COLLOQUIUM MATHEMATICUM

VOL. 94 2002 NO. 1

A TRANSVECTION DECOMPOSITION IN $\mathrm{GL}(n,2)$

BY

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Abstract. An algorithm is given to decompose an automorphism of a finite vector space over $\mathbb{Z}_2$ into a product of transvections. The procedure uses partitions of the indexing set of a redundant base. With respect to tents, i.e. finite $\mathbb{Z}_2$-representations generated by a redundant base, this is a decomposition into base changes.

1. Introduction. Let V be a vector space of dimension $m-1$ over the field $\mathbb{Z}_2$ with 2 elements. There is a well-known procedure to decompose an automorphism α of V into a product of *transvections* (automorphisms fixing a hyperplane pointwise). Given a basis $(v_1,\ldots,v_{m-1})$ of V, α is represented as an invertible $(m-1)\times(m-1)$ matrix over $\mathbb{Z}_2$; the classical algorithm (summing columns or rows in order to get strategically placed zeros) uses transvections that fix a coordinate hyperplane, represented by matrices which coincide with the identity matrix but for one column.

We describe in Sections 3 and 4 a different algorithm, based on attaching to V a redundant base B, and representing automorphisms accordingly. This brings in partitions with their order structure, and they change the game significantly. The decomposition we give might be shorter than the classical one, since the $m-2$ generating vectors of the hyperplanes are not chosen among the $m-1$ base vectors, but among the m vectors of B.

In Sections 5 and 6 we show a property of this decomposition with respect to finite $\mathbb{Z}_2$-representations $\mathscr{R}=(V;V_1,\ldots,V_n)$ in which all the subspaces V_i are generated from a redundant base B of V; such a representation is called a *tent* on B. An automorphism α of V is called a *base change* for $\mathscr{R}$ if $\alpha(\mathscr{R})=(V;\alpha(V_1),\ldots,\alpha(V_n))$ is again a tent on B. In general, if $\alpha=\beta\gamma$ is a base change, β and γ need not be base changes. We prove that the algorithm we describe in Section 3 is, for any tent, a decomposition of a base change into base changes.

2. Redundant bases. A *redundant base* for V is a family B of m vectors $(m-1)$-wise linearly independent: $B=(v_1,\ldots,v_m)$ where (say)

2000 *Mathematics Subject Classification*: 15Axx, 08Axx, 05Bxx, 06Bxx, 20Hxx.

Key words and phrases: $(0,1)$-matrices, transvection factorization, $\mathrm{GL}(n,2)$, $\mathbb{Z}_2$-representations, tents.

$(v_1, \ldots, v_{m-1})$ is a base of V and $v_m = v_1 + \ldots + v_{m-1}$. We set $I = \{1, \ldots, m\}$, and write $I \setminus E = E^{-1}$ for subsets E of I. Then $\sum\{v_i \mid i \in E\} = \sum\{v_i \mid i \in E^{-1}\}$; what determines that sum is the *bipartition* of I: $b_E = \{E, E^{-1}\}$; we will use the same symbol b_E for the sum itself:

$$b_E = \sum\{v_i \mid i \in E\} = \sum\{v_i \mid i \in E^{-1}\} \quad (= b_{E^{-1}}).$$

E.g., $b_I = b_\emptyset = 0$, $b_{\{i\}} = v_i$. A vector $v \in V$ coincides with b_E for some $E \subseteq I$, and will be represented as a column vector with m entries: either as the usual $(m-1)$-vector with 0 at the mth place, or as the vector obtained from that by switching zeros and ones.

A new redundant base is an m-tuple of vectors $\mathscr{E} = (b_{E_1}, \ldots, b_{E_m})$ with zero sum and $(m-1)$-wise linearly independent elements; if we write the elements as column vectors over B, then $\mathscr{E}$ becomes the matrix of an automorphism α of V. An $m \times m$ matrix representing an automorphism of a vector space of dimension $m-1$ with respect to a redundant base is called *admissible*. The inverse $\mathscr{F} = (b_{F_1}, \ldots, b_{F_m})$ of the admissible matrix $\mathscr{E}$, which is the matrix of α^{-1}, is defined by $\sum\{b_{E_i} \mid i \in F_j\} = b_{\{j\}}$ for all $j \in I$; we will denote it by $\mathscr{E}^{-1}$.

