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ON FIXED POINTS OF HOLOMORPHIC TYPE

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Abstract. We study a linearization of a real-analytic plane map in the neighborhood of its fixed point of holomorphic type. We prove a generalization of the classical Koenig theorem. To do that, we use the well known results concerning the local dynamics of holomorphic mappings in $\mathbb{C}^2$.

1. Introduction. Let $p \in \mathbb{C}$ and U be an open neighborhood of p. Assume that $f : U \to \mathbb{C}$ is an $\mathbb{R}$-analytic mapping such that $f(p) = p$ and $Jf(p) > 0$, where Jf denotes the jacobian of f at p. We have three possibilities:

1. $\left(\operatorname{Re} \dfrac{\partial f}{\partial z}(p)\right)^2 < Jf(p)$.

In this case the characteristic polynomial $P_f(p)$ of $Df(p)$ has only complex roots λ and $\overline{\lambda}$ with non-zero imaginary part. We shall say that p is a *fixed point* of f *of holomorphic type*.

2. $\left(\operatorname{Re} \dfrac{\partial f}{\partial z}(p)\right)^2 = Jf(p)$.

In this case $P_f(p)$ has a double real root. We shall say that p is a *fixed point of double type*.

3. $\left(\operatorname{Re} \dfrac{\partial f}{\partial z}(p)\right)^2 > Jf(p)$.

In this case $P_f(p)$ has two distinct real roots. We shall say that p is a *fixed point of real type*.

REMARK 1. Fixed points of holomorphic type are always isolated. If a fixed point p is not isolated, then at least one of the eigenvalues of $Df(p)$ must be equal to one.

In the present note we shall deal with case 1.

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Examples. (a) If f is holomorphic near p, then $Df(p)(z) = f'(p)\cdot z$. Hence p is a fixed point of holomorphic type if $\operatorname{Im} f'(p) \neq 0$, and is of double type otherwise.

(b) Let $f(z) = \alpha(2z+\overline{z}+|z|^2 z)$. The fixed point $p=0$ is: of holomorphic type if $|\operatorname{Re}\alpha| < |\operatorname{Im}\alpha|$, of double type if $|\operatorname{Re}\alpha| = |\operatorname{Im}\alpha|$, of real type if $|\operatorname{Re}\alpha| > |\operatorname{Im}\alpha|$.

(c) $f(x,y) = (y+\arctan x, -\delta x), \delta > 0$. The point $p=0$ is: of holomorphic type if $\delta > 1/4$, of double type if $\delta = 1/4$, of real type if $\delta < 1/4$.

(d) $f(z) = z(1/2-i)+\overline{z}/2$. The point $p=0$ is of holomorphic type since $\operatorname{Re}\partial f/\partial z \equiv 1/2$, $Jf \equiv 1$. However we have $f^{(6)} = f\circ f\circ f\circ f\circ f\circ f \equiv \mathrm{Id}$.

The aim of the present note is to prove the following.

Theorem 1. *Let $p \in U = \operatorname{int} U \subset \mathbb{C}$ and let $f : U \to \mathbb{C}$ be a real-analytic mapping such that $f(p) = p$, $Jf(p) > 0$ and $Jf(p) \neq 1$. Assume that p is a fixed point of holomorphic type. Then there exist $\lambda \in \mathbb{C}$ ($|\lambda| \neq 1$, $\operatorname{Im}\lambda \neq 0$), an open neighborhood V of p, and a real-analytic orientation preserving diffeomorphic map h from V onto some neighborhood of zero such that $h\circ f\circ h^{-1}(z) = \lambda z$ on $h(V)$.*

Remark 2. Theorem 1 is a generalization of the classical Koenig theorem for holomorphic mappings (see [1]).

The rest of the present paper will be devoted to the study of the cases $Jf = 1$ and $Df = 0$.

