EXAMPLES OF NONSEMISYMMETRIC RICCI-SEMISYMMETRIC HYPERSURFACES

BY

RYSZARD DESZCZ and MAŁGORZATA GŁOGOWSKA (Wrocław)

Abstract. We construct a class of nonsemisymmetric Ricci-semisymmetric warped products. Some manifolds of this class can be locally realized as hypersurfaces of a semi-Euclidean space \( \mathbb{E}_s^{n+1}, n \geq 5 \).

1. Quasi-Einstein manifolds. A semi-Riemannian manifold \((M, g), n = \text{dim} M \geq 3\), is called semisymmetric if on \( M \) we have

\[
R \cdot R = 0.
\]

For definitions of the symbols used, we refer to Section 2 of this paper. A review of results on semisymmetric semi-Riemannian manifolds is given in [10]. A semi-Riemannian manifold \((M, g), n \geq 3\), is said to be Ricci-semisymmetric if on \( M \) we have

\[
R \cdot S = 0.
\]

Manifolds of this class were investigated by several authors (see e.g. [3] and [21]). Every semisymmetric manifold is Ricci-semisymmetric. The converse is not true. The problem of the equivalence of (1) and (2), named the problem of P. J. Ryan (cf. [23]), was considered by several authors (see e.g. [1] and [7] and references therein). For instance, it is known that (1) and (2) are equivalent on hypersurfaces of 5-dimensional semi-Riemannian spaces of constant curvature. Ricci-semisymmetric hypersurfaces of Euclidean spaces were classified (locally) in [22]. A semi-Riemannian manifold \((M, g), n \geq 3\),


Key words and phrases: Ricci-semisymmetric manifold, quasi-Einstein manifold, warped product, hypersurface, Cartan hypersurface, P. J. Ryan problem.

Research supported by the Polish State Committee of Scientific Research (KBN) grant 2 P03A 006 17 for the first named author and by an Agricultural University of Wrocław (Poland) grant for the second named author.

The main results of this paper were presented by the first named author at the workshop Harmonic Maps and Curvature Properties of Submanifolds, 2 held in Leeds (England) in April 2000.
is said to be a *quasi-Einstein manifold* if at every point of $M$ we have

$$S = \alpha g + \beta w \otimes w, \quad w \in T^*_x M, \quad \alpha, \beta \in \mathbb{R}. \quad (3)$$

We refer to [15] for a review of results on quasi-Einstein manifolds. In particular, if $S = (\kappa/n) g$ on $M$ then $(M, g)$ is called an *Einstein manifold*.

Let $M$ be a hypersurface in a semi-Riemannian space $N^{n+1}_s(c)$ of constant curvature with signature $(s, n + 1 - s)$, $n \geq 4$, $c = \tau n/(n+1)$, where $\tau$ denotes the scalar curvature of the ambient space. Let $U_H$ be the subset of $M$ consisting of all points $x$ at which the transformation $A^2$ is not a linear combination of the shape operator $A$ and the identity transformation $\text{Id}$. If (3) is satisfied at $x \in M - U_H$ then the Weyl tensor $C$ of $M$ vanishes at $x$ or at this point the Ricci tensor $S$ of $M$ is proportional to the metric tensor $g$. Therefore we restrict our considerations to the set $U_H$. We have

**Theorem 1.1.** Let $M$ be a quasi-Einstein hypersurface of $E^{n+1}_s$, $n \geq 4$, and let (3) be satisfied on $U_H \subset M$.

(i) ([13, Theorem 5.1]) On $U_H$ the following three conditions are equivalent to each other:

$$\begin{align*}
(a) \quad & R \cdot S = 0, \\
(b) \quad & A^3 = \text{tr}(A)A^2 - \frac{\varepsilon \kappa}{n-1} A, \quad \varepsilon = \pm 1, \\
(c) \quad & A(W) = 0,
\end{align*} \quad (4)$$

where $w$ and $\alpha$ are defined by (3) and $W$ is related to $w$ by $g(W, X) = w(X)$, $X \in T_x M$.

(ii) ([9, Theorem 5.1]; [13, Corollary 5.2]) If at every $x \in U_H$ either (4)(a), (4)(b) or (4)(c) is satisfied then on $U_H$ we have

$$\begin{align*}
(a) \quad & \text{rank} \left( S - \frac{\kappa}{n-1} g \right) = 1, \\
(b) \quad & R \cdot C = Q(S, C), \\
(c) \quad & C \cdot S = 0.
\end{align*} \quad (5)$$

Semi-Riemannian manifolds, of dimension $n \geq 4$, satisfying at every point the condition: the tensors $R \cdot C$ and $Q(S, C)$ are linearly dependent, were investigated e.g. in [11] and [16]. This condition is equivalent to

$$R \cdot C = LQ(S, C) \quad (6)$$
on $U = \{ x \in M \mid Q(S, C) \neq 0 \text{ at } x \}$, where $L$ is some function on $U$. We denote by $U_L$ the set of all points of $U$ at which $L$ is nonzero. Evidently, (5)(b) is (6) with $L = \text{const} = 1$. Combining the main results of [14] with Theorem 1.1 we obtain

**Theorem 1.2** ([15, Theorem 1.3]). If $M$ is a hypersurface of $E^{n+1}_s$, $n \geq 5$, satisfying $R \cdot C = LQ(S, C)$ on $U \subset M$ then on $\overline{U} = U_H \cap U_L \subset M$
we have: (4), (5) and

\[ C \cdot R = \frac{n-3}{n-2} Q(S,R). \]

In Section 4 (see Theorem 4.1) we prove that if at a point \( x \in U_H \) of a Ricci-semisymmetric quasi-Einstein hypersurface \( M \) in \( \mathbb{E}_s^{n+1}, n \geq 4 \), the scalar curvature \( \kappa \) of \( M \) is nonzero and either (4)(a), (4)(b) or (4)(c) is satisfied then \( M \) is nonsemisymmetric. In our opinion, Theorems 1.1, 1.2 and 4.1 play an important role in the study of the problem of equivalence of (1) and (2) on quasi-Einstein hypersurfaces of semi-Euclidean spaces.

