3, [13, Proposition 2.13] and Proposition 2.6, we get the following:

PROPOSITION 3.5. *Let R be a ring. The following statements are equivalent*:

(i) *each ring contained between R and $R[X]$ is pseudo-polynomial*;
(ii) $(R, R[X])$ *is a lying-over pair*;
(iii) $(R, R[X])$ *is a survival pair*;
(iv) $\dim(R) = 0$.

The next example provides a survival extension of the form $R \subset R[u]$ (for some u in the quotient field of R) which is not lying-over.

EXAMPLE 3.6. With the same hypotheses as in Example 2.5, set $A = R[X]/P \simeq R[1/\alpha]$. We prove that the extension $R \subset A$ is not lying-over. Indeed, assume that there is a prime ideal Q/P of $R[X]/P$ such that $p = (Q/P) \cap R$. Since $R_p = B_q$, we see that $\alpha \in Q_p$. But $\alpha X - 1 \in Q_p$, implying that $1 \in Q_p$, a contradiction. We conclude that $R \subset A$ is not a lying over extension while it is a survival extension since $I.A = I[X]/P$. ∎

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REFERENCES

[1] D. D. Anderson, D. F. Anderson and M. Zafrullah, *Rings between $D[X]$ and $K[X]$*, Houston J. Math. 17 (1991), 109–129.

[2] D. F. Anderson, A. Bouvier, D. E. Dobbs, M. Fontana and S. Kabbaj, *On Jaffard domains*, Exposition. Math. 5 (1988), 145–175.

[3] A. Ayache, *Inégalité ou formule de la dimension et produits fibrés*, thèse de doctorat, Univ. d'Aix-Marseille III, 1991.

[4] A. Ayache et P.-J. Cahen, *Anneaux vérifiant absolument l'inégalité ou la formule de la dimension*, Boll. Un. Mat. Ital. B (7) 6 (1992), 39–65.

[5] —, —, *Radical valuatif et sous-extensions*, Comm. Algebra 26 (1998), 2767–2787.

[6] A. Ayache, P.-J. Cahen et O. Echi, *Anneaux quasi-prüfériens et P-anneaux*, Boll. Un. Mat. Ital. B (7) 10 (1996), 1–24.

[7] A. Ayache and A. Jaballah, *Residually algebraic pairs of rings*, Math. Z. 225 (1997), 49–65.

[8] A. Bouvier, D. E. Dobbs and M. Fontana, *Universally catenarian integral domains*, Adv. Math. 72 (1988), 211–238.

[9] J. W. Brewer, P. R. Montgomery, E. A. Rutter and W. J. Heinzer, *Krull dimension of polynomial rings*, in: Lecture Notes in Math. 311, Springer, 1972, 26–45.

[10] P.-J. Cahen, *Couple d'anneaux partageant un idéal*, Arch. Math. (Basel) 51 (1988), 505–514.

[11] —, *Construction B, I, D et anneaux localement ou résiduellement de Jaffard*, ibid. 54 (1990), 125–141.

[12] C. Chevalley, *La notion d'anneau de décomposition*, Nagoya Math. J. 7 (1954), 21–33.

[13] D. E. Dobbs, *Lying-over pairs of commutative rings*, Canad. J. Math. 33 (1981), 454–475.

[14] D. E. Dobbs and I. J. Papick, *Going-down: a survey*, Nieuw. Arch. Wisk. 26 (1978), 255–291.

[15] O. Echi, *Sur les hauteurs valuatives*, Boll. Un. Mat. Ital. B (7) 9 (1995), 281–297.

[16] M. Fontana, *Topologically defined classes of commutative rings*, Ann. Mat. Pura Appl. (4) 123 (1980), 331–355.

[17] M. Fontana, L. Izelgue et S. Kabbaj, *Quelques propriétés des chaînes d'idéaux premiers dans les anneaux $A + XB[X]$*, Comm. Algebra 22 (1994), 9–24.

[18] R. Gilmer, *Multiplicative Ideal Theory*, Dekker, New York, 1972.

[19] P. Jaffard, *Théorie de la dimension dans les anneaux de polynômes*, Mém. Sci. Math. 146, Gauthier-Villars, Paris, 1960.

[20] S. Kabbaj, *Sur les S-domaines forts de Kaplansky*, J. Algebra 137 (1991), 400–415.

[21] I. Kaplansky, *Commutative Rings*, rev. ed., Univ. of Chicago Press, Chicago, 1974.

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