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SPACES OF MULTIPLIERS AND THEIR PREDUALS FOR THE ORDER MULTIPLICATION ON $[0,1]$

BY

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Abstract. Let $I=[0,1]$ be the compact topological semigroup with max multiplication and usual topology. $C(I)$, $L^p(I)$, $1\le p\le\infty$, are the associated Banach algebras. The aim of the paper is to characterise $\mathrm{Hom}_{C(I)}(L^r(I),L^p(I))$ and their preduals.

1. Introduction. In [20], Wendel proved that the operators on the group algebra $L^1(G)$ which commute with convolution correspond in a natural way to the measure algebra $M(G)$. The more general situation of Banach module homomorphisms for Banach modules K over $L^1(G)$ has been considered by several authors. The operators from $L^1(G)$ into K, and from K into $L^\infty(G)$, which commute with module composition have been investigated by Gulick, Liu and van Rooij [10]. Investigations of various other kinds of module homomorphisms occur in Figà-Talamanca [6], Figà-Talamanca and Gaudry [7], Johnson [13] and Rieffel [17].

The extension of Wendel's result on multipliers on a group algebra $L^1(G)$ to locally compact semigroups has been carried out by, among others, Larsen [15], Todd [19], Dhar and Vasudeva [3], Johnson and Lahr [14] and Baker, Pym and Vasudeva [1]. Both Larsen [15] and Todd [19] obtain characterisations of multipliers for $L^1(S)$ where S is a totally ordered semigroup. The characterisations of multipliers for $L^1(S)$, where S is a partially ordered semigroup which is the product of totally ordered semigroups, have been obtained by Dhar and Vasudeva [3] and Johnson and Lahr [14]. A characterisation of multipliers of a convolution measure algebra in the measure algebra $M(S)$ of certain subsemilattices S of the cube I^n, where I denotes the unit interval with min multiplication, has been obtained in [2]. Baker, Pym and Vasudeva [1] obtain characterisations of multipliers from $L^p(I)$ to $L^r(I)$, $1\le r\le p$, where $I=[0,1]$ is the compact topological semigroup with max multiplication and the usual topology. Of course, the choice of max or min makes no difference to the theory.

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The aim of this paper is to study the multipliers or at least some version of them for the interval $[0,1]$ with max as multiplication. This is a compact semigroup, so that it is possible to define convolution of measures on it. It is also well known that $L^1(I)$ forms an algebra under convolution. It is not clear what should be the notion of multiplier in this context; the one we choose—$\mathrm{Hom}_{C(I)}(L^r(I), L^p(I))$, the space of $C(I)$-module homomorphisms from $L^r(I)$ to $L^p(I)$ where $C(I)$ acts by convolution—is at once interesting and natural. There are differences from the group case arising from the way in which the convolution action of $C(I)$ on $L^p(I)$ differs fundamentally from the dual action that $L^p(I)$ inherits from $L^{p'}(I)$.

Our methods are in the spirit of the work of Figà-Talamanca, Gaudry, Hörmander, Herz and Eymard ([7], [12], [11], [5]) in that we study the predual space (the so-called $A^{r,p}$ space) of the space of multipliers. Indeed in this situation we are able to give a complete description of $A^{r,p}$ as a space of measurable functions on I in the case when $r \le p$, though here also the lack of symmetry between the convolution action and its adjoint is manifested, and we have to study two different $A^{r,p}$ spaces corresponding to the two actions. This aspect of our work is described in Section 3. Our characterisation of these spaces makes heavy use of Hardy's inequality and interpolation theory. The regularities of behaviour that occur in the $A^{r,p}$ spaces in the group case are not as apparent here, and indeed are focussed in the vicinity of the origin, as is to be expected from the nature of the multiplication operation. In particular, there appears to be no analogue of Herz's theorem that the A^p spaces form an algebra under pointwise multiplication.

Section 2 is concerned with preliminary material. In Section 4 we consider the predual of the space of multipliers as an algebra under convolution. It is shown that this algebra has an approximate identity and we identify its maximal ideal space. Finally in Section 5, we use the knowledge we have obtained of the predual to obtain a characterisation of the multipliers in question.

2. Preliminaries. This section contains the preliminary material on which we shall draw throughout the rest of the paper. Henceforth, I will denote the closed interval $[0,1]$. We make I a semigroup by defining the product of x and y in I by $xy = \max\{x, y\}$. When I is endowed with the usual interval topology, I is a compact topological semigroup. For $p \in [1,\infty)$, we let, as is customary, $L^p(I)$ be the Banach space of Lebesgue measurable functions on I whose pth powers are absolutely integrable. $L^\infty(I)$ consists of the functions measurable and essentially bounded on I with respect to Lebesgue measure. An important subspace of $L^\infty(I)$ is the space of all continuous functions on I, which we denote by $C(I)$. For every positive number α, let α' denote the conjugate index of α, i.e. α' is such that $\alpha^{-1}+(\alpha')^{-1} = 1$.

If E is a Banach space, let E^* denote the topological dual of E. By identification, $(L^1(I))^* = L^\infty(I)$ and for $1 < p < \infty$, $(L^p(I))^* = L^{p'}(I)$.

Let A be a normed algebra over $\mathbb{C}$ and let V be a normed linear space over $\mathbb{C}$. Then V is said to be a *normed left* (resp. *right*) *A-module* if V is a left (resp. right) A-module and also satisfies $\|av\| \le k\|a\| \cdot \|v\|$ (resp. $\|va\| \le k\|v\| \cdot \|a\|$) for all $a \in A$, $v \in V$ and for some positive constant k. A normed left A-module is called a *Banach left A-module* if it is complete as a normed linear space.

If V and W are left (resp. right) Banach A-modules, then $\mathrm{Hom}_A(V, W)$ will denote the Banach space of all continuous A-module homomorphisms from V to W with the operator norm. The elements of $\mathrm{Hom}_A(V, W)$ are customarily called *multipliers* from V to W. If V is a left (resp. right) Banach A-module, then V^*, the dual of V, is a right (resp. left) Banach A-module under the adjoint action of A.