There is also a question of examples of hypersurfaces satisfying the assumptions of Theorems 1.1, 1.2 and 4.1. In Section 3 we present examples of nonsemisymmetric Ricci-semisymmetric warped products \( M \times_F N \) of a flat manifold \((M, g), p = \dim M \geq 1\) and an Einstein manifold \((N, \tilde{g}), n - p = \dim \tilde{N} \geq 4\), with some warping function \( F \). If \( p = 1 \) then \( M \times_F N \) is a quasi-Einstein manifold. The Ricci tensor of such a warped product satisfies (5)(a). The scalar curvature of these manifolds is nonzero. If \( p \geq 2 \) then \( M \times_F N \) is a nonquasi-Einstein manifold. In Section 4 we present an example of a nonsemisymmetric Ricci-semisymmetric hypersurface \( M \) of \( \mathbb{E}_s^{n+1}, n \geq 5 \), satisfying (4), (5) and (7) (see Example 4.2). We also present examples of nonquasi-Einstein and nonsemisymmetric Ricci-semisymmetric warped products which can be realized as hypersurfaces of \( \mathbb{E}_s^{n+1}, n \geq 5 \) (see Example 4.3). Ricci-pseudosymmetric warped products which can be locally realized as hypersurfaces of \( \mathbb{E}_s^{n+1}, n \geq 5 \), were investigated in [8].

2. Basic formulas. Let \((M, g), n \geq 3\), be a connected semi-Riemannian manifold of class \( C^\infty \) and let \( \nabla \) be its Levi-Civita connection. We define on \( M \) the endomorphisms \( X \wedge_A Y \), \( \mathcal{R}(X,Y) \) and \( \mathcal{C}(X,Y) \) by

\[
(X \wedge_A Y)Z = A(Y,Z)X - A(X,Z)Y,
\]

\[
\mathcal{R}(X,Y)Z = [\nabla_X, \nabla_Y]Z - \nabla_{[X,Y]}Z,
\]

\[
\mathcal{C}(X,Y) = \mathcal{R}(X,Y) - \frac{1}{n-2} \left( X \wedge g SY + SX \wedge g Y - \frac{\kappa}{n-1} X \wedge g Y \right),
\]

where the Ricci operator \( S \) is defined by \( g(X, SY) = S(X,Y) \), \( A \) is a symmetric \((0,2)\)-tensor, \( S \) the Ricci tensor, \( \kappa \) the scalar curvature and \( X,Y,Z \in \mathcal{X}(M) \), \( \mathcal{X}(M) \) being the Lie algebra of vector fields of \( M \). The Riemann–Christoffel curvature tensor \( R \) and the Weyl conformal curvature tensor \( C \) of \((M, g)\) are defined by \( R(X_1, X_2, X_3, X_4) = g(\mathcal{R}(X_1, X_2)X_3, X_4) \) and \( C(X_1, X_2, X_3, X_4) = g(\mathcal{C}(X_1, X_2)X_3, X_4) \), respectively. We refer to [9] (see also [14] or [16]) for the definitions of the tensors: \( R \cdot R, R \cdot C, R \cdot S, C \cdot R, C \cdot S, Q(g, R), Q(g, C), Q(g, S), Q(S, R) \) and \( Q(S, C) \). For symmetric \((0,2)\)-tensors
A and B we denote by \( A \wedge B \) their Kulkarni–Nomizu product. We have the identity (see e.g. [9])
\[
Q(S, g \wedge S) = -\frac{1}{2}Q(g, S \wedge S).
\]

A semi-Riemannian manifold \((M, g)\) is said to be pseudosymmetric ([6]) if at every point of \( M \) the tensors \( R \cdot R \) and \( Q(g, R) \) are linearly dependent. This is equivalent to \( R \cdot R = L_R Q(g, R) \) on \( \mathcal{U}_R = \{ x \in M \mid R - \frac{\kappa}{n-1}n G \neq 0 \} \), where \( L_R \) is some function on \( \mathcal{U}_R \). The \((0,4)\)-tensor \( G \) is defined by \( G = \frac{1}{2} g \wedge g \). Evidently, every semi-Riemannian semisymmetric manifold is pseudosymmetric. The converse is not true ([6]). It is easy to see that at every point of any pseudosymmetric manifold \((M, g)\) the tensors \( R \cdot S \) and \( Q(g, S) \) are linearly dependent. The converse is not true ([3]).

A semi-Riemannian manifold \((M, g)\) is called Ricci-pseudosymmetric if \( R \cdot S \) and \( Q(g, S) \) are linearly dependent at every point of \( M \). \((M, g)\) is Ricci-pseudosymmetric if and only if \( R \cdot S = L_S Q(g, S) \) on the set \( \mathcal{U}_S = \{ x \in M \mid S \neq (\kappa/n)g \} \), where \( L_S \) is some function on \( \mathcal{U}_S \). Examples of compact non-Einstein Ricci-pseudosymmetric manifolds which are nonpseudosymmetric were found in [17] and [18]. For instance, in [18, Theorem 1] it was shown that the Cartan hypersurfaces have that property. We recall that the Cartan hypersurface in the sphere \( S^{n+1}(c) \) is a compact, minimal hypersurface with constant principal curvatures \( -(3c)^{1/2}, 0, (3c)^{1/2} \) of the same multiplicity ([2]). On every Cartan hypersurface we have ([18, Proposition 1])
\[
\tilde{R} \cdot \tilde{S} = \frac{\tau}{n(n+1)} Q(\tilde{g}, \tilde{S}).
\]

For recent results on Ricci-pseudosymmetric hypersurfaces in \( N^{n+1}_{s}(c) \), \( n \geq 4 \), we refer to [8] and [19].

It is known that at every point of a hypersurface \( \tilde{N} \) of \( N^{n+1}_{s}(c) \), \( n \geq 4 \), the following condition is satisfied ([6, Section 5.5]): the tensors \( \tilde{R} \cdot \tilde{R} - Q(\tilde{S}, \tilde{R}) \) and \( Q(\tilde{g}, \tilde{C}) \) are linearly dependent. Precisely, on \( \tilde{N} \) we have
\[
\tilde{R} \cdot \tilde{R} - Q(\tilde{S}, \tilde{R}) = -\frac{(n-2)\tau}{n(n+1)} Q(\tilde{g}, \tilde{C}).
\]

In particular, if the ambient space is \( \mathbb{E}^{n+1}_{s} \) then (9) reduces to
\[
\tilde{R} \cdot \tilde{R} = Q(\tilde{S}, \tilde{R}).
\]

Every quasi-Einstein conformally flat manifold is a pseudosymmetric manifold satisfying (10) ([6, Section 6.3]).