2. The proof of Theorem 1. We shall start with the following crucial

Proposition 1. *Let $A(z) = az + b\overline{z}$ be an $\mathbb{R}$-linear mapping such that*

$$(\operatorname{Re} a)^2 < |a|^2 - |b|^2.$$

Let $\lambda, \overline{\lambda}$ be the two roots of the characteristic polynomial of A. Then $\operatorname{Im}\lambda \neq 0$ and there exists an $\mathbb{R}$-linear automorphism H of $\mathbb{C}$ ($\mathbb{C} = \mathbb{R}^2$) for which $H^{-1}AH(z) = \lambda z$.

If $b = 0$ there is nothing to prove. We shall assume that $b \neq 0$. Put $c = b/(\lambda - a)$. We have

$$|c|^2 \neq 1$$

since $\lambda \neq \overline{\lambda}$, $ac + b = \lambda\cdot c$, $|b|^2 = (\lambda - a)(\lambda - \overline{a}) = (\overline{\lambda}-\overline{a})(\overline{\lambda} - a)$ and $a + b\overline{c} = a + |b|^2/(\overline{\lambda}-\overline{a}) = \overline{\lambda}$.

Define an $\mathbb{R}$-linear mapping $H(z) = cz + \overline{z}$. We have

$$AH(z) = a(cz+\overline{z}) + b(\overline{cz} + z) = (ac+b)z + (a+b\overline{c})\overline{z} = \lambda cz + \overline{\lambda}\overline{z};$$

since

$$H^{-1}(w) = \frac{1}{1-|c|^2}\,(\overline{w} - \overline{c}w),$$

it follows that

$$H^{-1}AH(z)=\frac{1}{1-|c|^2}\,(\lambda z+\overline{\lambda}\overline{cz}-\lambda|c|^2z-\overline{c}\overline{\lambda}\overline{z})=\frac{1}{1-|c|^2}\,(\lambda z-\lambda|c|^2z)=\lambda z.$$

REMARK 3. We can take $\overline{\lambda}$ instead of λ and find H_1 conjugating A to $\overline{\lambda}z$ since $\overline{\lambda\overline{z}}=\overline{\lambda}z$; the map H_1 will have opposite orientation to H. Hence we can always find λ for which the mapping H is orientation preserving.

We now prove Theorem 1.

Assume that $p=0$ and that $f(z)=\sum_{i,j=0}^{\infty}a_{ij}z^i\overline{z}^j$ on U. Define the holomorphic function $F:U\times U\to\mathbb{C}^2$ by

$$F(z,w)=(f_1(z,w),f_2(z,w))=\Big(\sum_{i,j=0}^{\infty}a_{ij}z^iw^j,\sum_{i,j=0}^{\infty}\overline{a}_{ij}w^iz^j\Big).$$

Note that $f_1(z,\overline{z})=f(z)$ and $f_2(z,\overline{z})=\overline{f(z)}$.

Let $M=\{(z,\overline{z}):z\in\mathbb{C}\}$. We have $F(M\cap(U\times U))\subset M$. The derivative

$$DF(0)=\begin{bmatrix}\dfrac{\partial f}{\partial z}(0) & \dfrac{\partial f}{\partial\overline{z}}(0)\\ \overline{\dfrac{\partial f}{\partial\overline{z}}}(0) & \overline{\dfrac{\partial f}{\partial z}}(0)\end{bmatrix}=\begin{bmatrix}a_{10} & a_{01}\\ \overline{a}_{01} & \overline{a}_{10}\end{bmatrix}$$

has eigenvalues $\lambda,\overline{\lambda}$, which are the roots of the characteristic polynomial of $Df(0)$.

Since $Jf(0)\neq 1$ we have either $|\lambda|<1$ or $|\lambda|>1$. It suffices to consider the case $Jf(0)<1\Rightarrow|\lambda|<1$. If $Jf(0)>1$ we can consider f^{-1} and F^{-1}. In this case there exists a neighborhood $W_0=V_0\times V_0$ of zero such that $F(W_0)\subset W_0$ and $F^{(n)}(z,w)=F\circ\overset{n\text{ times}}{\cdots}\circ F(z,w)\to 0$ as $n\to\infty$.