We define the convolution $*$: $C(I) \times L^p(I) \to L^p(I)$ by

$$\int_0^1 (f * g)(t)\phi(t)\,dt = \int_0^1\int_0^1 \phi(\max(s,t)) f(t) g(s)\,dt\,ds \tag{1}$$

for $f \in C(I)$, $\phi \in C(I)$ and $g \in L^p(I)$. This leads to the following formula for the convolution of f and g:

$$(f * g)(t) = f(t)\int_0^t g(s)\,ds + g(t)\int_0^t f(s)\,ds \tag{2}$$

for almost all $t \in I$. Observe that $\|g\|_1 \le \|g\|_p$ $(g \in L^p(I))$, by Hölder's inequality. Also

$$|(f * g)(t)| \le \|f\|_\infty(\|g\|_1 + |g(t)|)$$

for almost all $t \in I$, $f \in C(I)$ and $g \in L^p(I)$. Consequently,

$$\|f * g\|_p \le \|f\|_\infty\|g\|_1 + \|f\|_\infty\|g\|_p \le 2\|f\|_\infty\|g\|_p$$

for $f \in C(I)$ and $g \in L^p(I)$. Thus $L^p(I)$ is a normed left $C(I)$-module and when so regarded, it is denoted by L^p_*. Indeed, L^p_* is a left Banach $C(I)$-module.

Next we investigate the adjoint action of $C(I)$ on the dual of L^p_*. We define the dual pairing between $L^p(I)$ and $L^{p'}(I)$ by

$$\langle f, g\rangle = \int_0^1 f(s)g(s)\,ds, \qquad f \in L^p(I),\ g \in L^{p'}(I),$$

and for $f \in L^r(I)$ and $g \in L^p(I)$, we define

$$f \circ g(s) = g(s)\int_0^s f(t)\,dt + \int_s^1 f(t)g(t)\,dt, \qquad s \in [0,1] \text{ and } \frac{1}{r} + \frac{1}{p} \le 1. \tag{3}$$

Then, for $\phi \in C(I)$, $f \in L^p(I)$ and $g \in L^{p'}(I)$, we have

$$\begin{aligned}\langle \phi * f, g\rangle &= \int_0^1 (\phi * f)(t)g(t)\,dt = \int_0^1 \Big\{\phi(t)\int_0^t f(s)\,ds + f(t)\int_0^t \phi(s)\,ds\Big\} g(t)\,dt\\ &= \int_0^1 f(s)\Big\{g(s)\int_0^s \phi(t)\,dt + \int_s^1 \phi(t)g(t)\,dt\Big\}\,ds = \langle f, g\circ\phi\rangle,\end{aligned}$$

by Fubini's Theorem.

Thus the adjoint action of an element $\phi \in C(I)$ on $L^{p'}(I)$ under which $L^{p'}(I)$ becomes a Banach right $C(I)$-module consists of the operation $\circ$ as defined in (3). The Banach right $C(I)$-module $L^{p'}(I)$ with the adjoint action will be denoted by $L^{p'}_{\circ}$.

In case $L^p(I)$ is regarded as a Banach right $C(I)$-module under the convolution action then $L^{p'}(I)$ becomes a Banach left $C(I)$-module under the "$\bullet$" action defined by $\phi \bullet g = g \circ \phi$.

The concept of tensor product for Banach modules was introduced by Rieffel [16]. We include the definition for completeness. Let V and W be, respectively, a left and right Banach A-module. Let $V \hat{\otimes} W$ denote the projective tensor product [9] of V and W as Banach spaces and let K be the closed linear subspace of $V \hat{\otimes} W$ which is spanned by all elements of the form

$$av \otimes w - v \otimes wa, \qquad a \in A,\ v \in V,\ w \in W.$$

Then the A-module tensor product, $V \hat{\otimes}_A W$, is defined to be the quotient Banach space $(V \hat{\otimes} W)/K$. Using the universal property of the projective tensor product with respect to bounded bilinear maps from $V \times W$, it is easily seen that $V \hat{\otimes}_A W$ has the desired universal property with respect to A-balanced bounded bilinear maps from $V \times W$.

We let $M^{r,p}_{\circ}$ denote $\mathrm{Hom}_{C(I)}(L^r_*, L^p_*)$ and $M^{r,p}_*$ denote $\mathrm{Hom}_{C(I)}(L^r_*, L^p_{\circ})$ for $1 \le r, p \le \infty$. Observe that $\mathrm{Hom}_{C(I)}(L^r_{\circ}, L^p_{\circ})$ may be identified with $\mathrm{Hom}_{C(I)}(L^{p'}_*, L^{r'}_*)$ by the adjoint map at least for $p, r \ne 1$. If we let $A^{r,p}_* = L^r_* \hat{\otimes}_{C(I)} L^{p'}_*$ and $A^{r,p}_{\circ} = L^r_* \hat{\otimes}_{C(I)} L^{p'}_{\circ}$, where the tensor product is the projective tensor product of Banach modules as defined in the paragraph above, then, by a theorem of Rieffel [17],

$$(A^{r,p}_*)^* = M^{r,p}_* \quad \text{and} \quad (A^{r,p}_{\circ})^* = M^{r,p}_{\circ}.$$

A key result in the understanding and characterisation of $A^{r,p}_*$ and $A^{r,p}_{\circ}$ is the following minor variant of an argument of Rieffel [17].

THEOREM 1. *Let D be a dense subalgebra of a Banach algebra A and let B be a Banach A-module. Suppose that A has an approximate identity (e_n) satisfying $\|e_n a\| \le C\|a\|$ for each n and all $a \in A$. Then the map*

$\pi : A\hat{\otimes} B \to B$ given by $a\otimes b \mapsto ab$ factors through $A\hat{\otimes}_D B$ and the resulting map $A \hat{\otimes}_D B \to B$ is injective.

Proof. Since the theorem is well known, only a brief outline of the proof is given below.

