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# EXAMPLES OF NONSEMISYMMETRIC RICCI-SEMISYMMETRIC HYPERSURFACES

BY

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*Dedicated to the memory of Professor Stanisław Gnot*

**Abstract.** We construct a class of nonsemisymmetric Ricci-semisymmetric warped products. Some manifolds of this class can be locally realized as hypersurfaces of a semi-Euclidean space $\mathbb{E}_s^{n+1}$, $n \geq 5$.

**1. Quasi-Einstein manifolds.** A semi-Riemannian manifold $(M, g)$, $n = \dim M \geq 3$, is called *semisymmetric* if on $M$ we have

$$R \cdot R = 0. \tag{1}$$

For definitions of the symbols used, we refer to Section 2 of this paper. A review of results on semisymmetric semi-Riemannian manifolds is given in [10]. A semi-Riemannian manifold $(M, g)$, $n \geq 3$, is said to be *Ricci-semisymmetric* if on $M$ we have

$$R \cdot S = 0. \tag{2}$$

Manifolds of this class were investigated by several authors (see e.g. [3] and [21]). Every semisymmetric manifold is Ricci-semisymmetric. The converse is not true. The problem of the equivalence of (1) and (2), named the *problem of P. J. Ryan* (cf. [23]), was considered by several authors (see e.g. [1] and [7] and references therein). For instance, it is known that (1) and (2) are equivalent on hypersurfaces of 5-dimensional semi-Riemannian spaces of constant curvature. Ricci-semisymmetric hypersurfaces of Euclidean spaces were classified (locally) in [22]. A semi-Riemannian manifold $(M, g)$, $n \geq 3$,

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is said to be a *quasi-Einstein manifold* if at every point of $M$ we have

$$S = \alpha g + \beta w \otimes w, \qquad w \in T_x^*M,\ \alpha, \beta \in \mathbb{R}. \tag{3}$$

We refer to [15] for a review of results on quasi-Einstein manifolds. In particular, if $S = (\kappa/n)g$ on $M$ then $(M, g)$ is called an *Einstein manifold.*

Let $M$ be a hypersurface in a semi-Riemannian space $N_s^{n+1}(c)$ of constant curvature with signature $(s, n+1-s)$, $n \geq 4$, $c = \frac{\tau}{n(n+1)}$, where $\tau$ denotes the scalar curvature of the ambient space. Let $U_H$ be the subset of $M$ consisting of all points $x$ at which the transformation $\mathcal{A}^2$ is not a linear combination of the shape operator $\mathcal{A}$ and the identity transformation Id. If (3) is satisfied at $x \in M - U_H$ then the Weyl tensor $C$ of $M$ vanishes at $x$ or at this point the Ricci tensor $S$ of $M$ is proportional to the metric tensor ([13, Lemma 4.1(iii)]). Therefore we restrict our considerations to the set $U_H$. We have

THEOREM 1.1. *Let $M$ be a quasi-Einstein hypersurface of $\mathbb{E}_s^{n+1}$, $n \geq 4$, and let* (3) *be satisfied on $U_H \subset M$.*

(i) ([13, Theorem 5.1]) *On $U_H$ the following three conditions are equivalent to each other*:

$$\begin{aligned} &\text{(a) } R \cdot S = 0, \qquad \text{(b) } \mathcal{A}^3 = \operatorname{tr}(\mathcal{A})\mathcal{A}^2 - \frac{\varepsilon\kappa}{n-1}\mathcal{A}, \quad \varepsilon = \pm 1, \\ &\text{(c) } \mathcal{A}(W) = 0, \end{aligned} \tag{4}$$

*where $w$ and $\alpha$ are defined by* (3) *and $W$ is related to $w$ by $g(W, X) = w(X)$, $X \in T_xM$.*

(ii) ([9, Theorem 5.1]; [13, Corollary 5.2]) *If at every $x \in U_H$ either* (4)(a), (4)(b) *or* (4)(c) *is satisfied then on $U_H$ we have*

$$\begin{aligned} &\text{(a) } \operatorname{rank}\left(S - \frac{\kappa}{n-1}g\right) = 1, \qquad \text{(b) } R \cdot C = Q(S, C), \\ &\text{(c) } C \cdot S = 0. \end{aligned} \tag{5}$$