3. Ricci-semisymmetric manifolds. In this section we present a family of nonsemisymmetric Ricci-semisymmetric quasi-Einstein warped products \( \tilde{M} \times_F \tilde{N}, \dim \tilde{M} = 1, \dim \tilde{N} = n - 1 \geq 3 \), satisfying at every point
$x \in \overline{M} \times \tilde{N}$ the following condition:

$$S = \frac{\kappa}{n-1} g + \beta w \otimes w, \quad w \in T^*_x(\overline{M} \times \tilde{N}), \quad \beta \in \mathbb{R}. \tag{11}$$

We also present a family of Ricci-semisymmetric nonquasi-Einstein warped products $\overline{M} \times F \tilde{N}$, $p = \dim \overline{M} \geq 2, n-p = \dim \tilde{N} \geq 3$. These constructions are related to the notion of a cone in the sense of [21].

**Proposition 3.1 (cf. [9, Proposition 3.1(ii)])**. If (11) holds at a point $x$ of a Ricci-semisymmetric semi-Riemannian manifold $(M, g), n \geq 4$, then at $x$ we have

$$R \cdot C - Q(S, C) = R \cdot R - Q(S, R). \tag{12}$$

As an immediate consequence of Proposition 3.1 and (10) we obtain

**Corollary 3.1 (cf. [9, Theorem 3.1])**. Let $M$ be a Ricci-semisymmetric hypersurface of $\mathbb{E}^{n+1}$, $n \geq 4$. If (11) holds at a point $x$ of $M$ then at $x$ we have $\tilde{R} \cdot \tilde{C} = Q(\tilde{S}, \tilde{C})$.

Let now $(\overline{M}, \overline{g})$ and $(\tilde{N}, \tilde{g})$, $p = \dim \overline{M}, n-p = \dim \tilde{N}, 1 \leq p < n$, be semi-Riemannian manifolds covered by systems of charts $\{U; x^a\}$ and $\{\tilde{V}; y^\alpha\}$, respectively. Let $F$ be a positive smooth function on $\overline{M}$. The warped product $\overline{M} \times F \tilde{N}$ is the product manifold $\overline{M} \times \tilde{N}$ with the metric $g = \overline{g} \times F \tilde{g}$, defined by $\overline{g} \times F \tilde{g} = \pi_1^* \overline{g} + (F \circ \pi_1) \pi_2^* \tilde{g}$, where $\pi_1 : \overline{M} \times \tilde{N} \rightarrow \overline{M}$ and $\pi_2 : \overline{M} \times \tilde{N} \rightarrow \tilde{N}$ are the natural projections. Let $\{\overline{U} \times \tilde{V}; x^1, \ldots, x^p, x^{p+1} = y^1, \ldots, y^{n-p}\}$ be a product chart for $\overline{M} \times \tilde{N}$. The local components of the metric $g = \overline{g} \times F \tilde{g}$ with respect to this chart are $g_{hk} = \overline{g}_{ab}$ if $h = a$ and $k = b$, $g_{hk} = \tilde{F} \tilde{g}_{\alpha\beta}$ if $h = \alpha$ and $k = \beta$, and $g_{hk} = 0$ otherwise, where $a, b, c, d \in \{1, \ldots, p\}$ and $\alpha, \beta \in \{p+1, \ldots, n\}$. We will mark by bars (resp., by tildes) tensors formed from $\overline{g}$ (resp., $\tilde{g}$).

It is known that the local components $R_{rstu}$ of the Riemann–Christoffel curvature tensor $R$ and the local components $S_{ls}$ of the Ricci tensor $S$ of $\overline{M} \times F \tilde{N}$ which may not vanish identically are the following (see e.g. [4], [5] or [16]):

$$R_{abcd} = \overline{R}_{abcd},$$

$$R_{\alpha\alpha\beta\beta} = -\frac{1}{2} T_{ab} \overline{g}_{\alpha\beta}, \tag{13}$$

$$R_{\alpha\beta\gamma\delta} = F \tilde{R}_{\alpha\beta\gamma\delta} - \frac{\Delta_1 F}{4} \tilde{G}_{\alpha\beta\gamma\delta},$$

$$S_{ab} = \overline{S}_{ab} - \frac{n-p}{2F} T_{ab}, \tag{14}$$

$$S_{\alpha\beta} = \tilde{S}_{\alpha\beta} - \frac{1}{2} \left( \text{tr} T + \frac{n-p-1}{2F} \Delta_1 F \right) \tilde{g}_{\alpha\beta}.\]
The \((0, 2)\)-tensor \(T_{ab}\), with local components \(T_{ab}^H\), is defined by
\[
T_{ab}^H = \nabla_b F^a - \frac{1}{2F} F_a F_b, \quad \text{tr} T = g^{ab} T_{ab}^H,
\]
\[
\Delta_1 F = \Delta_1 g F = g^{ab} F_a F_b, \quad F_a = \partial_a F = \frac{\partial F}{\partial x^a}.
\]
The scalar curvature \(\kappa\) of \(\tilde{M} \times_F \tilde{N}\) satisfies the relation
\[
\kappa = \kappa + \tilde{\kappa} \quad (\text{tr} T + \frac{n-p-1}{4F} \Delta_1 F).
\]

**Theorem 3.1.** Let \((\tilde{M}, \tilde{g})\), \(p = \dim \tilde{M}\), and \((\tilde{N}, \tilde{g})\), \(n-p = \dim \tilde{N}\), \(1 \leq p < n\), be semi-Riemannian manifolds and let \(F\) be a smooth positive function on \(\tilde{M}\).