If $\sum_i a_i \otimes b_i \in A \hat{\otimes} B$ then

$$\Big\|\sum_i a_i \otimes b_i - \sum_i e_n a_i \otimes b_i\Big\| \le \sum_i \|a_i - e_n a_i\| \cdot \|b_i\|$$
$$\le \sum_{i=1}^{M} \|a_i - e_n a_i\| \cdot \|b_i\| + (C+1) \sum_{i=M+1}^{\infty} \|a_i\| \cdot \|b_i\|.$$

Given $\varepsilon > 0$, choose M so large that the second sum on the right hand side is less than $\varepsilon/2$ independently of n and then by taking n large the first sum is less than $\varepsilon/2$. If $\pi(\sum_i a_i \otimes b_i) = 0$, then

$$\sum_i e_n a_i \otimes b_i = \sum_i (e_n a_i \otimes b_i - e_n \otimes a_i b_i)$$

is in $\operatorname{Ker}\pi$ as $\operatorname{Ker}\pi$ is closed and $A \hat{\otimes}_D B = A \hat{\otimes}_A B$. This completes the proof.

3. Description of the preduals. We define an operator

$$B : L^r(I) \hat{\otimes} L^{p'}(I) \to L^{p'} \qquad (1 \le r, p' < \infty)$$

by

$$B(f \otimes g)(s) = g(s) \int_0^s f(t)\, dt.$$

It will play a key role in our discussion of the preduals of the spaces of multipliers. We begin this section with characterisations of the image of B which we call $B^{r,p}$ in $L^{p'}$ when $1 < r, p < \infty$ and $B^{\infty,1}$ in L^∞ when $r = p' = \infty$. Let $I_n = [0, 1/2^n]$ and $J_n = [1/2^n, 1/2^{n-1}]$, $n = 1, 2, \ldots$, and let $|J|$ denote the length of an interval J. For a measurable function ϕ on I let $P_n(\phi)$ denote the function $\chi_{J_n}\phi$, $n = 1, 2, \ldots$ Define $e_n = 2^n \chi_{I_n}$, $n = 1, 2, \ldots$ Since $\int_0^1 e_n(s)\, ds = 1$ for each n, we can easily see that if $f \equiv 0$ on I_n then $B(e_n \otimes f) = f$. As $f = \sum_{n=1}^\infty P_n(f)$, we obtain

$$f = \sum_{n=1}^{\infty} B(e_n \otimes P_n(f)) = \sum_{n=1}^{\infty} e_n \circ P_n(f) = \sum_{n=1}^{\infty} e_n * P_n(f) \tag{4}$$

almost everywhere, for any measurable function f on I. We shall also need the following inequality:

$$\|e_n * f\|_r \le \frac{2r-1}{r-1} \|f\|_r, \qquad f \in L^r,\ 1 < r < \infty.$$

To see this, observe that

$$e_n * f(s) = \begin{cases} 2^n \int_0^s f(t)\,dt + 2^n s f(s), & 0 \le s \le 1/2^n, \\ f(s), & s > 1/2^n. \end{cases}$$

Consequently,

$$\|e_n * f\|_r \le \left\| \frac{1}{s} \int_0^s f(t)\,dt \right\|_r + \|f\|_r \le \frac{2r-1}{r-1}\,\|f\|_r \tag{5}$$

using Hardy's inequality.

THEOREM 2. *For $r \le p$,*

$$B^{r,p} = \Big\{ \phi : \phi \text{ is measurable},\ \sum_{n=1}^{\infty} 2^{n/r'} \|P_n \phi\|_{p'} < \infty \Big\}.$$

Proof. First note that if $\sum_{n=1}^{\infty} 2^{n/r'} \|P_n\phi\|_{p'} < \infty$ then we may write $\phi = \sum_{n=1}^{\infty} B(e_n \otimes P_n(\phi))$ and $\sum_{n=1}^{\infty} \|e_n\|_r \|P_n(\phi)\|_{p'} = \sum_{n=1}^{\infty} 2^{n/r'} \|P_n\phi\|_{p'} < \infty$. Thus

$$\psi = \sum_{n=1}^{\infty} e_n \otimes P_n(\phi) \in L^r \hat{\otimes} L^{p'}$$

and $B(\psi) = \phi$.

To prove the converse, we first show that there is a constant k such that if $\phi = B(f \otimes g)$, where $f \in L^r$, $g \in L^{p'}$ then

$$\sum_{n=1}^{\infty} 2^{n/r'} \|P_n(\phi)\|_{p'} \le k \|f\|_r \|g\|_{p'}.$$

Let

$$X = \sum_{n=1}^{\infty} 2^{n/r'} \|P_n(\phi)\|_{p'} = \sum_{n=1}^{\infty} 2^{n/r'} \Big(\int_0^1 \Big| P_n \Big(g(t) \int_0^t f(s)\,ds \Big) \Big|^{p'} dt \Big)^{1/p'}.$$

Without loss of generality, we may assume that $f, g \ge 0$ since we can replace them by their absolute values. Then

$$\begin{aligned} X &\le \sum_{n=1}^{\infty} 2^{n/r'} \int_0^{2^{-(n-1)}} f(s)\,ds\, \|P_n(g)\|_{p'} \\ &\le \Big(\sum_{n=1}^{\infty} \Big(2^{n/r'} \int_0^{2^{-(n-1)}} f(s)\,ds \Big)^p \Big)^{1/p} \|g\|_{p'} \end{aligned}$$

by Hölder's inequality. To show that $\sum_{n=1}^{\infty} (2^{n/r'} \int_0^{2^{-(n-1)}} f(s)\,ds)^p < \infty$ we define a map S from measurable functions to complex sequences by

$$S(f) = \Big(2^{n/r'} \int\limits_0^{2^{-(n-1)}} f(s)\,ds\Big)_{n\in\mathbb{N}}.$$

Its adjoint is given, formally at least, by

$$S^*((c_n)) = \sum_{n=1}^{\infty} 2^{n/r'} c_n \chi_{I_{n-1}}.$$

It is enough to prove that S^* maps $\ell^{p'}$ into $L^{r'}$, for then S will be its adjoint map from L^r to ℓ^p and we will have shown that

$$\sum_{n=1}^{\infty} \Big(2^{n/r'} \int\limits_0^{2^{-(n-1)}} f(s)\,ds\Big)^p < \infty.$$