Semi-Riemannian manifolds, of dimension $n \geq 4$, satisfying at every point the condition: the tensors $R \cdot C$ and $Q(S, C)$ are linearly dependent, were investigated e.g. in [11] and [16]. This condition is equivalent to

$$R \cdot C = LQ(S, C) \tag{6}$$

on $U = \{x \in M \mid Q(S, C) \neq 0 \text{ at } x\}$, where $L$ is some function on $U$. We denote by $U_L$ the set of all points of $U$ at which $L$ is nonzero. Evidently, (5)(b) is (6) with $L = \text{const} = 1$. Combining the main results of [14] with Theorem 1.1 we obtain

THEOREM 1.2 ([15, Theorem 1.3]). *If $M$ is a hypersurface of $\mathbb{E}_s^{n+1}$, $n \geq 5$, satisfying $R \cdot C = LQ(S, C)$ on $U \subset M$ then on $\overline{U} = U_H \cap U_L \subset M$*

*we have*: (4), (5) *and*

$$C \cdot R = \frac{n-3}{n-2}\, Q(S,R). \tag{7}$$

In Section 4 (see Theorem 4.1) we prove that if at a point $x \in U_H$ of a Ricci-semisymmetric quasi-Einstein hypersurface $M$ in $\mathbb{E}^{n+1}_s$, $n \geq 4$, the scalar curvature $\kappa$ of $M$ is nonzero and either (4)(a), (4)(b) or (4)(c) is satisfied then $M$ is nonsemisymmetric. In our opinion, Theorems 1.1, 1.2 and 4.1 play an important role in the study of the problem of equivalence of (1) and (2) on quasi-Einstein hypersurfaces of semi-Euclidean spaces.

There is also a question of examples of hypersurfaces satisfying the assumptions of Theorems 1.1, 1.2 and 4.1. In Section 3 we present examples of nonsemisymmetric Ricci-semisymmetric warped products $\overline{M} \times_F \widetilde{N}$ of a flat manifold $(\overline{M}, \overline{g})$, $p = \dim \overline{M} \geq 1$, and an Einstein manifold $(\widetilde{N}, \widetilde{g})$, $n - p = \dim \widetilde{N} \geq 4$, with some warping function $F$. If $p = 1$ then $\overline{M} \times_F \widetilde{N}$ is a quasi-Einstein manifold. The Ricci tensor of such a warped product satisfies (5)(a). The scalar curvature of these manifolds is nonzero. If $p \geq 2$ then $\overline{M} \times_F \widetilde{N}$ is a nonquasi-Einstein manifold. In Section 4 we present an example of a nonsemisymmetric Ricci-semisymmetric hypersurface $M$ of $\mathbb{E}^{n+1}_s$, $n \geq 5$, satisfying (4), (5) and (7) (see Example 4.2). We also present examples of nonquasi-Einstein and nonsemisymmetric Ricci-semisymmetric warped products which can be realized as hypersurfaces of $\mathbb{E}^{n+1}_s$, $n \geq 5$ (see Example 4.3). Ricci-pseudosymmetric warped products which can be locally realized as hypersurfaces of $\mathbb{E}^{n+1}_s$, $n \geq 5$, were investigated in [8].

**2. Basic formulas.** Let $(M, g)$, $n \geq 3$, be a connected semi-Riemannian manifold of class $C^\infty$ and let $\nabla$ be its Levi-Civita connection. We define on $M$ the endomorphisms $X \wedge_A Y$, $\mathcal{R}(X,Y)$ and $\mathcal{C}(X,Y)$ by

$$\begin{aligned}
(X \wedge_A Y)Z &= A(Y,Z)X - A(X,Z)Y,\\
\mathcal{R}(X,Y)Z &= [\nabla_X, \nabla_Y]Z - \nabla_{[X,Y]}Z,\\
\mathcal{C}(X,Y) &= \mathcal{R}(X,Y) - \frac{1}{n-2}\left(X \wedge_g \mathcal{S}Y + \mathcal{S}X \wedge_g Y - \frac{\kappa}{n-1}\, X \wedge_g Y\right),
\end{aligned}$$