(i) ([5, Theorem 1]) The condition \(R \cdot R = L R Q(g, R)\) is satisfied on \(U_R\) of \(\tilde{M} \times_F \tilde{N}\) if and only if on \(U_R\) we have
\[
(R \cdot \tilde{R})_{abcdefgh} = L R Q(\tilde{g}, R)_{abcdefgh},
\]
\[
H^f_{a} \tilde{R}_{fabc} = \frac{1}{2F} (T_{ac} H_{bd} - T_{ab} H_{cd}),
\]
\[
H_{ad} \left( \tilde{\Delta}_1 F \right) \tilde{G}_{\delta\alpha\beta\gamma} = -\frac{1}{2} T_{d} H^f_{a} \tilde{G}_{\delta\alpha\beta\gamma},
\]
\[
(R \cdot \tilde{R})_{\alpha\beta\gamma\delta\lambda\mu} = \left( FL_R + \frac{\Delta_1 F}{4F} \right) Q(\tilde{g}, \tilde{R})_{\alpha\beta\gamma\delta\lambda\mu},
\]
where
\[
H_{ad} = \frac{1}{2} T_{ad} + F L R_{ad}.
\]

(ii) ([5, Corollary 1]) Let \((\tilde{M}, \tilde{g})\), \(p \geq 2\), and \((\tilde{N}, \tilde{g})\), \(n-p \geq 2\), be semi-Riemannian spaces of constant curvature. The condition \(R \cdot R = L R Q(g, R)\) is satisfied on \(U_R\) of \(\tilde{M} \times_F \tilde{N}\) if and only if on \(U_R\) we have
\[
\frac{2\kappa}{p(p-1)} (\tilde{g}_{ab} H_{cd} - \tilde{g}_{ac} H_{bd}) = \frac{1}{F} (T_{ac} H_{bd} - T_{ab} H_{cd}),
\]
\[
H_{ad} \left( \frac{\kappa}{(n-p)(n-p-1)} - \frac{\Delta_1 F}{4F} \right) = -\frac{1}{2} T_{d} H^f_{a}.
\]

(iii) (cf. [4, Lemma 6]) The only local components of the Weyl tensor \(C\) of \(\tilde{M} \times_F \tilde{N}\), \(p = 1\) and \(n = 4\), which are not identically zero are
\[
C_{\alpha\beta11} = -\frac{1}{2} \left( \tilde{S}_{\alpha\beta} - \frac{\kappa}{3 \tilde{g}_{\alpha\beta}} \right),
\]
\[
C_{\alpha\beta\gamma\delta} = \frac{F}{2} \left( \tilde{g}_{\alpha\delta} \tilde{S}_{\beta\gamma} - \tilde{g}_{\alpha\beta} \tilde{S}_{\gamma\delta} + \tilde{g}_{\beta\gamma} \tilde{S}_{\alpha\delta} - \tilde{g}_{\beta\delta} \tilde{S}_{\alpha\gamma} \right) - \frac{F\kappa}{3} \tilde{G}_{\delta\alpha\beta\gamma}.
\]
(iv) ([3, Theorem 1]) The condition $R \cdot S = L_S Q(g, S)$ is satisfied on $U_S$ of $\tilde{M} \times_{\tilde{F}} \tilde{N}$ if and only if on $U_S$ we have

$$
(\tilde{R} \cdot \tilde{S})_{abcd} - L_S Q(\tilde{g}, \tilde{S})_{abcd} = \left(\frac{n-p}{F}\right)((\tilde{R} \cdot H)_{abcd} - L_S Q(\tilde{g}, H)_{abcd}),
$$

$$
H_{ab} \left( \tilde{S}_{\alpha\beta} - \frac{1}{2F} \left( \text{tr} T + \frac{(n-p-1)\Delta_1 F}{2F} \right) g_{\alpha\beta} \right)
$$

$$
= H_{cb} \left( \tilde{S}^c_a - \frac{n-p}{2F} T^c_a \right) g_{\alpha\beta},
$$

$$
(\tilde{R} \cdot \tilde{S})_{\alpha\beta\gamma\delta} = \left( F L_S + \frac{\Delta_1 F}{4F} \right) Q(\tilde{g}, \tilde{R})_{\alpha\beta\gamma\delta}.
$$

Example 3.1. Let $\eta$ and $\xi_1, \ldots, \xi_p$, $p \geq 1$, be constants such that $\xi_1^2 + \ldots + \xi_p^2 > 0$. Let $\tilde{M} = \{(x^1, \ldots, x^p)\}$ be a nonempty open connected subset of $\mathbb{R}^p$ such that the function $F$ defined by

$$
F(x^1, \ldots, x^p) = (\xi_a x^a + \eta)^2
$$

is positive on $\tilde{M}$, where $a, b, c, d \in \{1, \ldots, p\}$. Further, we set $\tilde{S}_{ab} = \varepsilon_1 = \pm 1$. Using now (15) and (29) we find

$$
T_{ab} = 0, \quad \frac{\Delta_1 F}{F} = \frac{1}{F} \sum_{b=1}^{p} \varepsilon_b F_b^2 = 4c_0, \quad c_0 = \sum_{b=1}^{p} \varepsilon_b \xi_b^2.
$$

Example 3.2. We consider the warped product $\tilde{M} \times_{\tilde{F}} \tilde{N}$ of the manifold $(\tilde{M}, \tilde{g})$ defined in Example 3.1 and an $(n-p)$-dimensional Einstein semi-Riemannian manifold $(\tilde{N}, \tilde{g})$, $n-p \geq 3$, with $F$ defined by (29). Now (13), (14) and (16) yield

$$
R_{abcd} = R_{\alpha\beta\gamma\delta} = 0, \quad R_{\alpha\beta\gamma\delta} = F(\tilde{R}_{\alpha\beta\gamma\delta} - c_0 \tilde{G}_{\alpha\beta\gamma\delta}),
$$

$$
S_{ab} = 0,
$$

$$
S_{\alpha\beta} = \tilde{\kappa} - (n-p-1)(n-p)c_0 \frac{1}{F} g_{\alpha\beta},
$$

$$
\kappa = \frac{\tilde{\kappa} - (n-p-1)(n-p)c_0}{F},
$$

respectively. Applying (34) in (33) we get

$$
S_{\alpha\beta} = \frac{\kappa}{n-p} g_{\alpha\beta}.
$$

(i) We assume that $p = 1$. The manifold $\tilde{M} \times_{\tilde{F}} \tilde{N}$ is quasi-Einstein. In fact, we have