Now

$$\begin{aligned}
\|S^*((c_n))\|_{r'}^{r'} &= \int\limits_0^1 \Big|\sum_{n=1}^{\infty} 2^{n/r'} c_n \chi_{I_{n-1}}(x)\Big|^{r'} dx \le \int\limits_0^1 \Big(\sum_{n=1}^{\infty} 2^{n/r'} |c_n| \sum_{k\ge n} \chi_{J_k}\Big)^{r'} dx \\
&= \int\limits_0^1 \sum_{k=1}^{\infty} \Big(\sum_{n\le k} 2^{n/r'} |c_n|\Big)^{r'} \chi_{J_k}(x)\,dx = \sum_{k=1}^{\infty} \Big(\sum_{n\le k} 2^{n/r'} |c_n|\Big)^{r'} \frac{1}{2^k} \\
&= \sum_{k=1}^{\infty} 2 \cdot \left(\frac{1}{2^{(k+1)/r'}} \sum_{n\le k} 2^{n/r'} |c_n|\right)^{r'}.
\end{aligned}$$

At this point it is appropriate to define a linear map V from $\ell^{p'}$ to the space of complex sequences by

$$V((c_n)) = \frac{1}{2^{(k+1)/r'}} \sum_{n\le k} 2^{n/r'} c_n$$

and observe that for (c_n) in ℓ^1,

$$\begin{aligned}
\|V((c_n))\|_1 &= \sum_{k=1}^{\infty} \Big|\frac{1}{2^{(k+1)/r'}} \sum_{n\le k} 2^{n/r'} c_n\Big| \le \sum_{k=1}^{\infty} \frac{1}{2^{(k+1)/r'}} \Big(\sum_{n\le k} 2^{n/r'} |c_n|\Big) \\
&= \sum_{n=1}^{\infty} |c_n| \sum_{k\ge n} 2^{(n-k-1)/r'} \le d \sum_{n} |c_n|
\end{aligned}$$

for some positive constant d. Similarly, for $(c_n) \in \ell^\infty$,

$$\|V((c_n))\|_\infty = \sup_k \Big|\frac{1}{2^{(k+1)/r'}} \sum_{n\le k} 2^{n/r'} c_n\Big| \le e\|c_n\|_\infty$$

where e is another constant. Thus V maps ℓ^1 into ℓ^1 and ℓ^∞ into ℓ^∞. Hence by the Riesz–Thorin Convexity Theorem [4], V maps $\ell^{p'}$ into $\ell^{p'} \subset \ell^{r'}$ and so the result follows in this case. It follows that

$$\sum_{n=1}^{\infty} 2^{n/r'} \|P_n(B(f \otimes g))\|_{p'} \le k\|f\|_r \|g\|_{p'}.$$

Now let $\psi = \sum_{i=1}^{\infty} f_i \otimes g_i$ where $\sum_{i=1}^{\infty} \|f_i\|_r \|g_i\|_{p'} < \infty$. For $\phi = B(\psi)$,

$$\sum_{n=1}^{\infty} 2^{n/r'} \|P_n(\phi)\|_{p'} \le \sum_{n=1}^{\infty} \sum_{i=1}^{\infty} 2^{n/r'} \|P_n B(f_i \otimes g_i)\|_{p'} \le \sum_{i=1}^{\infty} k\|f_i\|_r \|g_i\|_{p'} < \infty.$$

Therefore, B maps $L^r \hat{\otimes} L^{p'}$ into $B^{r,p}$.

We next characterise the predual $A_\circ^{p,p}$ of the multiplier space $M_\circ^{p,p}$. Let AC^0 be the space of absolutely continuous functions on $[0,1]$ which vanish at 1.

THEOREM 3. $A_\circ^{p,p} = B^{p,p} + AC^0$, $1 < p \le \infty$.

Proof. First we show that $B^{p,p}$ is contained in $A_\circ^{p,p}$. For $\phi \in B^{p,p}$, we can write $\phi = \sum_{n=1}^{\infty} e_n \circ P_n \phi$ and, by definition of $B^{p,p}$, $\sum_{n=1}^{\infty} \|e_n\|_p \|P_n(\phi)\|_{p'} < \infty$ so that $\phi \in A_\circ^{p,p}$.

Now for an arbitrary $\phi \in AC^0$, we can write $\phi(s) = \int_s^1 f(t)g(t)\,dt$, where $f \in L^p$ and $g \in L^{p'}$. Hence

$$(f \circ g)(s) = B(f \otimes g) + \int_s^1 f(t)g(t)\,dt = B(f \otimes g) + \phi(s).$$

Since $f \circ g$ and $B(f \otimes g)$ belong to $A_\circ^{p,p}$, so does ϕ. It is clear that any element of $A_\circ^{p,p}$ is a sum of functions of the form $\phi + \psi$, where $\phi \in B^{p,p}$ and $\psi \in AC^0$ so that we have the required result.

REMARK. In §5, it will be shown that $A_\circ^{r,p} = (0)$ if $r < p$.

THEOREM 4. $A_*^{r,p} = B^{r,p} + B^{p',r'}$, $1 \le r \le p \le \infty$.

Proof. It is enough to show that $B^{r,p}, B^{p',r'} \subset A_*^{r,p}$. If $\phi \in B^{r,p}$ we may write

$$\phi = \sum_{n=1}^{\infty} e_n * P_n(\phi)$$

and

$$\sum_{n=1}^{\infty} \|e_n\|_r \|P_n(\phi)\|_{p'} = \sum_{n=1}^{\infty} 2^{n/r'} \|P_n(\phi)\|_{p'} < \infty.$$

Thus $\phi \in A_*^{r,p}$. Similarly, if $\phi \in B^{p',r'}$, we may then write

$$\phi = \sum_{n=1}^{\infty} P_n(\phi) * e_n$$

and

$$\sum_{n=1}^{\infty} \|P_n(\phi)\|_r \|e_n\|_{p'} = \sum_{n=1}^{\infty} 2^{n/p} \|P_n(\phi)\|_r < \infty.$$

THEOREM 5. (a) $B^{p',r'} \subset B^{r,p}$ *if* $p \geq r \geq p'$ *and* (b) $B^{r,p} \subset B^{p',r'}$ *if* $r \leq \min(p, p')$.