where the Ricci operator $\mathcal{S}$ is defined by $g(X, \mathcal{S}Y) = S(X,Y)$, $A$ is a symmetric $(0,2)$-tensor, $S$ the Ricci tensor, $\kappa$ the scalar curvature and $X, Y, Z \in \varXi(M)$, $\varXi(M)$ being the Lie algebra of vector fields of $M$. The Riemann–Christoffel curvature tensor $R$ and the Weyl conformal curvature tensor $C$ of $(M,g)$ are defined by $R(X_1,X_2,X_3,X_4) = g(\mathcal{R}(X_1,X_2)X_3,X_4)$ and $C(X_1,X_2,X_3,X_4) = g(\mathcal{C}(X_1,X_2)X_3,X_4)$, respectively. We refer to [9] (see also [14] or [16]) for the definitions of the tensors: $R\cdot R$, $R\cdot C$, $R\cdot S$, $C\cdot R$, $C\cdot S$, $Q(g,R)$, $Q(g,C)$, $Q(g,S)$, $Q(S,R)$ and $Q(S,C)$. For symmetric $(0,2)$-tensors

$A$ and $B$ we denote by $A \wedge B$ their Kulkarni–Nomizu product. We have the identity (see e.g. [9]) $Q(S, g \wedge S) = -\frac{1}{2}Q(g, S \wedge S)$.

A semi-Riemannian manifold $(M, g)$ is said to be *pseudosymmetric* ([6]) if at every point of $M$ the tensors $R \cdot R$ and $Q(g, R)$ are linearly dependent. This is equivalent to $R \cdot R = L_R Q(g, R)$ on $U_R = \big\{x \in M \bigm| R - \frac{\kappa}{(n-1)n} G \neq 0$ at $x\big\}$, where $L_R$ is some function on $U_R$. The $(0, 4)$-tensor $G$ is defined by $G = \frac{1}{2} g \wedge g$. Evidently, every semi-Riemannian semisymmetric manifold is pseudosymmetric. The converse is not true ([6]). It is easy to see that at every point of any pseudosymmetric manifold $(M, g)$ the tensors $R \cdot S$ and $Q(g, S)$ are linearly dependent. The converse is not true ([3]).

A semi-Riemannian manifold $(M, g)$ is called *Ricci-pseudosymmetric* if $R \cdot S$ and $Q(g, S)$ are linearly dependent at every point of $M$. $(M, g)$ is Ricci-pseudosymmetric if and only if $R \cdot S = L_S Q(g, S)$ on the set $U_S = \{x \in M \mid S \neq (\kappa/n) g \text{ at } x\}$, where $L_S$ is some function on $U_S$. Examples of compact non-Einstein Ricci-pseudosymmetric manifolds which are nonpseudosymmetric were found in [17] and [18]. For instance, in [18, Theorem 1] it was shown that the Cartan hypersurfaces have that property. We recall that the Cartan hypersurface in the sphere $S^{n+1}(c)$ is a compact, minimal hypersurface with constant principal curvatures $-(3c)^{1/2}$, $0$, $(3c)^{1/2}$ of the same multiplicity ([2]). On every Cartan hypersurface we have ([18, Proposition 1])

$$\widetilde{R} \cdot \widetilde{S} = \frac{\tau}{n(n+1)} Q(\widetilde{g}, \widetilde{S}). \tag{8}$$

For recent results on Ricci-pseudosymmetric hypersurfaces in $N_s^{n+1}(c)$, $n \geq 4$, we refer to [8] and [19].

It is known that at every point of a hypersurface $\widetilde{N}$ of $N_s^{n+1}(c)$, $n \geq 4$, the following condition is satisfied ([6, Section 5.5]): the tensors $\widetilde{R} \cdot \widetilde{R} - Q(\widetilde{S}, \widetilde{R})$ and $Q(\widetilde{g}, \widetilde{C})$ are linearly dependent. Precisely, on $\widetilde{N}$ we have

$$\widetilde{R} \cdot \widetilde{R} - Q(\widetilde{S}, \widetilde{R}) = -\frac{(n-2)\tau}{n(n+1)} Q(\widetilde{g}, \widetilde{C}). \tag{9}$$

In particular, if the ambient space is $\mathbb{E}_s^{n+1}$ then (9) reduces to

$$\widetilde{R} \cdot \widetilde{R} = Q(\widetilde{S}, \widetilde{R}). \tag{10}$$

Every quasi-Einstein conformally flat manifold is a pseudosymmetric manifold satisfying (10) ([6, Section 6.3]).

