ON FIXED POINTS OF HOLOMORPHIC TYPE

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Abstract. We study a linearization of a real-analytic plane map in the neighborhood of its fixed point of holomorphic type. We prove a generalization of the classical Koenig theorem. To do that, we use the well known results concerning the local dynamics of holomorphic mappings in $\mathbb{C}^2$.

1. Introduction. Let $p \in \mathbb{C}$ and $U$ be an open neighborhood of $p$. Assume that $f : U \to \mathbb{C}$ is an $\mathbb{R}$-analytic mapping such that $f(p) = p$ and $Jf(p) > 0$, where $Jf$ denotes the jacobian of $f$ at $p$. We have three possibilities:

1. $\left(\text{Re} \frac{\partial f}{\partial z}(p)\right)^2 < Jf(p)$.

In this case the characteristic polynomial $P_f(p)$ of $Df(p)$ has only complex roots $\lambda$ and $\bar{\lambda}$ with non-zero imaginary part. We shall say that $p$ is a fixed point of $f$ of holomorphic type.

2. $\left(\text{Re} \frac{\partial f}{\partial z}(p)\right)^2 = Jf(p)$.

In this case $P_f(p)$ has a double real root. We shall say that $p$ is a fixed point of double type.

3. $\left(\text{Re} \frac{\partial f}{\partial z}(p)\right)^2 > Jf(p)$.

In this case $P_f(p)$ has two distinct real roots. We shall say that $p$ is a fixed point of real type.

Remark 1. Fixed points of holomorphic type are always isolated. If a fixed point $p$ is not isolated, then at least one of the eigenvalues of $Df(p)$ must be equal to one.

In the present note we shall deal with case 1.

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Examples. (a) If $f$ is holomorphic near $p$, then $Df(p)(z) = f'(p) \cdot z$. Hence $p$ is a fixed point of holomorphic type if $\text{Im} f'(p) \neq 0$, and is of double type otherwise.

(b) Let $f(z) = \alpha(2z + \bar{z} + |z|^2 z)$. The fixed point $p = 0$ is: of holomorphic type if $|\text{Re} \alpha| < |\text{Im} \alpha|$, of double type if $|\text{Re} \alpha| = |\text{Im} \alpha|$, of real type if $|\text{Re} \alpha| > |\text{Im} \alpha|$.

(c) $f(x, y) = (y + \arctan x, -\delta x)$. The fixed point $p = 0$ is: of holomorphic type if $\delta > 1/4$, of double type if $\delta = 1/4$, of real type if $\delta < 1/4$.

(d) $f(z) = z(1/2 - i) + \bar{z}/2$. The point $p = 0$ is of holomorphic type since $\text{Re} \partial f/\partial z \equiv 1/2$, $Jf \equiv 1$. However we have $f^{(6)} = f \circ f \circ f \circ f \circ f \circ f \equiv \text{Id}$.

The aim of the present note is to prove the following.

Theorem 1. Let $p \in U = \text{int} \ U \subset \mathbb{C}$ and let $f : U \to \mathbb{C}$ be a real-analytic mapping such that $f(p) = p$, $Jf(p) > 0$ and $Jf(p) \neq 1$. Assume that $p$ is a fixed point of holomorphic type. Then there exist $\lambda \in \mathbb{C}$ ($|\lambda| \neq 1$, $\text{Im} \lambda \neq 0$), an open neighborhood $V$ of $p$, and a real-analytic orientation preserving diffeomorphic map $h$ from $V$ onto some neighborhood of zero such that $h \circ f \circ h^{-1}(z) = \lambda z$ on $h(V)$.

Remark 2. Theorem 1 is a generalization of the classical Koenig theorem for holomorphic mappings (see [1]).

The rest of the present paper will be devoted to the study of the cases $Jf = 1$ and $Df = 0$.

2. The proof of Theorem 1. We shall start with the following crucial

Proposition 1. Let $A(z) = az + b\bar{z}$ be an $\mathbb{R}$-linear mapping such that $(\text{Re} a)^2 < |a|^2 - |b|^2$.