Proof. (a) Let $p \geq r \geq p'$. Then for $\phi \in B^{p',r'}$, we have

$$\sum_{n=1}^{\infty} 2^{n/r'} \|P_n(\phi)\|_{p'} \leq \sum_{n=1}^{\infty} 2^{n/r'} \Big(\int_0^1 |P_n(\phi)|^{p'r/p'}\Big)^{\frac{1}{p'}\cdot\frac{p'}{r}} \Big(\int_{2^{-n}}^{2^{-(n-1)}} 1\,dt\Big)^{\frac{1}{p'}[1-\frac{p'}{r}]}$$
$$= \sum_{n=1}^{\infty} 2^{n(1-1/p')} \|P_n(\phi)\|_r = \sum_{n=1}^{\infty} 2^{n/p} \|P_n(\phi)\|_r < \infty.$$

Thus $\phi \in B^{r,p}$. Now interchanging the roles of r and p' we obtain the other inclusion.

COROLLARY 6. $A_*^{r,p} = B^{r,p}$ *if* $p \geq r \geq p'$ *and* $A_*^{r,p} = B^{p',r'}$ *if* $r \leq \min(p, p')$.

4. The convolution algebras $A_*^{r,p}$ $(r \leq p)$. In this section we define a multiplication on $A_*^{r,p}$ $(r \leq p)$ and show that it is a normed algebra. The algebra has an approximate identity. We also identify the maximal ideal space of the algebra under discussion.

The norm of an element in $A_*^{r,p}$ $(r \leq p)$ will be denoted by $\|\cdot\|_{*,r,p}$. Since the tensor product norm and the $B^{r,p}$ norms are equivalent (Corollary 6), we use them interchangeably.

THEOREM 7. *If* $\phi, \psi \in A_*^{r,p}$ $(1 < r \leq p < \infty)$ *then* $\phi * \psi \in A_*^{r,p}$ *where*

$$\phi * \psi(t) = \phi(t) \int_0^t \psi(s)\,ds + \psi(t) \int_0^t \phi(s)\,ds$$

for almost all $t \in [0, 1]$. *The algebra* $A_*^{r,p}$ *is commutative and semisimple. It has an approximate identity. The maximal ideal space of* $A_*^{r,p}$ *is the interval* $(0, 1]$ *with the usual topology and the Gelfand transform of* $\phi \in A_*^{r,p}$ *is given by* $\widehat{\phi}(t) = \int_0^t \phi(u)\,du$.

Proof. Consider $L_*^r \hat{\otimes} L_*^{p'}$. Since L^q $(1 \leq q \leq \infty)$ is a Banach algebra under convolution, $L_*^r \hat{\otimes} L_*^{p'}$ is naturally organised as a Banach algebra. Having so done

$$K = \operatorname{clspan}\{f * g \otimes h - g \otimes h * f : f \in C(I),\ g \in L_*^r,\ h \in L_*^{p'}\}$$

becomes a closed ideal. Hence $A_*^{r,p} = (L_*^r \otimes L_*^{p'})/K$ is a Banach algebra. Using the identification from Theorem 1, we see that the algebra multiplication can be realised as convolution of functions.

To prove that $A_*^{r,p}$ has an approximate identity it is enough to show that L_*^r $(1 < r < \infty)$ has an approximate identity. Note that for $e_n = 2^n \chi_{I_n}$, $n = 1, 2, \ldots$, and $f \in L_*^r$,

$$e_n * f(x) = \begin{cases} 2^n \int_0^x f(s)\,ds + 2^n x f(x), & 0 \le x \le 1/2^n, \\ f(x), & x > 1/2^n. \end{cases}$$

Now

$$\|e_n * f - f\|_r \le \Big(\int_0^{2^{-n}} \Big| 2^n \int_0^x f(s)\,ds \Big|^r dx \Big)^{1/r} + \Big(\int_0^{2^{-n}} 2^{nr} |f(x)|^r (2^{-n} - x)^r dx \Big)^{1/r}.$$

The second term on the right tends to zero as $n \to \infty$ whereas the first term is bounded by

$$\Big(\int_0^{2^{-n}} \Big| \frac{1}{x} \int_0^x f(s)\,ds \Big|^r dx \Big)^{1/r}.$$

By Hardy's inequality $(1/x)\int_0^x f(s)\,ds$ belongs to $L^r(I)$ since $f \in L^r(I)$. Hence

$$\Big(\int_0^{2^{-n}} \Big| \frac{1}{x} \int_0^x f(s)\,ds \Big|^r dx \Big)^{1/r} \to 0 \quad \text{as } n \to \infty.$$

It follows that $\|e_n * f - f\|_r \to 0$ as $n \to \infty$ for $f \in L^r(I)$. Indeed, there exists a constant $C = (2r-1)/(r-1)$, $r > 1$, such that $\|e_n * f\|_r \le C\|f\|_r$ by (5). Thus for $\phi = \sum_{i=1}^\infty f_i * g_i \in A_*^{r,p}$, we have

$$\|e_n * \phi - \phi\|_{*,r,p} \le \sum_{i=1}^M \|e_n * f_i - f_i\|_r \|g_i\|_{p'} + (C+1) \sum_{i=M+1}^\infty \|f_i\|_r \|g_i\|_{p'}.$$

We may choose M large so that the second term on the right is arbitrarily small independently of n and then choose n large so that the first term is arbitrarily small to see that $\|e_n * \phi - \phi\|_{*,r,p} \to 0$ as $n \to \infty$.

It remains to show that the maximal ideal space of $A_*^{r,p}$ is just $(0,1] = I_\circ$ and the Gelfand transform of $\phi \in A_*^{r,p}$ is given by $\widehat{\phi}(t) = \int_0^t \phi(u)\,du$. To see this note that the mapping $* : L^r \hat{\otimes} L^{p'}$ to $A_*^{r,p}$ given by $f \otimes g \mapsto f * g$ is a Banach algebra homomorphism and so the maximal ideal space of $A_*^{r,p}$ is embedded in the maximal ideal space of $L^r \hat{\otimes} L^{p'}$ which is $I_\circ \times I_\circ$. (See Gelbaum [8] and Baker, Pym and Vasudeva [1].) Now for $f, g \in L^r \cap L^{p'}$

and $\chi \in \Delta(A_*^{r,p})$, where $\Delta(A_*^{r,p})$ denotes the maximal ideal space of $A_*^{r,p}$,

$$\chi(f * g) = \widehat{f}(x)\widehat{g}(y) = \widehat{g}(x)\widehat{f}(y)$$

for some $x, y \in I_\circ$. Since there are enough elements in $L^r \cap L^{p'}$ to separate points of $I_\circ \times I_\circ$, it follows that $x = y$, so the embedding is just the diagonal. It is easy to verify that the Gelfand topology on $I_\circ$ is the usual topology inherited from the reals.