**3. Ricci-semisymmetric manifolds.** In this section we present a family of nonsemisymmetric Ricci-semisymmetric quasi-Einstein warped products $\overline{M} \times_F \widetilde{N}$, $\dim \overline{M} = 1$, $\dim \widetilde{N} = n - 1 \geq 3$, satisfying at every point

$x \in \overline{M} \times \widetilde{N}$ the following condition:

$$S = \frac{\kappa}{n-1}\, g + \beta w \otimes w, \qquad w \in T_x^*(\overline{M} \times \widetilde{N}),\ \beta \in \mathbb{R}. \tag{11}$$

We also present a family of Ricci-semisymmetric nonquasi-Einstein warped products $\overline{M} \times_F \widetilde{N}$, $p = \dim \overline{M} \geq 2$, $n-p = \dim \widetilde{N} \geq 3$. These constructions are related to the notion of a cone in the sense of [21].

Proposition 3.1 (cf. [9, Proposition 3.1(ii)]). *If* (11) *holds at a point $x$ of a Ricci-semisymmetric semi-Riemannian manifold* $(M, g)$, $n \geq 4$, *then at $x$ we have*

$$R \cdot C - Q(S, C) = R \cdot R - Q(S, R). \tag{12}$$

As an immediate consequence of Proposition 3.1 and (10) we obtain

Corollary 3.1 (cf. [9, Theorem 3.1]). *Let $M$ be a Ricci-semisymmetric hypersurface of* $\mathbb{E}_s^{n+1}$, $n \geq 4$. *If* (11) *holds at a point $x$ of $M$ then at $x$ we have* $\widetilde{R} \cdot \widetilde{C} = Q(\widetilde{S}, \widetilde{C})$.

Let now $(\overline{M}, \overline{g})$ and $(\widetilde{N}, \widetilde{g})$, $p = \dim \overline{M}$, $n - p = \dim \widetilde{N}$, $1 \leq p < n$, be semi-Riemannian manifolds covered by systems of charts $\{\overline{U}; x^a\}$ and $\{\widetilde{V}; y^\alpha\}$, respectively. Let $F$ be a positive smooth function on $\overline{M}$. The *warped product* $\overline{M} \times_F \widetilde{N}$ is the product manifold $\overline{M} \times \widetilde{N}$ with the metric $g = \overline{g} \times_F \widetilde{g}$, defined by $\overline{g} \times_F \widetilde{g} = \pi_1^* \overline{g} + (F \circ \pi_1) \pi_2^* \widetilde{g}$, where $\pi_1 : \overline{M} \times \widetilde{N} \to \overline{M}$ and $\pi_2 : \overline{M} \times \widetilde{N} \to \widetilde{N}$ are the natural projections. Let $\{\overline{U} \times \widetilde{V}; x^1, \ldots, x^p, x^{p+1} = y^1, \ldots, x^n = y^{n-p}\}$ be a product chart for $\overline{M} \times \widetilde{N}$. The local components of the metric $g = \overline{g} \times_F \widetilde{g}$ with respect to this chart are $g_{hk} = \overline{g}_{ab}$ if $h = a$ and $k = b$, $g_{hk} = F \widetilde{g}_{\alpha\beta}$ if $h = \alpha$ and $k = \beta$, and $g_{hk} = 0$ otherwise, where $a, b, c, d \in \{1, \ldots, p\}$ and $\alpha, \beta \in \{p+1, \ldots, n\}$. We will mark by bars (resp., by tildes) tensors formed from $\overline{g}$ (resp., $\widetilde{g}$).

It is known that the local components $R_{rstu}$ of the Riemann–Christoffel curvature tensor $R$ and the local components $S_{ts}$ of the Ricci tensor $S$ of $\overline{M} \times_F \widetilde{N}$ which may not vanish identically are the following (see e.g. [4], [5] or [16]):

$$\begin{aligned} R_{abcd} &= \overline{R}_{abcd}, \\ R_{\alpha ab\beta} &= -\frac{1}{2}\, T_{ab} \widetilde{g}_{\alpha\beta}, \\ R_{\alpha\beta\gamma\delta} &= F \widetilde{R}_{\alpha\beta\gamma\beta} - \frac{\Delta_1 F}{4}\, \widetilde{G}_{\alpha\beta\gamma\delta}, \end{aligned} \tag{13}$$

$$\begin{aligned} S_{ab} &= \overline{S}_{ab} - \frac{n-p}{2F}\, T_{ab}, \\ S_{\alpha\beta} &= \widetilde{S}_{\alpha\beta} - \frac{1}{2}\left(\operatorname{tr} T + \frac{n-p-1}{2F}\, \Delta_1 F\right) \widetilde{g}_{\alpha\beta}. \end{aligned} \tag{14}$$