$$
S_{ij} = \frac{\kappa}{n-1} g_{ij} + \gamma \varepsilon_1 w_i w_j,
$$

where $w_i = \delta^1_i$, $\gamma = -\kappa/(n-1)$ and $i, j \in \{1, \ldots, n\}$. From Theorem 3.1(iv) it follows that $\tilde{M} \times_{\tilde{F}} \tilde{N}$ is a Ricci-semisymmetric manifold. If $\text{dim} \tilde{N} = ...
n - 1 = 3 then, in view of Theorem 3.1(i), (iii), \( \tilde{M} \times_F \tilde{N} \) is a semisymmetric conformally flat manifold. Assume now that \( \dim \tilde{N} = n - 1 \geq 4 \). Using Theorem 3.1(i) we can deduce that if \((\tilde{N}, \tilde{g})\), \( \dim \tilde{N} \geq 4 \), is an Einstein semisymmetric manifold not of constant curvature then \( \tilde{M} \times_F \tilde{N} \) is a nonsemisymmetric Ricci-semisymmetric manifold. If the constant \( c_2 \) defined by \( c_2 = \tilde{\kappa} - (n - 1)(n - 2)c_0 \) is nonzero then the scalar curvature \( \kappa \) of \( \tilde{M} \times_F \tilde{N} \) is also nonzero.

(ii) We assume that \( p \geq 2 \). First of all, we note that

\[
S_{ij} = \frac{\kappa}{n - p} g_{ij} + \varrho \sum_{a=1}^{p} \varepsilon_a w_{ai} w_{aj},
\]

where \( w_{ai} = \delta_{ai}, \varrho = -\kappa/(n - p) \) and \( i, j \in \{1, \ldots, n\} \). From Theorem 3.1(iv) it follows that \( \tilde{M} \times_F \tilde{N} \) is a Ricci-semisymmetric manifold. Using (31) and (32) we find

\[
C_{abcd} = \frac{\kappa}{(n - 2)(n - 1)} G_{abcd}.
\]

Thus \( \tilde{M} \times_F \tilde{N} \) is a nonconformally flat manifold with nonconstant scalar curvature \( \kappa \), provided that the constant \( c_2 \) defined by

\[
c_2 = \tilde{\kappa} - (n - p - 1)(n - p)c_0
\]

is nonzero. If \( n - p = 3 \) then, in view of Theorem 3.1(i), \( \tilde{M} \times_F \tilde{N} \) is a semisymmetric manifold. Assume now that \( n - p \geq 4 \). Using again Theorem 3.1(i) we can deduce that if \((\tilde{N}, \tilde{g})\) is an Einstein semisymmetric manifold not of constant curvature then \( \tilde{M} \times_F \tilde{N} \) is a nonsemisymmetric Ricci-semisymmetric manifold. Since \( \tilde{M} \times_F \tilde{N} \) is a Ricci-semisymmetric manifold, we have \( R \cdot C = R \cdot R \). Thus the \((0, 6)\)-tensor \( T \) defined by

\[
T = R \cdot C - Q(S, C) - R \cdot R + Q(S, R)
\]

takes the form

\[
T = -Q(S, C) + Q(S, R) = -Q(S, C - R)
\]

\[
= \frac{1}{n - 2} Q \left( S, g \wedge S - \frac{\kappa}{n - 1} G \right)
\]

\[
= - \frac{1}{n - 2} Q \left( g, \frac{1}{2} S \wedge S \right) - \frac{\kappa}{(n - 2)(n - 1)} Q(S, G).
\]

Applying now (35) we get

\[
T_{a\beta\gamma\delta ab} = \frac{(p - 1)\kappa^2}{(n - p)^2(n - 2)(n - 1)} g_{ab} G_{a\beta\gamma\delta}.
\]

If the constant \( c_2 \) defined by (37) is nonzero then (12) is not satisfied on \( \tilde{M} \times_F \tilde{N} \).
Proposition 3.1 yields immediately

**Corollary 3.2.** The warped product defined in Example 3.2(i) satisfies (12).

**Example 3.3.** (i) Let $(\tilde{\mathcal{N}}, \tilde{g})$, \(\dim \tilde{\mathcal{N}} = n - p \geq 3, p \geq 1\), be a non-Einstein Ricci-pseudosymmetric semi-Riemannian manifold such that on \(U_{\tilde{\mathcal{N}}}\) we have \(\tilde{R} \cdot \tilde{S} = L_{\tilde{\mathcal{N}}}Q(\tilde{g}, \tilde{S})\) and \(L_{\tilde{\mathcal{N}}} = \text{const} \neq 0\). We consider the warped product \(\tilde{\mathcal{M}} \times F \tilde{\mathcal{N}}\) with \((\mathcal{M}, \tilde{g})\) and \(F\) defined in Example 3.1. Let \(c_0 = L_{\tilde{\mathcal{N}}}\). Now, in view of Theorem 3.1(iv), \(\tilde{\mathcal{M}} \times F \tilde{\mathcal{N}}\) is a Ricci-semisymmetric nonsemisymmetric manifold. If \((\tilde{\mathcal{N}}, \tilde{g})\) is nonpseudosymmetric then so is \(\tilde{\mathcal{M}} \times F \tilde{\mathcal{N}}\).

(ii) Let \((\tilde{\mathcal{N}}, \tilde{g})\), \(\dim \tilde{\mathcal{N}} = n - p = 3, 6, 12\) or 24, \(p \geq 1\), be a Riemannian manifold locally isometric to the Cartan hypersurface of dimension 3, 6, 12 or 24, respectively. We note that (8) holds on \((\tilde{\mathcal{N}}, \tilde{g})\). We consider the warped product \(\tilde{\mathcal{M}} \times F \tilde{\mathcal{N}}\) with \((\mathcal{M}, \tilde{g})\) and \(F\) as in (i). Let \(c_0 = \tau/(n(n + 1))\). Now, as in (i), \(\tilde{\mathcal{M}} \times F \tilde{\mathcal{N}}\) is a nonsemisymmetric Ricci-semisymmetric manifold provided that \(n - p = 6, 12\) or 24. If \(n - p = 3\) then \(\tilde{\mathcal{M}} \times F \tilde{\mathcal{N}}\) is semisymmetric. This is a consequence of Theorem 3.1(ii) and the fact that the 3-dimensional Cartan hypersurface is pseudosymmetric.