5. The spaces of multipliers. This section is devoted to the study of multipliers, namely, $M_*^{r,p} = \mathrm{Hom}_{C(I)}(L_*^r, L_\circ^p)$ and $M_\circ^{r,p} = \mathrm{Hom}_{C(I)}(L_*^r, L_*^p)$ where $r \le p$. $M_*^{r,p}$ can be regarded as the dual of $A_*^{r,p}$. We deal with the case $r \ge p'$. The other case is obtained by identifying $A_*^{r,p}$ and $A_*^{p',r'}$.

THEOREM 8. *For $p \ge r \ge p'$,*

$$M_*^{r,p} = \{t : t \text{ is measurable and } \sup_n 2^{-n/r'} \|P_n t\|_p\} < \infty.$$

This is effectively already proved. We merely need to observe that for $p \ge r \ge p'$, $A_*^{r,p} = B^{r,p}$, and note that the duality between $M_*^{r,p}$ and $A_*^{r,p}$ is given by

$$\langle t, \phi \rangle = \int_0^1 t(s)\phi(s)\, ds, \qquad t \in M_*^{r,p},\ \phi \in A_*^{r,p}.$$

To clarify how a function serves as a multiplier from L_*^r to $L_\circ^p$, write M_t, for $t \in M_*^{r,p}$, to denote the operator. Then

$$\langle t, f * g \rangle = \langle M_t(f), g \rangle, \qquad f \in L^r,\ g \in L^{p'},$$

where the pairing on the right is the usual L^p-$L^{p'}$ duality. It then follows that

$$M_t(f)(s) = t(s)\int_0^s f(u)\, du + \int_s^1 f(u)t(u)\, du, \qquad \text{a.e.},\ f \in L^r.$$

This completes the proof.

Now we look at $\mathrm{Hom}_{C(I)}(L_*^p, L_*^p)$, i.e. $(L^p \hat{\otimes}_{C(I)} L^{p'})^*$, where the action of $C(I)$ on $L^p(I)$ is by $*$ and on $L^{p'}(I)$ is by $\circ$. The map

$$\circ : L^p \hat{\otimes} L^{p'} \to L^{p'}$$

factors through $L^p \hat{\otimes}_{C(I)} L^{p'}$. (Note that $C(I)$ can be replaced by another dense subalgebra contained in $L^p \cap L^{p'}$.) Theorem 1 shows that the map

$$L_*^p \hat{\otimes}_{C(I)} L_\circ^{p'} \to L^p \circ L^{p'}$$

induced by $\circ$ is one-to-one.

Thus $L^p \circ L^{p'}$ can be identified with $L_*^p \hat{\otimes}_{C(I)} L_\circ^{p'}$, i.e. the predual of $\mathrm{Hom}_{C(I)}(L_*^p, L_*^p)$.

Lemma 9. *Let $\phi \in A_*^{p,p}$. Then $\|\phi\|_{\circ,p,p} \le C\|\phi\|_{*,p,p}$, $1 < p \le \infty$.*

Proof. First note that if $|||\phi||| = \sum_{n=1}^{\infty} 2^{n/p'}\|P_n(\phi)\|_{p'}$ then $|||\phi|||$ and $\|\phi\|_{*,p,p}$ are equivalent. Thus

$$\|\phi\|_{*,p,p} \ge C|||\phi||| = C\sum_{n=1}^{\infty} \|e_n\|_p \|P_n(\phi)\|_{p'}$$

and $\sum e_n \circ P_n\phi = \phi$ and so

$$\sum \|e_n\|_p\|P_n(\phi)\|_{p'} \ge \|\phi\|_{\circ,p,p}.$$

This proves the result.

Lemma 10. *If $\phi \in AC^\circ$, then $\|\phi\|_{\circ,p,p} \le C\|\phi\|_{AC^\circ}$.*

Proof. We show that the embedding $\theta : \phi \mapsto \phi$ from AC° to $A_\circ^{p,p}$ is continuous and use the closed graph theorem to do this. Suppose $f_n \to f$ in AC° and $\theta(f_n) \to g$ in $A_\circ^{p,p}$. Then $f_n \to f$ uniformly and $\theta(f_n) = f_n \to g$ in $L^{p'}$. Hence $f = g$.

Theorem 11. *For $1 < p \le \infty$, $M_\circ^{p,p} = \{\beta : \beta \in L^\infty$ and has an almost everywhere derivative h which satisfies $\sup_n 2^{-n/p'}\|P_n h\|_p < \infty\}$.*

Proof. Suppose $\mu \in (A_\circ^{p,p})^*$. Then $\mu|_{AC^\circ} \in (AC^\circ)^*$ and $\mu|_{B^{p,p}} \in (B^{p,p})^*$. Note that $(AC^\circ)^* = L^\infty(I)$, by the pairing

$$\langle \mu, \phi\rangle = \int_0^1 \beta(s)f(s)\,ds \tag{6}$$

where $\phi(s) = \int_s^1 f(t)\,dt$ is in AC° and μ corresponds to $\beta \in L^\infty(I)$. Since $\mu|_{B^{p,p}} \in (B^{p,p})^*$, it follows that $\beta \in (B^{p,p})^*$. For any $\varepsilon > 0$, $L^{p'}(I_\varepsilon) \subset B^{p,p}$ ($I_\varepsilon = [\varepsilon, 1]$) and so $\mu|_{L^{p'}(I_\varepsilon)}$ corresponds to an $L^p(I)$ function h_ε such that for $\phi \in L^{p'}(I_\varepsilon)$,

$$\langle \mu, \phi\rangle = \int_\varepsilon^1 h_\varepsilon(s)\phi(s)\,ds = \int_0^1 h_\varepsilon(s)\phi(s)\,ds$$

where ϕ is taken to be zero on $[0, \varepsilon)$. It is clear that h_ε's are compatible, i.e., $h_{\varepsilon'} = h_\varepsilon$ on $[\varepsilon, 1]$ if $\varepsilon' < \varepsilon$. Moreover, for $\phi(s) = \int_s^1 f(t)\,dt$ in $L^{p'}(I_\varepsilon)$, we have

$$\langle \mu, \phi\rangle = \int_0^1 h_\varepsilon(s)\int_s^1 f(t)\,dt\,ds = \int_0^1 f(t)\int_\varepsilon^t h_\varepsilon(s)\,ds\,dt.$$