If an automorphism α of V is represented over a linear base by a matrix M, we obtain the admissible matrix $\mathscr{E}$ representing α over the associated redundant base B by lining M with a row of zeros and with a corresponding column which is the sum of all columns of M; conversely, if we have $\mathscr{E}$, to get a traditional invertible matrix we first switch zeros and ones in columns until we have a row of zeros, then cancel it together with the corresponding column.

In particular, if α is a non-identical transvection fixing a (redundant base) coordinate hyperplane, its matrix $\mathscr{S}$ (which is admissible and self-inverse) has exactly two columns different from base vectors, and it is easy to verify that they look like this:

$$\begin{aligned}\mathscr{S} &= \mathscr{S}(r, s, X) \\ &= (b_{\{1\}}, \ldots, b_{\{r-1\}}, b_{\{r\} \cup X}, b_{\{r+1\}}, \ldots, b_{\{s-1\}}, b_{\{s\} \cup X}, b_{\{s+1\}}, \ldots, b_{\{m\}})\end{aligned}$$

for suitable $r, s \in I$ and $X \subseteq \{r, s\} \setminus I$. For example, if $X = \emptyset$, then $\mathscr{S}$ is the identity; if $X = \{r, s\} \setminus I$, then $\mathscr{S}$ is the transposition permuting v_r and v_s.

We recall that partitions on a set form a lattice, with the order "$\leq$" = "finer"; if $\{\mathscr{A}_\lambda \mid \lambda \in \Lambda\}$ is a family of partitions of a set I, then $\bigwedge_{\lambda \in \Lambda} \mathscr{A}_\lambda$ is the coarsest partition $\leq \mathscr{A}_\lambda$, while $\bigvee_{\lambda \in \Lambda} \mathscr{A}_\lambda$ is the finest partition $\geq \mathscr{A}_\lambda$, for all $\lambda \in \Lambda$. In particular, for $E, F \subseteq I$,

$$(*) \qquad b_E \wedge b_F = \{E \cap F, E^{-1} \cap F, E \cap F^{-1}, E^{-1} \cap F^{-1}\}$$

is in general a *quadripartition*, which reduces to a *tripartition* if one of E, E^{-1} contains one of F, F^{-1}; and to a bipartition iff $b_E = b_F$. For example,

$b_{\{i\}} \wedge b_F$ is a tripartition whenever $b_{\{i\}} \neq b_F$. Moreover

$$(**)\qquad \begin{aligned} b_E + b_F &= b_{(E\cap F)\cup(E^{-1}\cap F^{-1})} = b_{E\cap F} + b_{E^{-1}\cap F^{-1}},\\ b_E &= b_{(E\cap F)} + b_{(E\cap F^{-1})} = b_{E^{-1}\cap F} + b_{E^{-1}\cap F^{-1}},\\ b_F &= b_{(E\cap F)} + b_{(E^{-1}\cap F)} = b_{E\cap F^{-1}} + b_{E^{-1}\cap F^{-1}}. \end{aligned}$$

3. The algorithm. Here we outline the algorithm; proofs will follow in the next section. Let $\mathscr{E} = (b_{E_1}, \dots, b_{E_m})$ be an admissible matrix.

PROPOSITION A. *If $b_{E_r} \wedge b_{E_s}$ is a tripartition for all $s \in \{r\} \setminus I$, then $b_{E_r} = b_{\{k\}}$ for some $k \in I$.*

Start then with $r = 1$. If all $b_{E_1} \wedge b_{E_2}$, $b_{E_1} \wedge b_{E_3}, \dots, b_{E_1} \wedge b_{E_m}$ are tripartitions, then $b_{E_1} = b_{\{k_1\}}$; possibly performing a transposition, without loss of generality $k_1 = 1$, and we proceed with $r = 2, 3, \dots$ until we find an $s > r$ such that $b_{E_r} \wedge b_{E_s}$ is a quadripartition; we then have $\mathscr{E} = (b_{\{1\}}, b_{\{2\}}, \dots, b_{\{r-1\}}, b_{E_r}, \dots, b_{E_s}, \dots, b_{E_m})$.