The $(0,2)$-tensor $T$, with local components $T_{ab}$, is defined by

$$\begin{aligned} T_{ab} &= \overline{\nabla}_b F_a - \frac{1}{2F}\, F_a F_b, \qquad \operatorname{tr} T = \overline{g}^{ab} T_{ab}, \\ \Delta_1 F &= \Delta_{1\overline{g}} F = \overline{g}^{ab} F_a F_b, \qquad F_a = \partial_a F = \frac{\partial F}{\partial x^a}. \end{aligned} \tag{15}$$

The scalar curvature $\kappa$ of $\overline{M} \times_F \widetilde{N}$ satisfies the relation

$$\kappa = \overline{\kappa} + \frac{\widetilde{\kappa}}{F} - \frac{n-p}{F}\left(\operatorname{tr} T + \frac{n-p-1}{4F}\, \Delta_1 F\right). \tag{16}$$

THEOREM 3.1. *Let* $(\overline{M}, \overline{g})$, $p = \dim \overline{M}$, *and* $(\widetilde{N}, \widetilde{g})$, $n-p = \dim \widetilde{N}$, $1 \le p < n$, *be semi-Riemannian manifolds and let* $F$ *be a smooth positive function on* $\overline{M}$.

(i) ([5, Theorem 1]) *The condition* $R \cdot R = L_R Q(g,R)$ *is satisfied on* $U_R$ *of* $\overline{M} \times_F \widetilde{N}$ *if and only if on* $U_R$ *we have*

$$(\overline{R} \cdot \overline{R})_{abcdef} = L_R Q(\overline{g}, \overline{R})_{abcdef}, \tag{17}$$

$$H^f{}_d \overline{R}_{fabc} = \frac{1}{2F}\,(T_{ac} H_{bd} - T_{ab} H_{cd}), \tag{18}$$

$$H_{ad}\left(\widetilde{R}_{\delta\alpha\beta\gamma} - \frac{\Delta_1 F}{4F}\, \widetilde{G}_{\delta\alpha\beta\gamma}\right) = -\frac{1}{2}\, T_{fd} H^f{}_a \widetilde{G}_{\delta\alpha\beta\gamma}, \tag{19}$$

$$(\widetilde{R} \cdot \widetilde{R})_{\alpha\beta\gamma\delta\lambda\mu} = \left(F L_R + \frac{\Delta_1 F}{4F}\right) Q(\widetilde{g}, \widetilde{R})_{\alpha\beta\gamma\delta\lambda\mu}, \tag{20}$$

*where*

$$H_{ad} = \frac{1}{2}\, T_{ad} + F L_R \overline{g}_{ad}. \tag{21}$$

(ii) ([5, Corollary 1]) *Let* $(\overline{M}, \overline{g})$, $p \ge 2$, *and* $(\widetilde{N}, \widetilde{g})$, $n-p \ge 2$, *be semi-Riemannian spaces of constant curvature. The condition* $R \cdot R = L_R Q(g,R)$ *is satisfied on* $U_R$ *of* $\overline{M} \times_F \widetilde{N}$ *if and only if on* $U_R$ *we have*

$$\frac{2\overline{\kappa}}{p(p-1)}\,(\overline{g}_{ab} H_{cd} - \overline{g}_{ac} H_{bd}) = \frac{1}{F}\,(T_{ac} H_{bd} - T_{ab} H_{cd}), \tag{22}$$

$$H_{ad}\left(\frac{\widetilde{\kappa}}{(n-p)(n-p-1)} - \frac{\Delta_1 F}{4F}\right) = -\frac{1}{2}\, T_{fd} H^f{}_a. \tag{23}$$

(iii) (cf. [4, Lemma 6]) *The only local components of the Weyl tensor* $C$ *of* $\overline{M} \times_F \widetilde{N}$, $p = 1$ *and* $n = 4$, *which are not identically zero are*

$$C_{\alpha 11 \beta} = -\frac{1}{2}\left(\widetilde{S}_{\alpha\beta} - \frac{\widetilde{\kappa}}{3}\, \widetilde{g}_{\alpha\beta}\right), \tag{24}$$

$$C_{\alpha\beta\gamma\delta} = \frac{F}{2}\,(\widetilde{g}_{\alpha\delta} \widetilde{S}_{\beta\gamma} - \widetilde{g}_{\alpha\gamma} \widetilde{S}_{\beta\delta} + \widetilde{g}_{\beta\gamma} \widetilde{S}_{\alpha\delta} - \widetilde{g}_{\beta\delta} \widetilde{S}_{\alpha\gamma}) - \frac{F\widetilde{\kappa}}{3}\, \widetilde{G}_{\delta\alpha\beta\gamma}. \tag{25}$$