We also have $|\lambda|^2<|\overline{\lambda}|=|\lambda|<1$. Rosay and Rudin proved in [5] (see also [2, 4.4]) that $G=\lim_{n\to\infty}(DF(0))^{-n}\circ F^{(n)}$ is a well defined holomorphic mapping from W_0 into $\mathbb{C}^2$, $DG(0)=\mathrm{Id}$ and

$$(*)\qquad G\circ F=\widetilde{A}\circ G,\qquad \widetilde{A}=DF(0).$$

In our case $G(W_0\cap M)\subset M$ since $DF(0)(M)=M$ and $(DF(0))^{-1}(M)=M$. If $G=(g_1,g_2)$, then $\overline{g_1(z,\overline{z})}=g_2(z,\overline{z})$. We have $g_1(f(z),\overline{f(z)})=Df(0)(g_1(z,\overline{z}))$ by $(*)$.

Proposition 1 implies that $Df(0)=H\lambda H^{-1}$ where, by Remark 3, λ is chosen such that H is orientation preserving. Since $DG(0)=\mathrm{Id}$, we have $(\partial/\partial z)g_1(z,\overline{z})(0)=1$ and $(\partial/\partial\overline{z})g_1(z,\overline{z})(0)=0$. Thus $Jg_1(z,\overline{z})=1$ and $g(z)=g_1(z,\overline{z})$ is invertible in a neighborhood V of zero. Now it suffices to put $h=H^{-1}\circ g$. This ends the proof of Theorem 1.

3. The case of $Jf(p)=1$. Let f be a real-analytic map on the neighborhood of p and let p be a fixed point of holomorphic type. We can assume that $p=0$. If $Jf(0)=1$, then $|\lambda|=|\overline{\lambda}|=1$. Since the fixed point 0 is of holomorphic type, we have $\operatorname{Im}\lambda\neq 0$. Hence Proposition 1 is valid and we can assume that f has the form $f(z)=\lambda z+\sum_{i+j=2}^{\infty}a_{ij}z^i\overline{z}^j$.

Thus

$$F(z,w)=\Big(\lambda z+\sum_{i+j=2}^{\infty}a_{ij}z^iw^j,\overline{\lambda}w+\sum_{i+j=2}^{\infty}\overline{a}_{ij}w^iz^j\Big)$$

and $DF(0,0)(z,w)=(\lambda z,\overline{\lambda}w)$.

Since $DF(0)$ is unitary, we can apply Theorem 4.24 of [2, Ch. 4, 4.4] to obtain

THEOREM 2. *Let f be as above. The following conditions are equivalent*:

(a) *There exists a neighborhood of zero in $\mathbb{C}^2$ on which the sequence of iterates $F^{(n)}=F\circ\overset{n\ times}{\ldots}\circ F$, $n=1,2,\ldots,$ is uniformly bounded.*

(b) *There exists a neighborhood U of zero and a real-analytic diffeomorphism $h:U\to\mathbb{C}$ with $h(0)=0$ such that $h\circ f\circ h^{-1}(z)=\lambda z$ on $h(U)$.*

Proof. (a)⇒(b). Condition (a) implies that the point 0 belongs to the Fatou set of F. Thus by Theorem 4.24 of [2], there exists a biholomorphic map G defined on a neighborhood of zero in $\mathbb{C}^2$ such that $G\circ F\circ G^{-1}=DF(0)$. Hence we have the same situation as in the proof of Theorem 1.

(b)⇒(a). We have $h(z)=\sum_{k,j=0}^{\infty}c_{kj}z^k\overline{z}^j$ near zero. Define

$$H(z,w)=\Big(\sum_{k,j=1}^{\infty}c_{kj}z^kw^j,\sum_{k,j=1}^{\infty}c_{kj}w^kz^j\Big).$$

Since $JH(0)\neq 0$, H is biholomorphic on some neighborhood of zero in $\mathbb{C}^2$. We have $H\circ F\circ H^{-1}=DF(0)$ on the set $\{(z,\overline{z})\}_{z\in\mathrm{nbh\,of\,0}}$, by the very definition of H and F. The above set is a set of uniqueness for holomorphic functions and thus $H\circ F\circ H^{-1}=DF(0)$ on some neighborhood of zero in $\mathbb{C}^2$. The family $\{(DF(0))^{(n)}\}$ of mappings is uniformly bounded $((DF(0))^{(n)}(z,w)=(\lambda^nz,\overline{\lambda}^nw))$. This implies that so is the family $\{F^{(n)}\}$.