PROPOSITION B. *If $b_{E_r} \wedge b_{E_s}$ is a quadripartition, then exactly one (call it $\mathscr{E}_1$) of the two matrices obtained from $\mathscr{E}$ by replacing b_{E_r}, b_{E_s} either with $b_{E_r\cap E_s}$, $b_{E_r^{-1}\cap E_s^{-1}}$, or with $b_{E_r^{-1}\cap E_s}$, $b_{E_r\cap E_s^{-1}}$, is admissible; in fact $\mathscr{S}_1 = \mathscr{E}_1\mathscr{E}^{-1}$ is a transvection.*

We thus have $\mathscr{E} = \mathscr{S}_1\mathscr{E}_1$, and proceed with $\mathscr{E}_1$. Finite induction is ensured by

PROPOSITION C. *The number of quadripartitions of $\mathscr{E}_1$ is strictly smaller than the one of $\mathscr{E}$.*

OBSERVATION. The construction of $\mathscr{E}_1$ and $\mathscr{S}_1$, as shown in the proof of B and in the example, does not require the computation of the inverse $\mathscr{E}^{-1} = (b_{F_1}, \dots, b_{F_m})$ of $\mathscr{E}$, which would mean solving $m-1$ equations $b_{\{i\}} = \sum\{b_{E_j} \mid j \in F_i\}$ in the unknowns $F_i \subseteq I$; it requires the solution of only one equation (per induction step).

4. The proofs

Proof of A. For each $s \neq r$ we have (possibly switching the names of E_s and E_s^{-1}) either $E_r \subseteq E_s$ or $E_r^{-1} \subseteq E_s$. If, say, each E_s contains E_r, then E_r must be a singleton, otherwise the matrix would have two equal rows (of zeros), hence would not be admissible. Say then by contradiction that some of the E_s contain E_r, some E_r^{-1}. Observe that the partition $\bigwedge\{b_{E_s} \mid E_r \subseteq E_s\}$ has E_r as a block, while $\bigwedge\{b_{E_s} \mid E_r^{-1} \subseteq E_s\}$ has E_r^{-1} in a block; therefore

$$\left(\bigwedge\{b_{E_s} \mid E_r \subseteq E_s\}\right) \vee \left(\bigwedge\{b_{E_s} \mid E_r^{-1} \subseteq E_s\}\right) \geq b_{E_r}.$$

On the other hand $\bigwedge\{b_{E_s} \mid E_r \subseteq E_s\} \leq \sum\{b_{E_s} \mid E_r \subseteq E_s\}$ (this is a bipartition!), and $\bigwedge\{b_{E_s} \mid E_r^{-1} \subseteq E_s\} \leq \sum\{b_{E_s} \mid E_r^{-1} \subseteq E_s\}$; but by the zero-sum condition on the b_{E_i} the two sums are equal; hence $b_{E_r} \leq \sum\{b_{E_s} \mid E_r \subseteq E_s\}$. Thus the sum is either the bipartition b_{E_r} or zero ($= b_I$). It cannot be zero, because the columns it adds would then be linearly dependent, and by the initial observation they are less than m in number. If it is b_{E_r}, then $\sum\{b_{E_s} \mid E_r \subset E_s\} = 0$, against admissibility. ∎

Proof of B. Without loss of generality let $\{r, s\} = \{1, 2\}$. Note that the bipartitions we choose as replacements are made from the blocks of $b_{E_1} \wedge b_{E_2}$. By the zero-sum condition, we may only pick pairs whose sum is equal to $b_{E_1} + b_{E_2}$; in the following we make repeated use of $(**)$.

Since $\mathscr{E}$ is a generating set there is a subset X of I such that

$$b_{E_1 \cap E_2} = \sum\{b_{E_i} \mid i \in X\};$$

by zero-sum, we may choose X not containing 1. Then

$$b_{E_1 \cap E_2^{-1}} = b_{E_1} + b_{E_1 \cap E_2} = \sum\{b_{E_i} \mid i \in X \cup \{1\}\}. \tag{1}$$

Note trivially that exactly one of b_X, $b_{X \cup \{1\}}$ separates 1 and 2; suppose it is $b_{X \cup \{1\}}$, that is, $X \cap \{1, 2\} = \emptyset$. We also have

$$\begin{aligned} b_{E_1^{-1} \cap E_2^{-1}} &= b_{E_1 \cap E_2} + b_{E_1} + b_{E_2} = \sum\{b_{E_i} \mid i \in X \cup \{1, 2\}\}, \\ b_{E_1^{-1} \cap E_2} &= b_{E_1 \cap E_2^{-1}} + b_{E_1} + b_{E_2} = \sum\{b_{E_i} \mid i \in X \cup \{2\}\}. \end{aligned}$$