Let $\lambda, \bar{\lambda}$ be the two roots of the characteristic polynomial of $A$. Then $\text{Im} \lambda \neq 0$ and there exists an $\mathbb{R}$-linear automorphism $H$ of $\mathbb{C}$ ($\mathbb{C} = \mathbb{R}^2$) for which $H^{-1}AH(z) = \lambda z$.

If $b = 0$ there is nothing to prove. We shall assume that $b \neq 0$. Put $c = b/(\lambda - a)$. We have

$$|c|^2 \neq 1$$

since $\lambda \neq \bar{\lambda}$, $ac + b = \lambda \cdot c$, $|b|^2 = (\lambda - a)(\lambda - \bar{\lambda}) = (\bar{\lambda} - \bar{a})(\lambda - a)$ and $a + b\bar{c} = a + |b|^2/(\lambda - \bar{a}) = \bar{\lambda}$.

Define an $\mathbb{R}$-linear mapping $H(z) = cz + \bar{z}$. We have

$$AH(z) = a(cz + \bar{z}) + b(\bar{cz} + z) = (ac + b)z + (a + b\bar{c})\bar{z} = \lambda cz + \bar{\lambda}z;$$

since

$$H^{-1}(w) = \frac{1}{1 - |c|^2}(\bar{w} - \bar{c}w),$$
it follows that
\[ H^{-1}AH(z) = \frac{1}{1-|c|^2} (\lambda z + \bar{\lambda}c z - \lambda |c|^2 z - \bar{c}\bar{\lambda}z) = \frac{1}{1-|c|^2} (\lambda z - \lambda |c|^2 z) = \lambda z. \]

Remark 3. We can take $\bar{\lambda}$ instead of $\lambda$ and find $H_1$ conjugating $A$ to $\bar{\lambda}z$ since $\bar{\lambda}\bar{z} = \bar{\lambda}z$; the map $H_1$ will have opposite orientation to $H$. Hence we can always find $\lambda$ for which the mapping $H$ is orientation preserving.

We now prove Theorem 1.

Assume that $p = 0$ and that $f(z) = \sum_{i,j=0}^{\infty} a_{ij} z^i \bar{z}^j$ on $U$. Define the holomorphic function $F : U \times U \to \mathbb{C}^2$ by
\[ F(z, w) = (f_1(z, w), f_2(z, w)) = \left( \sum_{i,j=0}^{\infty} a_{ij} z^i w^j, \sum_{i,j=0}^{\infty} \bar{a}_{ij} w^i \bar{z}^j \right). \]

Note that $f_1(z, \bar{z}) = f(z)$ and $f_2(z, \bar{z}) = \overline{f(z)}$.

Let $M = \{(z, \bar{z}) : z \in \mathbb{C}\}$. We have $F(M \cap (U \times U)) \subset M$. The derivative
\[ DF(0) = \begin{bmatrix} \frac{\partial f}{\partial z}(0) & \frac{\partial f}{\partial \bar{z}}(0) \\ \frac{\partial f}{\partial \bar{z}}(0) & \frac{\partial f}{\partial z}(0) \end{bmatrix} = \begin{bmatrix} a_{10} & a_{01} \\ \bar{a}_{01} & \bar{a}_{10} \end{bmatrix} \]

has eigenvalues $\lambda, \bar{\lambda}$, which are the roots of the characteristic polynomial of $DF(0)$.

Since $Jf(0) \neq 1$ we have either $|\lambda| < 1$ or $|\lambda| > 1$. It suffices to consider the case $Jf(0) < 1 \Rightarrow |\lambda| < 1$. If $Jf(0) > 1$ we can consider $f^{-1}$ and $F^{-1}$. In this case there exists a neighborhood $W_0 = V_0 \times V_0$ of zero such that $F(W_0) \subset W_0$ and $F^{(n)}(z, w) = F \circ n \text{times} \circ F(z, w) \to 0$ as $n \to \infty$.