(iv) ([3, Theorem 1]) *The condition $R\cdot S = L_S Q(g,S)$ is satisfied on $U_S$ of $\overline{M}\times_F \widetilde{N}$ if and only if on $U_S$ we have*

$$(\overline{R}\cdot\overline{S})_{abcd} - L_S Q(\overline{g},\overline{S})_{abcd} = \frac{n-p}{F}\,((\overline{R}\cdot H)_{abcd} - L_S Q(\overline{g},H)_{abcd}), \tag{26}$$

$$H_{ad}\Big(\widetilde{S}_{\alpha\beta} - \frac{1}{2F}\Big(\operatorname{tr} T + \frac{(n-p-1)\Delta_1 F}{2F}\Big) g_{\alpha\beta}\Big) = H_{cb}\Big(\overline{S}^{c}{}_{a} - \frac{n-p}{2F}\,T^{c}{}_{a}\Big) g_{\alpha\beta}, \tag{27}$$

$$(\widetilde{R}\cdot\widetilde{S})_{\alpha\beta\gamma\delta} = \Big(F L_S + \frac{\Delta_1 F}{4F}\Big) Q(\widetilde{g},\widetilde{R})_{\alpha\beta\gamma\delta}. \tag{28}$$

Example 3.1. Let $\eta$ and $\xi_1,\ldots,\xi_p$, $p\geq 1$, be constants such that $\xi_1^2 + \ldots + \xi_p^2 > 0$. Let $\overline{M} = \{(x^1,\ldots,x^p)\}$ be a nonempty open connected subset of $\mathbb{R}^p$ such that the function $F$ defined by

$$F(x^1,\ldots,x^p) = (\xi_a x^a + \eta)^2 \tag{29}$$

is positive on $\overline{M}$, where $a,b,c,d\in\{1,\ldots,p\}$. Further, we set $\overline{g}_{ab} = \varepsilon_a = \pm 1$. Using now (15) and (29) we find

$$T_{ab} = 0, \qquad \frac{\Delta_1 F}{F} = \frac{1}{F}\sum_{b=1}^{p}\varepsilon_b F_b^2 = 4c_0, \qquad c_0 = \sum_{b=1}^{p}\varepsilon_b \xi_b^2. \tag{30}$$

Example 3.2. We consider the warped product $\overline{M}\times_F\widetilde{N}$ of the manifold $(\overline{M},\overline{g})$ defined in Example 3.1 and an $(n-p)$-dimensional Einstein semi-Riemannian manifold $(\widetilde{N},\widetilde{g})$, $n-p\geq 3$, with $F$ defined by (29). Now (13), (14) and (16) yield

$$R_{abcd} = R_{\alpha ab\beta} = 0, \qquad R_{\alpha\beta\gamma\delta} = F(\widetilde{R}_{\alpha\beta\gamma\delta} - c_0\widetilde{G}_{\alpha\beta\gamma\delta}), \tag{31}$$

$$S_{ab} = 0, \tag{32}$$

$$S_{\alpha\beta} = \frac{\widetilde{\kappa} - (n-p-1)(n-p)c_0}{(n-p)F}\, g_{\alpha\beta}, \tag{33}$$

$$\kappa = \frac{\widetilde{\kappa} - (n-p-1)(n-p)c_0}{F}, \tag{34}$$

respectively. Applying (34) in (33) we get

$$S_{\alpha\beta} = \frac{\kappa}{n-p}\, g_{\alpha\beta}. \tag{35}$$

(i) We assume that $p=1$. The manifold $\overline{M}\times_F\widetilde{N}$ is quasi-Einstein. In fact, we have

$$S_{ij} = \frac{\kappa}{n-1}\, g_{ij} + \gamma\varepsilon_1 w_i w_j,$$

where $w_i = \delta_i^1$, $\gamma = -\kappa/(n-1)$ and $i,j\in\{1,\ldots,n\}$. From Theorem 3.1(iv) it follows that $\overline{M}\times_F\widetilde{N}$ is a Ricci-semisymmetric manifold. If $\dim\widetilde{N} =$