The matrix

$$\begin{aligned} \mathscr{E}' &= (b_{E_1 \cap E_2}, b_{E_1^{-1} \cap E_2^{-1}}, b_{E_3}, \ldots, b_{E_m}) \\ &= (\sum\{b_{E_i} \mid i \in X\}, \sum\{b_{E_i} \mid i \in X \cup \{1, 2\}\}, b_{E_3}, \ldots, b_{E_m}) \end{aligned}$$

is not admissible, because the sum of its first vector with those indexed in X is zero (and $1 \notin X$). On the other hand

$$\begin{aligned} \mathscr{E}_1 &= (b_{E_1 \cap E_2^{-1}}, b_{E_1^{-1} \cap E_2}, b_{E_3}, \ldots, b_{E_m}) \\ &= (\sum\{b_{E_i} \mid i \in X \cup \{1\}\}, \sum\{b_{E_i} \mid i \in X \cup \{2\}\}, b_{E_3}, \ldots, b_{E_m}) \end{aligned}$$

is admissible, being the product of $\mathscr{E}$ by

$$\mathscr{S}_1 = \mathscr{S}(1, 2, X) = (b_{\{1\} \cup X}, b_{\{2\} \cup X}, b_{\{3\}}, \ldots, b_{\{m\}}).$$

If it had been b_X separating 1 and 2, the proof would hold for the matrix $\mathscr{E}_1'$ obtained by replacing X with $X' = X \setminus \{2\}$. ∎

Proof of C. Together with the trivial observation that for $\alpha, \beta, \varepsilon, \eta = \pm 1$ all of $(b_{E_r^\alpha \cap E_s^\beta}) \wedge (b_{E_r^\varepsilon \cap E_s^\eta})$ are at most tripartitions, while $b_{E_r} \wedge b_{E_s}$ is a quadripartition, the result rests on the following, easily (if cumbersomely) verifiable assertions for subsets C, D, E of I (here C stands for E_r, D stands for E_s, E for the variable E_i ($i \in \{r, s\} \setminus I$)):

a) If $b_E \wedge b_C$ is a tripartition, then one of $b_E \wedge b_{C^{-1}\cap D}$, $b_E \wedge b_{C\cap D^{-1}}$ is a tripartition. Hence if both $b_E \wedge b_{C^{-1}\cap D}$, $b_E \wedge b_{C\cap D^{-1}}$ are quadripartitions then both $b_E \wedge b_C$ and $b_E \wedge b_D$ are quadripartitions.

b) If both $b_E \wedge b_C$, $b_E \wedge b_D$ are tripartitions while $b_C \wedge b_D$ is a quadripartition, then both $b_E \wedge b_{C\cap D^{-1}}$ and $b_E \wedge b_{C\cap D^{-1}}$ are tripartitions. ∎

Example. Consider the 5×5 matrix over $\mathbb{Z}_2$:

$$M = \begin{pmatrix} 0 & 1 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 0 \end{pmatrix}.$$

Then

$$\mathscr{E} = (b_{E_1}, \ldots, b_{E_6}) = \begin{pmatrix} 0 & 1 & 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$= (b_{235}, b_{135}, b_{45}, b_{1234}, b_{124}, b_{12345}) = (b_{235}, b_{135}, b_{45}, b_{56}, b_{124}, b_6).$$

The first quadripartition we meet is

$$\begin{aligned} b_{E_1} \wedge b_{E_2} &= \{\{2,3,5\},\{1,4,6\}\} \wedge \{\{1,3,5\},\{2,4,6\}\} \\ &= \{\{3,5\},\{2\},\{1\},\{4,6\}\}. \end{aligned}$$

The bipartitions on its blocks are b_{35}, b_2, b_1, b_{46}. We must choose as substitutes of b_{E_1} and b_{E_2} either b_1, b_2 or b_{35}, b_{46} (both have sum equal to $b_{E_1} + b_{E_2} = b_{12}$). But clearly, if we put b_{46} in b_{E_1} or in b_{E_2}, then its sum with b_{45} and b_{56}, the terms in the third and fourth places, is 0, against admissibility; so the right choice is (up to permutation)

$$\mathscr{E}_1 = (b_{E'_1}, \ldots, b_{E'_6}) = (b_1, b_2, b_{45}, b_{56}, b_{124}, b_6).$$