We also have $|\lambda|^2 < |\bar{\lambda}| = |\lambda| < 1$. Rosay and Rudin proved in [5] (see also [2, 4.4]) that $G = \lim_{n \to \infty} (DF(0))^{-n} \circ F^{(n)}$ is a well defined holomorphic mapping from $W_0$ into $\mathbb{C}^2$, $DG(0) = Id$ and
\[ (*) \quad G \circ F = \tilde{A} \circ G, \quad \tilde{A} = DF(0). \]

In our case $G(W_0 \cap M) \subset M$ since $DF(0)(M) = M$ and $(DF(0))^{-1}(M) = M$. If $G = (g_1, g_2)$, then $g_1(z, \bar{z}) = g_2(z, \bar{z})$. We have $g_1(f(z), f(z)) = Df(0)(g_1(z, \bar{z}))$ by $(*)$.

Proposition 1 implies that $Df(0) = H\lambda H^{-1}$ where, by Remark 3, $\lambda$ is chosen such that $H$ is orientation preserving. Since $DG(0) = Id$, we have $(\partial / \partial z)g_1(z, \bar{z})(0) = 1$ and $(\partial / \partial \bar{z})g_1(z, \bar{z})(0) = 0$. Thus $Jg_1(z, \bar{z}) = 1$ and $g(z) = g_1(z, \bar{z})$ is invertible in a neighborhood $V$ of zero. Now it suffices to put $h = H^{-1} \circ g$. This ends the proof of Theorem 1.

3. The case of $Jf(p) = 1$. Let $f$ be a real-analytic map on the neighborhood of $p$ and let $p$ be a fixed point of holomorphic type. We can assume
that $p = 0$. If $Jf(0) = 1$, then $|\lambda| = |\overline{\lambda}| = 1$. Since the fixed point 0 is of holomorphic type, we have $\text{Im} \lambda \neq 0$. Hence Proposition 1 is valid and we can assume that $f$ has the form $f(z) = \lambda z + \sum_{i+j=2}^{\infty} a_{ij} z^i \overline{z}^j$.

Thus

$$F(z, w) = (\lambda z + \sum_{i+j=2}^{\infty} a_{ij} z^i w^j, \overline{\lambda} w + \sum_{i+j=2}^{\infty} \overline{a}_{ij} w^i z^j)$$

and $DF(0, 0)(z, w) = (\lambda z, \overline{\lambda} w)$.

Since $DF(0)$ is unitary, we can apply Theorem 4.24 of [2, Ch. 4, 4.4] to obtain

THEOREM 2. Let $f$ be as above. The following conditions are equivalent:

(a) There exists a neighborhood of zero in $\mathbb{C}^2$ on which the sequence of iterates $F^{(n)} = F \circ \cdots \circ F$, $n = 1, 2, \ldots$, is uniformly bounded.

(b) There exists a neighborhood $U$ of zero and a real-analytic diffeomorphism $h : U \rightarrow \mathbb{C}$ with $h(0) = 0$ such that $h \circ f \circ h^{-1}(z) = \lambda z$ on $h(U)$.

Proof. (a)⇒(b). Condition (a) implies that the point 0 belongs to the Fatou set of $F$. Thus by Theorem 4.24 of [2], there exists a biholomorphic map $G$ defined on a neighborhood of zero in $\mathbb{C}^2$ such that $G \circ F \circ G^{-1} = DF(0)$. Hence we have the same situation as in the proof of Theorem 1.

(b)⇒(a). We have $h(z) = \sum_{k,j=0}^{\infty} c_{kj} z^k \overline{z}^j$ near zero. Define

$$H(z, w) = \left( \sum_{k,j=1}^{\infty} c_{kj} z^k w^j, \sum_{k,j=1}^{\infty} c_{kj} w^k \overline{z}^j \right).$$

Since $JH(0) \neq 0$, $H$ is biholomorphic on some neighborhood of zero in $\mathbb{C}^2$. We have $H \circ F \circ H^{-1} = DF(0)$ on the set $\{(z, \overline{z}) \in \text{nbhd of } 0\}$, by the very definition of $H$ and $F$. The above set is a set of uniqueness for holomorphic functions and thus $H \circ F \circ H^{-1} = DF(0)$ on some neighborhood of zero in $\mathbb{C}^2$. The family $\{(DF(0))^{(n)}(z, w) = (\lambda^n z, \overline{\lambda}^n \overline{w})\}$ of mappings is uniformly bounded. This implies that so is the family $\{F^{(n)}\}$.