Comparing this with (6), we get

$$\beta(t) = \int_\varepsilon^t h_\varepsilon(s)\,ds, \qquad t > \varepsilon.$$

Evidently, $\beta'(t) = h_\varepsilon(t)$ a.e. on $[\varepsilon, 1]$ and so we have proved that there exists h measurable on $(0,1]$ such that $h \in L^p(I_\varepsilon)$ for every $\varepsilon > 0$ and $\beta'(t) = h(t)$ a.e. on $(0,1]$. If we take $\phi \in B^{p,p}$, then

$$\langle \mu, \phi \rangle = \int_0^1 h(t)\phi(t)\,dt$$

exists and is finite. Hence $\sup_n 2^{-n/p'} \|P_n(h)\|_p < \infty$. This completes the proof.

We must calculate the effect of such a β as a multiplier from L^p_* to L^p_*. Let M_β denote the multiplier corresponding to β. Then for $f \in C(I)$ and $g \in L^{p'}(I)$,

$$\begin{aligned}\langle M_\beta(f), g\rangle &= \langle \beta, f \circ g\rangle \\ &= \Big\langle \beta, g(\cdot)\int_0^{\cdot} f(t)\,dt + \int_{\cdot}^1 g(t)f(t)\,dt\Big\rangle \\ &= \int_0^1 \Big\{ h(t)g(t)\int_0^t f(s)\,ds + g(t)f(t)\beta(t)\Big\}\,dt,\end{aligned}$$

and this equals

$$\int_0^1 g(t)\Big\{kf(t) - f(t)\int_t^1 h(s)\,ds + h(t)\int_0^t f(s)\,ds\Big\}\,dt.$$

Thus

$$M_\beta(f) = kf(t) - f(t)\int_t^1 h(s)\,ds + h(t)\int_0^t f(s)\,ds$$

for $f \in C(I)$ and consequently for $L^p(I)$. Here $k = \beta(1)$.

THEOREM 12. *If $r < p$ then $M_\circ^{r,p} = (0)$. Hence $A_\circ^{r,p} = (0)$.*

Proof. Let $\mu \in M_\circ^{r,p}$ and so it can be identified with an element of $\mathrm{Hom}_{C(I)}(L^r_*, L^p_*)$ and consequently with an element of $\mathrm{Hom}_{C(I)}(L^r_*, L^r_*)$. Let M_μ denote the multiplier corresponding to μ. In view of Theorem 11 and the paragraph following it, we have

$$M_\mu(f) = kf(t) - f(t)\int_t^1 h(s)\,ds + h(t)\int_0^t f(s)\,ds, \qquad f \in L^r_*,$$

where μ corresponds to β in $L^\infty(I)$, $\beta'(s) = h(s)$ a.e. on $(0,1]$ and h satisfies $\sup 2^{-n/r'}\|P_n h\|_r < \infty$.

If $k - \int_s^1 h(t)\,dt = 0$ for every $s \in (0,1]$, then $\int_{s_1}^{s_2} h(t)\,dt = 0$ for every $s_1, s_2 \in (0,1]$ and this implies $h \equiv 0$ a.e. Thus $kf \in L^p$ for every $f \in L^r$. This is impossible unless $k = 0$.

Suppose $\alpha = |k - \int_s^1 h(t)\,dt| > 0$ for some $s \in (0,1]$. Then there exists a neighbourhood $N = (s-\varepsilon, s+\varepsilon)$ of s such that $|k - \int_x^1 h(t)\,dt| > \frac{1}{2}\alpha$ for every $x \in N$. Therefore,

$$(M_\mu(f))(x) = \Big[k - \int_x^1 h(t)\,dt\Big] f(x) + h(x)\int_0^x f(t)\,dt, \qquad f \in L_*^r.$$

For $x \in N$ and $f \in L_*^r$,

$$|(M_\mu(f))(x)| > \frac{1}{2}\alpha|f(x)| - |h(x)|\Big|\int_0^x f(t)\,dt\Big|,$$

i.e.,

$$\frac{1}{2}\alpha|f(x)| < |(M_\mu(f))(x)| + |h(x)|\Big|\int_0^x f(t)\,dt\Big|. \tag{7}$$

Now choose $f = 0$ on $(s-\varepsilon, 1]$ such that $\int_0^1 f(t)\,dt = 1$. Then for $x \in N$,

$$M_\mu(f)(x) = h(x)\int_0^x f(t)\,dt = h(x) \in L^p(N).$$

Using (7), we obtain $f \in L^p(N)$ and this is a contradiction.

6. Comparison with earlier work. In this section we compare the results obtained in §5 with those obtained in [1]. That the characterisations of $M_\circ^{r,p}$ $(r \le p)$ in §5 and [1] are identical is the content of the following theorem.

THEOREM 13. *The following are equivalent for* $1 < p \le \infty$.

(i) $T \in \mathrm{Hom}_{C(I)}(L_*^p, L_*^p)$;

(ii) $Tf(x) = \Big(k - \int_x^1 h(y)\,dy\Big) f(x) + h(x)\int_0^x f(y)\,dy, \quad f \in L^p,$

where h is a measurable function on $(0,1]$ *such that*

(a) $x \mapsto \int_x^1 h(y)\,dy$ *is a bounded function, and*

(b) $(x \mapsto h(x)\int_0^x f(y)\,dy) \in L^p$ *for all* $f \in L^p$.