The transvection $\mathscr{S}_1$ is obtained by expressing $b_{E'_1} = b_1$ as a sum of b_{E_i}'s: that is, $b_1 = \sum\{b_{E_i} \mid i \in X_1 \cup \{1\}\}$; since $b_1 = b_{235} + b_{45} + b_{56} = b_{E_1} + b_{E_3} + b_{E_4}$ we have $X_1 = \{3,4\}$, hence

$$\mathscr{S}_1 = \mathscr{S}(1,2,\{3,4\}) = (b_{134}, b_{234}, b_3, b_4, b_5, b_6), \qquad \mathscr{E} = \mathscr{S}_1\mathscr{E}_1.$$

In $\mathscr{E}_1$ the first quadripartition we meet is

$$b_{E'_3} \wedge b_{E'_4} = b_{45} \wedge b_{56} = \{\{1,2,3\},\{4\},\{5\},\{6\}\},$$

hence the candidate substitutes are b_{123}, b_5 or b_4, b_6. But if we put b_6 in the third (or fourth) place of $\mathscr{E}_1$, then its sum with the sixth term is 0, violating admissibility. Thus

$$\mathscr{E}_2 = (b_{E''_1}, \ldots, b_{E''_6}) = (b_1, b_2, b_{123}, b_5, b_{124}, b_6),$$

and since $b_{E_3''} = b_{123} = b_{45} + b_6 = b_{E_3'} + b_{E_6'} = \sum\{b_{E_i'} \mid i \in \{3,6\}\}$, we have $X_2 = \{6\}$, hence

$$\mathscr{S}_2 = \mathscr{S}(3,4,\{6\}) = (b_1, b_2, b_{36}, b_{46}, b_5, b_6), \qquad \mathscr{E}_1 = \mathscr{S}_2\mathscr{E}_2.$$

In $\mathscr{E}_2$ we put b_5 in its rightful fifth place by the transposition

$$\mathscr{S}_3 = (b_1, b_2, b_3, b_5, b_4, b_6) = \mathscr{S}(4,5,\{1,2,3,6\}),$$

so $\mathscr{E}_2 = \mathscr{S}_3\mathscr{E}_3$ (for $b_5 = b_{12346} = b_{4\cup\{1236\}}$).

But $\mathscr{E}_3 = (b_1, b_2, b_{123}, b_{124}, b_5, b_6)$ is the transvection $\mathscr{S}_4 = \mathscr{S}(3,4,\{1,2\})$. Thus $\mathscr{E} = \mathscr{S}_1\mathscr{S}_2\mathscr{S}_3\mathscr{S}_4$.

5. Tents. Let $V_1, \ldots, V_n$ be subspaces of V generated by subsets of the redundant base B; we will say they are generated *from* B. The representation $\mathscr{R} = (V; V_1, \ldots, V_n)$ (see [1]) is then called a *tent* on B. Tents are structures involved in the study of a class of torsionfree Abelian groups of finite rank called *Butler* $B^{(1)}$*-groups*. Tents and their transformations are investigated in depth in [3, 5].

If V' is generated from B, that is, $V' = \langle v_j \mid j \in A\rangle$ with $A \subseteq I$, we will write $V' = V(A)$, with $0 = V(\emptyset)$; for suitable subsets A_r of I we set $V_r = V(A_r)$ for each $r = 1, \ldots, n$. Observe that if $\geq m-1$ then $V(A_r) = V$; while if $< m-1$ then $(v_j \mid j \in A_r)$ is a (non-redundant) base of V_r.

An automorphism α of V is called a *base change* for $\mathscr{R}$ if $\alpha(\mathscr{R}) = (V; \alpha(V_1), \ldots, \alpha(V_n))$ is again a tent on B, or equivalently if $\mathscr{R}$ is a tent on $\alpha(B)$. If $\alpha = \beta\gamma$ is a base change, then β and γ need not be base changes: e.g., even if β is not a base change (see $\mathscr{R}''$ in the next example), the identity $\beta\beta^{-1}$ is. We will show that if α is a base change for $\mathscr{R}$, the transvection decomposition described in Section 3 is a base change decomposition of α.