**Remark 3.1.** Using (13) and (14) and the definitions of the tensors \(R \cdot R\) and \(Q(S, R)\) we can easily check that the warped product defined in Example 3.2 satisfies \(R \cdot R = Q(S, R)\).

**Lemma 3.1.** Let \((M, g), n \geq 4\), be a semi-Riemannian manifold.

(i) If at \(x \in M\), (2), (11) and \(R \cdot R = Q(S, R)\) are satisfied then at \(x\) we have

\[R \cdot C = Q(S, C).\]  

(ii) If at \(x \in M\), (11), \(R \cdot R = Q(S, R), \kappa \neq 0\) and \(C \neq 0\) are satisfied then \(M\) is nonsemisymmetric.

**Proof.** (i) is a consequence of our assumptions and Proposition 3.1.

(ii) We suppose that \(R \cdot R = 0\) at \(x\). Thus also \(Q(S, R)\) vanishes at \(x\). Further, by our assumptions, \(\text{rank} S > 1\) at \(x\). Now the equation \(Q(S, R) = 0\), in view of Lemma 3.4 of [12], implies \(\lambda R = \frac{1}{2} S \wedge S, \lambda \in \mathbb{R} - \{0\}\), which yields

\[\lambda C = \frac{1}{2} S \wedge S - \frac{\lambda}{n - 2} g \wedge S + \frac{\lambda \kappa}{(n - 2)(n - 1)} G.\]  

This leads to

\[\lambda C = \frac{1}{2} \left( S - \frac{\lambda}{n - 2} g \right) \wedge \left( S - \frac{\lambda}{n - 2} g \right) + \frac{\lambda}{n - 2} \left( \frac{\kappa}{n - 1} - \frac{\lambda}{n - 2} \right) G,\]
whence, by an application of (11), we get
\[ \lambda C = \left( \frac{\kappa}{n-1} - \frac{\lambda}{n-2} \right) \left( \frac{\kappa}{n-1} G + \beta g \wedge w \otimes w \right). \]
By suitable contraction this gives
\[ \left( \frac{\kappa}{n-1} - \frac{\lambda}{n-2} \right) ((\kappa + \beta\|w\|^2)g + (n-1)\beta \wedge w \otimes w) = 0. \]
Since \((\kappa + \beta\|w\|^2)g + (n-1)\beta \wedge w \otimes w\) is a nonzero tensor, the above relation leads to \(\frac{\kappa}{n-1} - \frac{\lambda}{n-2} = 0\). Therefore (39) implies \(C = 0\), a contradiction.

**Remark 3.2** ([6, Section 9]). If at a point of a semi-Riemannian manifold \((M, g)\), \(n \geq 5\), the tensors \(C\) and \(R \cdot R\) are nonzero then the tensor \(R \cdot C\) is also nonzero at this point.

4. **Ricci-semisymmetric hypersurfaces.** Let \(\tilde{N}\), \(n = \text{dim} \tilde{N} \geq 3\), be a connected hypersurface isometrically immersed in a semi-Riemannian manifold \((N, g)\). We denote by \(\tilde{g}\) the metric tensor induced on \(\tilde{N}\) from \(g\). Further, we denote by \(\tilde{\nabla}\) and \(\nabla\) the Levi-Civita connections corresponding to \(\tilde{g}\) and \(g\), respectively. Let \(\xi\) be a local unit normal vector field on \(\tilde{N}\) in \(N\) and let \(\varepsilon = \tilde{g}(\xi, \xi) = \pm 1\). We can write the *Gauss formula* and the *Weingarten formula* of \(\tilde{N}\) in \(N\) in the following form: \(\nabla_X Y = \tilde{\nabla}_X Y + \varepsilon \tilde{H}(X, Y)\xi\) and \(\nabla_X \xi = -A X\), respectively, where \(X, Y\) are vector fields tangent to \(\tilde{N}\), \(\tilde{H}\) is the *second fundamental tensor* of \(\tilde{N}\) in \(N\), \(A\) is the *shape operator* of \(\tilde{N}\) in \(N\) and \(\tilde{H}^k(X, Y) = g(A^k X, Y)\), \(\text{tr} (\tilde{H}^k) = \text{tr}(A^k)\), \(k \geq 1\), \(\tilde{H}^1 = \tilde{H}\) and \(A^1 = A\). We denote by \(\bar{R}\) and \(R\) the Riemann–Christoffel curvature tensors of \(\tilde{N}\) and \(N\), respectively. We let \(U_{\tilde{H}}\) be the set of all \(x \in \tilde{N}\) at which \(A^2\) is not a linear combination of \(A\) and \(\text{Id}\). Note that \(U_{\tilde{H}} \subset U_{\tilde{G}}\). The *Gauss equation* of \(\tilde{N}\) in \(N\) has the form
\[ \bar{R}(X_1, \ldots, X_4) = R(X_1, \ldots, X_4) + \frac{\varepsilon}{2} (\tilde{H} \wedge \tilde{H})(X_1, \ldots, X_4), \]
where \(X_1, \ldots, X_4\) are vector fields tangent to \(\tilde{N}\). Let \(x^r = x^r(y^\alpha)\) be the local parametric expression of \(\tilde{N}\) in \((N, g)\), where \(y^\alpha\) and \(x^r\) are local coordinates of \(\tilde{N}\) and \(N\), respectively, and \(\alpha, \beta, \gamma, \delta \in \{1, \ldots, n\}\) and \(r \in \{1, \ldots, n+1\}\).

Let the ambient space \((N, g)\) be a semi-Riemannian space \(N^{n+1}_{\text{c}}(c)\) of constant curvature. Now (40) reads
\[ \bar{R}_{\alpha\beta\gamma\delta} = \varepsilon (\tilde{H}_{\alpha\delta} \tilde{H}_{\beta\gamma} - \tilde{H}_{\alpha\gamma} \tilde{H}_{\beta\delta}) + \frac{\tau}{n(n+1)} \tilde{G}_{\alpha\beta\gamma\delta}, \]
where \(\tilde{R}_{\alpha\beta\gamma\delta}\), \(\tilde{G}_{\alpha\beta\gamma\delta}\) and \(\tilde{H}_{\alpha\delta}\) are the local components of \(\bar{R}\), \(\tilde{G} = \frac{1}{2} \tilde{g} \wedge \tilde{g}\), and \(\tilde{H}\), respectively.