$n-1=3$ then, in view of Theorem 3.1(i), (iii), $\overline{M}\times_F\widetilde{N}$ is a semisymmetric conformally flat manifold. Assume now that $\dim\widetilde{N}=n-1\geq 4$. Using Theorem 3.1(i) we can deduce that if $(\widetilde{N},\widetilde{g})$, $\dim\widetilde{N}\geq 4$, is an Einstein semisymmetric manifold not of constant curvature then $\overline{M}\times_F\widetilde{N}$ is a nonsemisymmetric Ricci-semisymmetric manifold. If the constant $c_2$ defined by $c_2=\widetilde{\kappa}-(n-1)(n-2)c_0$ is nonzero then the scalar curvature $\kappa$ of $\overline{M}\times_F\widetilde{N}$ is also nonzero.

(ii) We assume that $p\geq 2$. First of all, we note that

$$S_{ij}=\frac{\kappa}{n-p}\,g_{ij}+\varrho\sum_{a=1}^{p}\varepsilon_a w_{ai}w_{aj}, \tag{36}$$

where $w_{ai}=\delta_{ai}$, $\varrho=-\kappa/(n-p)$ and $i,j\in\{1,\ldots,n\}$. From Theorem 3.1(iv) it follows that $\overline{M}\times_F\widetilde{N}$ is a Ricci-semisymmetric manifold. Using (31) and (32) we find

$$C_{abcd}=\frac{\kappa}{(n-2)(n-1)}\,G_{abcd}.$$

Thus $\overline{M}\times_F\widetilde{N}$ is a nonconformally flat manifold with nonconstant scalar curvature $\kappa$, provided that the constant $c_2$ defined by

$$c_2=\widetilde{\kappa}-(n-p-1)(n-p)c_0 \tag{37}$$

is nonzero. If $n-p=3$ then, in view of Theorem 3.1(i), $\overline{M}\times_F\widetilde{N}$ is a semisymmetric manifold. Assume now that $n-p\geq 4$. Using again Theorem 3.1(i) we can deduce that if $(\widetilde{N},\widetilde{g})$ is an Einstein semisymmetric manifold not of constant curvature then $\overline{M}\times_F\widetilde{N}$ is a nonsemisymmetric Ricci-semisymmetric manifold. Since $\overline{M}\times_F\widetilde{N}$ is a Ricci-semisymmetric manifold, we have $R\cdot C=R\cdot R$. Thus the $(0,6)$-tensor $T$ defined by

$$T=R\cdot C-Q(S,C)-R\cdot R+Q(S,R)$$

takes the form

$$\begin{aligned}T&=-Q(S,C)+Q(S,R)=-Q(S,C-R)\\&=\frac{1}{n-2}\,Q\Big(S,g\wedge S-\frac{\kappa}{n-1}\,G\Big)\\&=-\frac{1}{n-2}\,Q\Big(g,\frac{1}{2}\,S\wedge S\Big)-\frac{\kappa}{(n-2)(n-1)}\,Q(S,G).\end{aligned}$$

Applying now (35) we get

$$T_{a\beta\gamma\delta\alpha b}=\frac{(p-1)\kappa^2}{(n-p)^2(n-2)(n-1)}\,g_{ab}G_{\alpha\beta\gamma\delta}.$$

If the constant $c_2$ defined by (37) is nonzero then (12) is not satisfied on $\overline{M}\times_F\widetilde{N}$.

Proposition 3.1 yields immediately

COROLLARY 3.2. *The warped product defined in Example* 3.2(i) *satisfies* (12).

EXAMPLE 3.3. (i) Let $(\widetilde{N},\widetilde{g})$, $\dim\widetilde{N} = n-p \geq 3$, $p \geq 1$, be a non-Einstein Ricci-pseudosymmetric semi-Riemannian manifold such that on $U_{\widetilde{S}}$ we have $\widetilde{R}\cdot\widetilde{S} = L_{\widetilde{S}}Q(\widetilde{g},\widetilde{S})$ and $L_{\widetilde{S}} = \mathrm{const} \neq 0$. We consider the warped product $\overline{M}\times_F\widetilde{N}$ with $(\overline{M},\overline{g})$ and $F$ defined in Example 3.1. Let $c_0 = L_{\widetilde{S}}$. Now, in view of Theorem 3.1(iv), $\overline{M}\times_F\widetilde{N}$ is a Ricci-semisymmetric nonsemisymmetric manifold. If $(\widetilde{N},\widetilde{g})$ is nonpseudosymmetric then so is $\overline{M}\times_F\widetilde{N}$.