We start by determining all transvections that are base changes for a tent $\mathscr{R}$. This is a first step into the main unsolved problem on tents, which is the determination of all base changes of a given tent. The converse problem, of determining all tents for which a given automorphism is a base change, has been solved in [4], and the solution will be used in the proof of Proposition 1.

LEMMA. *Let $A \subseteq I$, $V' = V(A)$. The automorphism α induced by the transvection $\mathscr{S} = \mathscr{S}(r,s,X)$ is a base change for $\mathscr{R}' = (V; V')$ if and only if one of the following occurs*:

(i) $\{r,s\} \cap A = \emptyset$;
(ii) $X \subseteq A$;
(iii) $\overline{X} = (\{r,s\} \cup X) \setminus I \subseteq A$.

Proof. Recall that α fixes all vectors of B (in particular, all v_i with $i \in X$) except for v_r, v_s which are transformed into $b_{\{r\}\cup X}$, $b_{\{s\}\cup X}$.

For sufficiency we need to show that in the given cases $\alpha(V')$ is generated from B. If (i) holds this is true, for $\alpha(V') = V'$. If $v_r \in V'$, hence $b_{\{r\}\cup X} \in$

$\alpha(V')$, in case (ii) $\alpha(V')$ contains $\alpha(v_i) = v_i$ for all $i \in X$, hence also v_r; case (iii) works similarly because $b_{\{r\}\cup X} = b_{\{s\}\cup \overline{X}}$. An analogous argument for $v_s \in V'$ lets us conclude that $\alpha(V')$ is generated from B.

For necessity, let $r \in A$; then $\alpha(v_r) = b_{\{r\}\cup X} \in \alpha(V')$, hence it must be a sum of elements of B contained in $\alpha(V')$. This can only occur if either $\{v_i \mid i \in \{r\} \cup X\} \subseteq \alpha(V')$, in which case $X \subseteq A$, or $\{v_i \mid i \in \{s\} \cup \overline{X}\} \subseteq \alpha(V')$, in which case $\overline{X} \subseteq A$. ∎

Set now, for each $A \subseteq I$, $A\hat{} = A \setminus \{r, s\}$. For $J \subseteq I\hat{}$, let p_J be the "pointed" partition consisting of all the singletons in J plus the block $I\hat{} \setminus J$. Then the previous lemma can be reworded as follows:

Lemma 1. *The transvection $\mathscr{S} = \mathscr{S}(r, s, X)$ is a base change for $\mathscr{R}' = (V; V')$ if and only if either $A = A\hat{}$ or for the partition $\{X, \overline{X}\}$ of $I\hat{}$ we have $\{X, \overline{X}\} \geq p_{A\hat{}}$.* ∎

Example. Let $m = 6$, $I = \{1, \ldots, 6\}$, $\mathscr{S} = \mathscr{S}(1, 5, \{2, 6\})$, so $X = \{2, 6\}$, $\overline{X} = \{3, 4\}$, $I\hat{} = \{2, 3, 4, 6\}$. Then

$$V' = \langle v_1, v_3, v_4\rangle = V(A), \qquad V'' = \langle v_1, v_2, v_3\rangle = V(C)$$

for $A = \{1, 3, 4\}$ and $C = \{1, 2, 3\}$, and

$$\begin{aligned} A\hat{} &= \{3, 4\}, \qquad p_{A\hat{}} = p_{\{3,4\}} = \{\{3\}, \{4\}, \{2, 6\}\},\\ C\hat{} &= \{2, 3\}, \qquad p_{C\hat{}} = p_{\{2,3\}} = \{\{2\}, \{3\}, \{4, 6\}\}.\end{aligned}$$

Thus if α is the automorphism of V induced by $\mathscr{S}$ we have

$$\alpha(V') = \langle b_{\{1,2,6\}}, v_3, v_4\rangle = \langle b_{\{3,4,5\}}, v_3, v_4\rangle = \langle v_5, v_3, v_4\rangle,$$

hence α is a base change for $\mathscr{R}' = (V; V')$; in fact,

$$\{X, \overline{X}\} = \{\{2, 6\}, \{3, 4\}\} \geq p_{\{3,4\}} = \{\{3\}, \{4\}, \{2, 6\}\};$$

while

$$\alpha(V'') = \langle b_{\{1,2,6\}}, v_2, v_3\rangle = \langle v_1 + v_6, v_2, v_3\rangle$$

is not generated from B, hence α is not a base change for $\mathscr{R}'' = (V; V'')$; in fact, $\{X, \overline{X}\} = \{\{2, 6\}, \{3, 4\}\} \not\geq p_{\{2,3\}} = \{\{2\}, \{3\}, \{4, 6\}\}$. ∎