As a consequence of Theorem 1.1, Lemma 3.1 and Remark 3.2 we have the following
Proposition 4.1. Let $M$ be a hypersurface of $\mathbb{E}_s^{n+1}$, $n \geq 5$.

(i) If at $x \in M$ condition (11) is satisfied then (38) holds at $x$.

(ii) If at $x \in M$ the following conditions are satisfied: (11), $\kappa \neq 0$ and $C \neq 0$, then at $x$ we have $R \cdot R \neq 0$, $R \cdot C \neq 0$ and $Q(S,C) \neq 0$.

(iii) If at $x \in \overline{U} = U_H \cap U_L \subset M$ condition (3) is satisfied, $\kappa \neq 0$, and either (4)(a), (4)(b), or (4)(c) holds, then $M$ is nonsemisymmetric.

We now present examples of nonsemisymmetric Ricci-semisymmetric hypersurfaces in $\mathbb{E}_s^{n+1}$, $n \geq 5$.

Proposition 4.2. Let $(\mathcal{M}, \mathfrak{g})$, dim $\mathcal{M} = p \geq 2$, be as defined in Example 3.1 and let $(\tilde{\mathcal{N}}, \tilde{\mathfrak{g}})$, dim $\tilde{\mathcal{N}} = n - p \geq 1$, be a semi-Riemannian manifold isometric to a hypersurface of $\mathcal{N}_s^{n-p+1}(c)$. Let $\mathcal{M} \times_F \tilde{\mathcal{N}}$ be the warped product with $F$ and $c_0$ defined by (29) and (30), respectively, and

$$c_0 = \frac{\tau}{(n-p)(n-p+1)},$$

where $\tau$ is the scalar curvature of $\mathcal{N}_s^{n-p+1}(c)$. Then $\mathcal{M} \times_F \tilde{\mathcal{N}}$ can be realized locally as a hypersurface of $\mathbb{E}_s^{n+1}$.

Proof. (i) Let $(\mathcal{M}, \mathfrak{g})$ be a hypersurface of $\mathcal{N}_s^{n-p+1}(c)$. Thus (41) yields

$$\tilde{R} = \frac{\varepsilon}{2} \tilde{H} \wedge \tilde{H} + \frac{\tau}{(n-p)(n-p+1)} \tilde{G},$$

where $\tau$ is the scalar curvature of the ambient space and $\tilde{R}$ and $\tilde{H}$ are the curvature tensor and the second fundamental tensor of $\mathcal{M}$, respectively. By making use of (30), (42) and (43), we can write the formulas (5), (6) and (7) of [4] in the form

$$R_{abcd} = 0,$$

$$R_{\alpha a\beta} = -\frac{1}{2} T_{ab} \tilde{g}_{\alpha\beta} = 0,$$

$$R_{\alpha \beta \gamma \delta} = F \tilde{R}_{\alpha \beta \gamma \delta} - \frac{\Delta_1 F}{4} \tilde{G}_{\alpha \beta \gamma \delta}$$

$$= \varepsilon (\sqrt{F} \tilde{H}_{\alpha \delta} \sqrt{F} \tilde{H}_{\beta \gamma} - \sqrt{F} \tilde{H}_{\alpha \gamma} \sqrt{F} \tilde{H}_{\beta \delta})$$

$$+ F \left( \frac{\tau}{(n-p)(n-p+1)} - \frac{\Delta_1 F}{4F} \right) \tilde{G}_{\alpha \beta \gamma \delta}$$

$$= \varepsilon (H_{\alpha \delta} H_{\beta \gamma} - H_{\alpha \gamma} H_{\beta \delta}),$$

respectively, where $H_{\alpha \delta} = \sqrt{F} \tilde{H}_{\alpha \delta}$. Let $H$ be the symmetric $(0,2)$-tensor $H$ on $\mathcal{M} \times_F \tilde{\mathcal{N}}$ with local components $H_{ab} = 0$, $H_{a\delta} = 0$ and $H_{a\delta} = \sqrt{F} \tilde{H}_{a\delta}$. Using the fact that $\tilde{\nabla}_a \tilde{H}_{\beta \delta} = \tilde{\nabla}_\beta \tilde{H}_{a\delta}$, we can easily check that $H$ is a Codazzi tensor on $\mathcal{M} \times_F \tilde{\mathcal{N}}$. Thus the semi-Riemannian manifold $\mathcal{M} \times_F \tilde{\mathcal{N}}$ can be locally realized as a hypersurface of $\mathbb{E}_s^{n+1}$. Our proposition is thus proved.
Proposition 4.1 and Example 3.2 imply

**Corollary 4.1.** If \((\tilde{N}, \tilde{g})\), \(n - p \geq 4\), is a semisymmetric Einstein manifold not of constant curvature then the warped product \(\bar{M} \times_F \tilde{N}\) defined in Proposition 4.1 is a nonsemisymmetric Ricci-semisymmetric manifold which can be locally realized as a hypersurface of \(\mathbb{E}^{n+1}_s\).

Proposition 4.1 and Example 3.3 yield

**Corollary 4.2.** If \((\tilde{N}, \tilde{g})\), \(U_{\tilde{S}} = \tilde{N}\), \(n - p \geq 4\), is a non-Einstein Ricci-pseudosymmetric manifold such that on \(U_{\tilde{S}}\),

\[\tilde{R} \cdot \tilde{R} = \bar{L}_{\tilde{S}}Q(\tilde{g}, \tilde{S}) \quad \text{and} \quad \bar{L}_{\tilde{S}} = \frac{\tau}{(n - p)(n - p + 1)},\]

then the warped product \(\bar{M} \times_F \tilde{N}\) defined in Proposition 4.1 is a nonsemisymmetric Ricci-semisymmetric manifold which can be locally realized as a hypersurface of \(\mathbb{E}^{n+1}_s\).

**Remark 4.1.** The above result is also true when \(p = 1\). However, in that case we must additionally assume that the space \(N^{n-p+1}(c)\) is nonflat.