REMARK 4. The assumption that $F^{(n)}$ are uniformly bounded on some neighborhood of zero in $\mathbb{C}^2$ cannot be replaced by the assumption that $f^{(n)}$ are uniformly bounded on some neighborhood of zero in $\mathbb{C}$.

Let us consider the following

EXAMPLE. Let $\varphi(z) = z/(1+z^2)$. The point zero belongs to the Julia set of $\varphi$ and therefore the family $\{\varphi^{(n)}\}$ cannot be uniformly bounded on any neighborhood of zero. However the family $\{\varphi^{(n)}|_{\mathbb{R}}\}$ is uniformly bounded on the whole real axis $\mathbb{R}$. Put $f(z) = -i \varphi(\text{Re } z) + \text{Im } z$. Since $\lambda = -i$, zero is a fixed point of $f$ of holomorphic type. The family $\{f^{(n)}\}$ is uniformly
bounded on the whole complex plane. We have
\[ F(z, w) = \left( -i\varphi\left( \frac{z + w}{2} \right) - i\frac{z - w}{2} \right) + i\varphi\left( \frac{z + w}{2} \right) - i\frac{z - w}{2} \].

The first component of the map \( F^{(2n)} \) is equal to
\[ (F^{(2n)})_1(z, w) = -i\varphi^{(n)}\left( \frac{z - w}{2i} \right) + \varphi^{(n)}\left( \frac{z + w}{2} \right) \]
and the first component of the map \( F^{(2n+1)} \) is equal to
\[ (F^{(2n+1)})_1(z, w) = -i\varphi^{(n+1)}\left( \frac{z + w}{2} \right) + \varphi^{(n)}\left( \frac{z - w}{2i} \right) \].

Thus the family \( \{F^{(k)}\} \) cannot be bounded on any neighborhood of zero in \( \mathbb{C}^2 \).

However zero is an attracting fixed point of \( f \). The basin of attraction of zero is equal to the whole plane.

4. The case of \( Df(p) \equiv 0 \). In the previous parts of this note we found an analogue of the classical Koenig theorem and a condition for the existence of a Siegel disc. The natural question arises: Is it possible to find an analogue of the Böttcher theorem? A slight modification of the Hubbard–Papadopol example [3] shows that the answer is negative.

**Example.** Let \( f(z) = z^2 + \bar{z}^3 \). We shall show that there is no real-analytic diffeomorphism \( h \) of a neighborhood of zero such that \( h \circ f \circ h^{-1}(z) = z^2 \).

We have \( F(z, w) = (z^2 + w^3, w^2 + z^3) \). Suppose that there exists a real-analytic diffeomorphism \( h \) conjugating \( f \) to \( z^2 \) on a neighborhood of zero in \( \mathbb{C} \). As before for \( h(z) = \sum_{k+j=1}^{\infty} a_{kj} z^k \bar{z}^j \) define
\[ H(z, w) = \left( \sum_{k+j=1}^{\infty} a_{kj} z^k w^j, \sum_{k+j=1}^{\infty} \bar{a}_{kj} w^k z^j \right) \].

Since \( h \circ f \circ h^{-1}(z) = z^2 \), we have \( H \circ F \circ H^{-1}(z, w) = (z^2, w^2) \). However as observed by Hubbard and Papadopol, this is impossible, because, by a theorem of Mumford [4], no local homeomorphism of \( \mathbb{C}^2 \) near the origin can map a smooth curve to a singular curve. Note that \( F \) maps the curve \( \{z = 0\} \) onto the singular curve \( \{z^2 = w^3\} \) and the curve \( \{w = 0\} \) to the curve \( \{z^3 = w^2\} \).

**REFERENCES**


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