REMARK 4. The assumption that $F^{(n)}$ are uniformly bounded on some neighborhood of zero in $\mathbb{C}^2$ cannot be replaced by the assumption that $f^{(n)}$ are uniformly bounded on some neighborhood of zero in $\mathbb{C}$.

Let us consider the following

EXAMPLE. Let $\varphi(z)=z/(1+z^2)$. The point zero belongs to the Julia set of φ and therefore the family $\{\varphi^{(n)}\}$ cannot be uniformly bounded on any neighborhood of zero. However the family $\{\varphi^{(n)}|_{\mathbb{R}}\}$ is uniformly bounded on the whole real axis $\mathbb{R}$. Put $f(z)=-i\varphi(\operatorname{Re}z)+\operatorname{Im}z$. Since $\lambda=-i$, zero is a fixed point of f of holomorphic type. The family $\{f^{(n)}\}$ is uniformly bounded on the whole complex plane. We have

$$F(z,w)=\left(-i\varphi\left(\frac{z+w}{2}\right)-i\,\frac{z-w}{2},+i\varphi\left(\frac{z+w}{2}\right)-i\,\frac{z-w}{2}\right).$$

The first component of the map $F^{(2n)}$ is equal to

$$(F^{(2n)})_1(z,w)=-i\varphi^{(n)}\left(\frac{z-w}{2i}\right)+\varphi^{(n)}\left(\frac{z+w}{2}\right)$$

and the first component of the map $F^{(2n+1)}$ is equal to

$$(F^{(2n+1)})_1(z,w)=-i\varphi^{(n+1)}\left(\frac{z+w}{2}\right)+\varphi^{(n)}\left(\frac{z-w}{2i}\right).$$

Thus the family $\{F^{(k)}\}$ cannot be bounded on any neighborhood of zero in $\mathbb{C}^2$.

However zero is an attracting fixed point of f. The basin of attraction of zero is equal to the whole plane.

4. The case of $Df(p)\equiv 0$. In the previous parts of this note we found an analogue of the classical Koenig theorem and a condition for the existence of a Siegel disc. The natural question arises: Is it possible to find an analogue of the Böttcher theorem? A slight modification of the Hubbard–Papadopol example [3] shows that the answer is negative.

EXAMPLE. Let $f(z)=z^2+\overline{z}^3$. We shall show that there is no real-analytic diffeomorphism h of a neighborhood of zero such that $h\circ f\circ h^{-1}(z)=z^2$.

We have $F(z,w)=(z^2+w^3,w^2+z^3)$. Suppose that there exists a real-analytic diffeomorphism h conjugating f to z^2 on a neighborhood of zero in $\mathbb{C}$. As before for $h(z)=\sum_{k+j=1}^{\infty}a_{kj}z^k\overline{z}^j$ define

$$H(z,w)=\Big(\sum_{k+j=1}^{\infty}a_{kj}z^kw^j,\ \sum_{k+j=1}^{\infty}\overline{a}_{kj}w^kz^j\Big).$$

Since $h\circ f\circ h^{-1}(z)=z^2$, we have $H\circ F\circ H^{-1}(z,w)=(z^2,w^2)$. However as observed by Hubbard and Papadopol, this is impossible, because, by a theorem of Mumford [4], no local homeomorphism of $\mathbb{C}^2$ near the origin can map a smooth curve to a singular curve. Note that F maps the curve $\{z=0\}$ onto the singular curve $\{z^2=w^3\}$ and the curve $\{w=0\}$ to the curve $\{z^3=w^2\}$.

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