Clearly, α is a base change for $\mathscr{R} = (V; V_1, \ldots, V_n)$ if and only if it is a base change for each $\mathscr{R}_i = (V; V_i)$. Introducing the partition

$$\mathscr{A} = \bigvee\{p_{A_i\hat{}} \mid A_i \cap \{r, s\} \neq \emptyset,\ i = 1, \ldots, n\}$$

we can apply Lemma 1 to conclude with the following

Theorem 1. *The automorphism α induced by the transvection $\mathscr{S} = \mathscr{S}(r, s, X)$ is a base change for the tent $\mathscr{R} = (V; V_1, \ldots, V_n)$ if and only if $\{X, \overline{X}\} \geq \mathscr{A}$.* ∎

This computation becomes very simple if we describe the tent by its incidence table, as is shown in the next example.

EXAMPLE. Let $m = 6$, $\mathscr{R} = (V; V_1, \dots, V_5)$, given in the obvious way by the following incidence table (e.g., $V_2 = \langle v_2, v_3, v_4, v_6 \rangle$):

	V_1	V_2	V_3	V_4	V_5
v_1	0	0	0	1	1
v_2	0	1	1	0	1
v_3	1	1	1	1	0
v_4	0	1	0	1	0
v_5	0	0	1	0	1
v_6	1	1	1	1	1

In order to compute all exchanges $\mathscr{S}(1, 2, X)$ that are base changes for $\mathscr{R}$, to comply with (i) eliminate the first column. Then, on the last four rows, string together zeros horizontally and vertically. Then pull: rows strung together constitute blocks of $\mathscr{A}$: here $\mathscr{A} = \{\{3, 4\}, \{5\}, \{6\}\}$. X can be any union of blocks of $\mathscr{A}$; for instance, $\mathscr{S}(1, 2, \{3, 4\})$ and $\mathscr{S}(1, 2, \{3, 4, 5\})$ are base changes for $\mathscr{R}$.

6. Transvection decomposition of base changes. Let $\mathscr{E} = (b_{E_1}, \dots, b_{E_m})$ be the matrix of an automorphism α of V performing a base change of the tent $\mathscr{R} = (V; V_1, \dots, V_n)$. Let $\mathscr{E} = \mathscr{S}_1 \mathscr{E}_1$, where $\mathscr{S}_1$ is a transvection computed as in Proposition B of the algorithm in Section 3. Then we have

PROPOSITION 1. *$\mathscr{S}_1$ is a base change for $\mathscr{R}$.*

Proof. Without loss of generality let $\mathscr{S}_1 = \mathscr{S}(1, 2, X)$. By Theorem 1, we need to prove that $\{X, \overline{X}\} \geq p_{A_{\hat{i}}}$ whenever $A_i \cap \{1, 2\} \neq \emptyset$.

Let $\mathscr{F} = (b_{F_1}, \dots, b_{F_m})$ be the matrix of α^{-1}, that is, $\sum\{b_{E_i} \mid i \in F_j\} = b_{\{j\}}$ for all $j \in I$. From (1) in the proof of Proposition B we have

$$b_{E_1 \cap E_2} = \sum\{b_{E_i} \mid i \in X\} = \alpha(b_X),$$

hence

$$b_X = \alpha^{-1}(b_{E_1 \cap E_2}) = \sum\{b_{F_i} \mid i \in E_1 \cap E_2\};$$

thus, since $X \cap \{1, 2\} = \emptyset$,

$$b_X \wedge b_{\{1\}} \wedge b_{\{2\}} = \{\{1\}, \{2\}, X, \overline{X}\}. \tag{2}$$