Proof. In view of Theorem 11, it is enough to show that

$$h(x)\int_0^x f(y)\,dy \in L^p \text{ for every } f \in L^p \;\Leftrightarrow\; \sup_n 2^{-n/p'}\|P_n h\|_p < \infty.$$

Suppose $\sup_n 2^{-n/p'}\|P_n h\|_p < \infty$. Let $g \in L^{p'}$. If $\widehat{f}(x)$ denotes $\int_0^x f(y)\,dy$,

$f \in L^p$, then

$$\begin{aligned}
|\langle h\widehat{f}, g\rangle| &= \Big|\int_0^1 (h\widehat{f})(x)g(x)\,dx\Big| = \Big|\sum_n \int_{J_n} (h\widehat{f})(x)g(x)\,dx\Big| \\
&\le \sum_n \int_{J_n} |h(x)|\cdot|(g\widehat{f})(x)|\,dx \\
&\le \sum_n \|P_n h\|_p \|P_n(g\widehat{f})\|_{p'} = \sum_n 2^{-n/p'}\|P_n h\|_p 2^{n/p'}\|P_n(g\widehat{f})\|_{p'} \\
&\le \sup_n 2^{-n/p'}\|P_n h\|_p \sum_n 2^{n/p'}\|P_n(g\widehat{f})\|_{p'} < \infty,
\end{aligned}$$

since $g\widehat{f} \in B^{p,p}$. Thus $h\widehat{f} \in L^p$ for all $f \in L^p$.

On the other hand suppose that $h\widehat{f} \in L^p$ for all $f \in L^p$. We shall show that

$$\sup_n 2^{-n/p'}\|P_n h\|_p = \sup_{\phi\in B^{p,p}} \frac{\|h\phi\|_1}{\|\phi\|_{*,p,p}}.$$

Indeed,

$$\begin{aligned}
\|h\phi\|_1 &\le \sum_n \|P_n h\|_p \|P_n(\phi)\|_{p'} = \sum_n 2^{-n/p'}\|P_n h\|_p 2^{n/p'}\|P_n\phi\|_{p'} \\
&\le \sup_n 2^{-n/p'}\|P_n h\|_p \sum_n 2^{n/p'}\|P_n\phi\|_p = \sup_n 2^{-n/p'}\|P_n h\|_p \|\phi\|_{*,p,p}.
\end{aligned}$$

On setting $\phi = \chi_{J_n}|h|^{p-1}$, we get

$$\frac{\|h\phi\|_1}{\|\phi\|_{*,p,p}} = \frac{\|P_n(h)\|_p^p}{2^{n/p'}\|P_n(h)\|_p^{p/p'}} = 2^{-n/p'}\|P_n h\|_p.$$

Using $\|\phi\|_{p'} \le \|\phi\|_{*,p,p}$ for $\phi \in B^{p,p}$, we get

$$\begin{aligned}
\|h\phi\|_1 &= \int_0^1 |h(x)\phi(x)|\,dx = \int_0^1 |(x+1)^{1/p'}h(x)(x+1)^{-1/p'}\phi(x)|\,dx \\
&\le \|(x+1)^{1/p'}h\|_p \|(x+1)^{-1/p'}\phi\|_{p'} \le \|(x+1)^{1/p'}h\|_p \|\phi\|_{*,p,p}.
\end{aligned}$$

Thus

$$\sup_n \frac{\|h\phi\|_1}{\|\phi\|_{*,p,p}} \le \|(x+1)^{1/p'}h\|_p < \infty,$$

since $f(x) = (x+1)^{-1/p} \in L^p$. This completes the proof.

Remark. The characterisation for $\mathrm{Hom}_{C(I)}(L_*^r, L_*^p)$ for $r < p$, obtained in §5, namely $\mathrm{Hom}_{C(I)}(L_*^r, L_*^p) = (0)$, is identical with the one obtained in [1].

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REFERENCES

[1] J. W. Baker, J. S. Pym, and H. L. Vasudeva, *Totally ordered measure spaces and their L^p algebras*, Mathematika 29 (1982), 42–54.
[2] —, —, —, *Multipliers for some measure algebras on compact semilattices*, *recent developments in the algebraic*, *analytical and topological theory of semigroups*, in: Lecture Notes in Math. 998, Springer, 1983, 8–30.
[3] R. K. Dhar and H. L. Vasudeva, *Characterisations of multipliers of $L^1(R)$*, Rev. Roumaine Math. Pures Appl. 30 (1985), 325–332.
[4] N. Dunford and J. T. Schwartz, *Linear Operators I*, Wiley, New York, 1958.
[5] P. Eymard, *L'algèbre de Fourier d'un groupe localement compact*, Bull. Soc. Math. France 92 (1964), 181–236.
[6] A. Figà-Talamanca, *Translation invariant operations on L^p*, Duke Math. J. 32 (1965), 452–502.
[7] A. Figà-Talamanca and G. I. Gaudry, *Density and representation theorems for multipliers of type (p, q)*, J. Austral. Math. Soc. (A) 7 (1967), 1–6.
[8] B. R. Gelbaum, *Tensor products of Banach algebras*, Canad. J. Math. 11 (1959), 297–310.
[9] A. Grothendieck, *Produits tensoriels topologiques et espaces nucléaires*, Mem. Amer. Math. Soc. 16 (1955).
[10] S. L. Gulick, T. S. Liu, and A. C. M. van Rooij, *Group algebra modules*, *I*, Canad. J. Math. 19 (1967), 133–150.
[11] C. Herz, *The theory of p-spaces with an application to convolution operators*, Trans. Amer. Math. Soc. 154 (1971), 69–82.
[12] L. Hörmander, *Estimates for translation invariant operators in L^p spaces*, Acta Math. 104 (1960), 93–140.
[13] B. E. Johnson, *An introduction to the theory of centralizers*, Proc. London Math. Soc. 14 (1964), 299–320.
[14] D. L. Johnson and C. D. Lahr, *Multipliers of L^1 algebras with order convolution*, Publ. Math. Debrecen 28 (1981), 153–161.
[15] R. Larsen, *The multipliers of $L^1([0, 1])$ with order convolution*, ibid. 23 (1976), 239–248.
[16] M. A. Rieffel, *Induced representations*, J. Funct. Anal. 1 (1967), 443–491.
[17] —, *Multipliers and tensor products of L^p-spaces of locally compact groups*, Studia Math. 33 (1969), 71–82.
[18] R. Schatten, *A Theory of Cross-Spaces*, Princeton Univ. Press, 1950.
[19] D. G. Todd, *Multipliers of certain convolution algebras over locally compact semigroups*, Math. Proc. Cambridge Philos. Soc. 87 (1980), 51–59.
[20] J. G. Wendel, *Left centralizers and isomorphisms of group algebras*, Pacific J. Math. 2 (1952), 251–261.

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