**Proposition 4.3.** Let \(\tilde{N}\) be a hypersurface of \(N^{n+1}(c)\), \(n \geq 4\). If at a point \(x\) the following conditions are satisfied:

\[\tilde{R} \cdot \tilde{R} = 0, \quad \bar{S} = \frac{\kappa}{n} \tilde{g} \quad \text{and} \quad \tilde{R} - \frac{\kappa}{(n - 1)n} \tilde{G} \neq 0,\]

then at \(x\) we have

\[\kappa = \frac{n - 2}{n + 1} \tau. \quad (44)\]

**Proof.** (9), by our assumptions, turns into

\[\left(\kappa - \frac{n - 2}{n + 1} \tau\right)Q(\tilde{g}, \tilde{R}) = 0.\]

Since \(Q(\tilde{g}, \tilde{R}) \neq 0\) if and only if \(\tilde{R} - \frac{\kappa}{(n - 1)n} \tilde{G} \neq 0\), the last equality implies (44), which completes the proof.

As an immediate consequence of the last proposition we have the following

**Corollary 4.3.** Let \(\tilde{N}\) be a semisymmetric Einstein hypersurface of \(N^{2k+1}(c)\), \(k \geq 2\). Then on \(U_{\tilde{R}} \subset \tilde{N}\) we have

\[\tilde{R}_{\alpha \beta \gamma \delta} = \varepsilon(\tilde{H}_{\alpha \delta} \tilde{H}_{\beta \gamma} - \tilde{H}_{\alpha \gamma} \tilde{H}_{\beta \delta}) + \frac{\kappa}{4k(k - 1)} \tilde{G}_{\alpha \beta \gamma \delta}. \quad (45)\]

**Example 4.1 ([20]).** We now present examples of semisymmetric Einstein hypersurfaces. Namely, in [20] Einstein hypersurfaces with \(\mathcal{A}^2 = \)
$-b^2 \text{Id}$, $b \neq 0$, were classified. They are complex spheres, either $\mathbb{C}S^k(1/b)$ or $\mathbb{C}S^k(i/b)$, where, for any $\gamma \in \mathbb{C}$,

$$\mathbb{C}S^k(\gamma) = \{(z_1, \ldots, z_{k+1}) \in \mathbb{C}^{k+1} : z_1^2 + \ldots + z_{k+1}^2 = \gamma^2\}, \quad k \geq 2.$$  

It is known that $\mathbb{C}S^k$ is an irreducible symmetric space (see [20] and references therein). $\mathbb{C}S^k(1/b)$ is a hypersurface in the indefinite sphere $S^{2k+1}_{k+1}(1/b)$ and $\mathbb{C}S^k(i/b)$ is a hypersurface in the indefinite hyperbolic space $H^{2k+1}_{k+1}(1/b)$, defined by

$$S^{2k+1}_{k+1}(1/b) = \left\{(x_1, y_1, \ldots, x_{k+1}, y_{k+1}) \in \mathbb{R}^{2k+2} : \sum_{j=1}^{k+1} x_j^2 - y_j^2 = 1/b^2 \right\},$$

$$H^{2k+1}_{k+1}(1/b) = \left\{(x_1, y_1, \ldots, x_{k+1}, y_{k+1}) \in \mathbb{R}^{2k+2} : \sum_{j=1}^{k+1} x_j^2 - y_j^2 = -1/b^2 \right\}.$$

$S^{2k+1}_{k+1}(1/b)$ has constant curvature $b^2$ and signature $(k+1, k)$, while $H^{2k+1}_{k+1}(1/b)$ has constant curvature $-b^2$ and signature $(k, k+1)$. By identifying $(x_1 + iy_1, \ldots, x_{k+1}+iy_{k+1}) \in \mathbb{C}^{k+1}$ with $(x_1, y_1, \ldots, x_{k+1}, y_{k+1}) \in \mathbb{R}^{2k+2}$, the complex spheres can be viewed as hypersurfaces defined by the single equation $\sum_j x_j y_j = 0$ in the sphere and hyperbolic space. It is clear that (45) holds on these hypersurfaces.

**Example 4.2.** Let $(\tilde{\mathcal{N}}, \tilde{g})$, dim $\tilde{\mathcal{N}} = n - p = 2k \geq 4$, $p \geq 1$, be a semi-Riemannian manifold isometric to an open nonempty part of the semisymmetric Einstein hypersurface of Example 4.1. We now consider the warped product $\tilde{M} \times_F \tilde{\mathcal{N}}$ where $(\tilde{M}, \tilde{g})$ and $F$ are defined in Example 3.1, and the constant $c_0$ satisfies (42). In view of Corollary 4.1, $\tilde{M} \times_F \tilde{\mathcal{N}}$ can be locally realized as a nonsymmetric Ricci-semisymmetric hypersurface of $\mathbb{E}^{n+1}_s$, $n \geq 5$. From Corollary 4.3 it follows that $c_0 = \tau/(2k(2k+1)) = \tilde{\kappa}/(4k(k-1))$, where $\tilde{\kappa}$ is the scalar curvature of $(\tilde{\mathcal{N}}, \tilde{g})$. Thus on $\tilde{M} \times_F \tilde{\mathcal{N}}$ we have $\kappa = -\tilde{\kappa}/((n-p-2)F)$.

**Example 4.3.** Using Corollary 4.2, in the same way as in Example 4.2, we can show that the warped product $\tilde{M} \times_F \tilde{\mathcal{N}}$, dim $\tilde{M} = p \geq 1$, dim $\tilde{\mathcal{N}} = n - p = 3, 6, 12$ or 24, defined in Example 3.3(ii) can be locally realized as a Ricci-semisymmetric hypersurface in $\mathbb{E}^{n+1}_s$, $n \geq 4$. If $n - p = 6, 12$ or 24 the hypersurface is nonsemisymmetric.

**Remark 4.2.** (i) An example of a semisymmetric quasi-Einstein hypersurface of $\mathbb{E}^{n+1}_s$, $n \geq 4$, was given in [9, Example 5.1].

(ii) An example of a nonsemisymmetric Ricci-semisymmetric quasi-Einstein hypersurface of the Euclidean space $\mathbb{E}^{n+1}$, $n \geq 5$, was found recently in [1].
REFERENCES


Department of Mathematics
Agricultural University of Wrocław
Grunwaldzka 53
50-357 Wrocław, Poland
E-mail: rysz@ozi.ar.wroc.pl
mglog@ozi.ar.wroc.pl

Received 29 June 2000;
revised 21 March 2002