Setting $\bigwedge\{b_{F_i} \mid i \in E_j\} = \mathscr{A}_j$, $\bigwedge\{b_{F_i} \mid i \in E_j^{-1}\} = \mathscr{C}_j$, we have

$$\sum\{b_{F_i} \mid i \in E_1 \cap E_2\} \geq \bigwedge\{b_{F_i} \mid i \in E_1 \cap E_2\} \geq \bigwedge\{b_{F_i} \mid i \in E_1\} = \mathscr{A}_1,$$

and $\mathscr{A}_1 \geq \mathscr{A}_1 \wedge b_{\{1\}} \wedge b_{\{2\}}$; analogously

$$\sum\{b_{F_i} \mid i \in E_1 \cap E_2^{-1}\} \geq \bigwedge\{b_{F_i} \mid i \in E_1 \cap E_2^{-1}\} \geq \bigwedge\{b_{F_i} \mid i \in E_2^{-1}\} \\ = \mathscr{C}_2 \geq \mathscr{C}_2 \wedge b_{\{1\}} \wedge b_{\{2\}}.$$

Since

$$\sum\{b_{F_i} \mid i \in E_1 \cap E_2\} + \sum\{b_{F_i} \mid i \in E_1 \cap E_2^{-1}\} \\ = \sum\{b_{F_i} \mid i \in E_1\} = b_{\{1\}},$$

the two vectors $\sum\{b_{F_i} \mid i \in E_1 \cap E_2\}$ and $\sum\{b_{F_i} \mid i \in E_1 \cap E_2^{-1}\}$ differ only on the first coordinate, thus also

$$\sum\{b_{F_i} \mid i \in E_1 \cap E_2\} \geq \mathscr{C}_2 \wedge b_{\{1\}} \wedge b_{\{2\}},$$

hence

$$b_X = \sum\{b_{F_i} \mid i \in E_1 \cap E_2\} \geq (\mathscr{A}_1 \wedge b_{\{1\}} \wedge b_{\{2\}}) \vee (\mathscr{C}_2 \wedge b_{\{1\}} \wedge b_{\{2\}}).$$

If we repeat the process with $E_1^{-1} \cap E_2$, $E_1^{-1} \cap E_2^{-1}$, we get

$$\begin{aligned} b_X &\geq (\mathscr{A}_1 \wedge b_{\{1\}} \wedge b_{\{2\}}) \vee (\mathscr{C}_1 \wedge b_{\{1\}} \wedge b_{\{2\}}) \\ &\quad \vee (\mathscr{A}_2 \wedge b_{\{1\}} \wedge b_{\{2\}}) \vee (\mathscr{C}_2 \wedge b_{\{1\}} \wedge b_{\{2\}}) \\ &= \mathscr{L} = \{\{1\},\{2\},L_1,\dots,L_k\}. \end{aligned} \tag{3}$$

Let $\mathscr{L}^\wedge = \{L_1,\dots,L_k\}$, a partition of $I^\wedge = \{1,2\} \setminus I$; then from (2) and (3) we have $\{X,\overline{X}\} \geq \mathscr{L}^\wedge$. Thus for $A_i \cap \{1,2\} \neq \emptyset$ we will get $\{X,\overline{X}\} \geq p_{A_i^\wedge}$ if we show $\mathscr{L}^\wedge \geq p_{A_i^\wedge}$. This inequality follows from [4, Rule 3.3], which, applied to our case (our $\mathscr{A}_j$ is called $\mathscr{C}_{1j}$ there, our $\mathscr{C}_j$ is $\mathscr{C}_{2j}$ there) states that if $\mathscr{E}$ is a base change for $\mathscr{R} = (V; V(A_1),\dots,V(A_n))$ and if $j \in A_i$, then A_i^{-1} is contained in a block either of $\mathscr{A}_j$ or of $\mathscr{C}_j$. We apply it to $j = 1,2$. Since A_i^{-1} is the non-singleton block of $p_{A_i^\wedge}$, this means $p_{A_i^\wedge} \geq \mathscr{A}_j$ or $p_{A_i^\wedge} \geq \mathscr{C}_j$, hence $p_{A_i^\wedge} \geq \mathscr{L}^\wedge$, as desired. ∎

THEOREM 2. *The algorithm described in Section* 3 *yields a transvection decomposition of* α *into base changes.*

Proof. The composite of two base changes is a base change. Since the transvection $\mathscr{S}_1$ of Proposition 1 is self-inverse, $\mathscr{E}_1 = \mathscr{S}_1\mathscr{E}$ is a base change. Finite induction yields the required result. ∎

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Received 3 October 2001;
revised 26 February 2002 (4123)