(ii) Let $(\widetilde{N},\widetilde{g})$, $\dim\widetilde{N} = n-p = 3,6,12$ or $24$, $p \geq 1$, be a Riemannian manifold locally isometric to the Cartan hypersurface of dimension $3,6,12$ or $24$, respectively. We note that (8) holds on $(\widetilde{N},\widetilde{g})$. We consider the warped product $\overline{M}\times_F\widetilde{N}$ with $(\overline{M},\overline{g})$ and $F$ as in (i). Let $c_0 = \tau/(n(n+1))$. Now, as in (i), $\overline{M}\times_F\widetilde{N}$ is a nonsemisymmetric Ricci-semisymmetric manifold provided that $n-p = 6,12$ or $24$. If $n-p = 3$ then $\overline{M}\times_F\widetilde{N}$ is semisymmetric. This is a consequence of Theorem 3.1(ii) and the fact that the 3-dimensional Cartan hypersurface is pseudosymmetric.

REMARK 3.1. Using (13) and (14) and the definitions of the tensors $R\cdot R$ and $Q(S,R)$ we can easily check that the warped product defined in Example 3.2 satisfies $R\cdot R = Q(S,R)$.

LEMMA 3.1. *Let* $(M,g)$, $n \geq 4$, *be a semi-Riemannian manifold.*

(i) *If at* $x\in M$, (2), (11) *and* $R\cdot R = Q(S,R)$ *are satisfied then at* $x$ *we have*

$$R\cdot C = Q(S,C). \tag{38}$$

(ii) *If at* $x\in M$, (11), $R\cdot R = Q(S,R)$, $\kappa \neq 0$ *and* $C \neq 0$ *are satisfied then* $M$ *is nonsemisymmetric.*

*Proof.* (i) is a consequence of our assumptions and Proposition 3.1.

(ii) We suppose that $R\cdot R = 0$ at $x$. Thus also $Q(S,R)$ vanishes at $x$. Further, by our assumptions, $\operatorname{rank} S > 1$ at $x$. Now the equation $Q(S,R) = 0$, in view of Lemma 3.4 of [12], implies $\lambda R = \frac{1}{2}S\wedge S$, $\lambda\in\mathbb{R}-\{0\}$, which yields

$$\lambda C = \frac{1}{2}\,S\wedge S - \frac{\lambda}{n-2}\,g\wedge S + \frac{\lambda\kappa}{(n-2)(n-1)}\,G.$$

This leads to

$$\lambda C = \frac{1}{2}\Big(S - \frac{\lambda}{n-2}g\Big)\wedge\Big(S - \frac{\lambda}{n-2}g\Big) + \frac{\lambda}{n-2}\Big(\frac{\kappa}{n-1} - \frac{\lambda}{n-2}\Big)G,$$

whence, by an application of (11), we get

$$\lambda C = \left(\frac{\kappa}{n-1} - \frac{\lambda}{n-2}\right)\left(\frac{\kappa}{n-1}\, G + \beta g \wedge w \otimes w\right). \tag{39}$$

By suitable contraction this gives

$$\left(\frac{\kappa}{n-1} - \frac{\lambda}{n-2}\right)((\kappa + \beta\|w\|^2)g + (n-1)\beta \wedge w \otimes w) = 0.$$

Since $(\kappa+\beta\|w\|^2)g+(n-1)\beta\wedge w\otimes w$ is a nonzero tensor, the above relation leads to $\frac{\kappa}{n-1} - \frac{\lambda}{n-2} = 0$. Therefore (39) implies $C = 0$, a contradiction.

REMARK 3.2 ([6, Section 9]). If at a point of a semi-Riemannian manifold $(M, g)$, $n \geq 5$, the tensors $C$ and $R \cdot R$ are nonzero then the tensor $R \cdot C$ is also nonzero at this point.

**4. Ricci-semisymmetric hypersurfaces.** Let $\widetilde{N}$, $n = \dim \widetilde{N} \geq 3$, be a connected hypersurface isometrically immersed in a semi-Riemannian manifold $(N, g)$. We denote by $\widetilde{g}$ the metric tensor induced on $\widetilde{N}$ from $g$. Further, we denote by $\widetilde{\nabla}$ and $\nabla$ the Levi-Civita connections corresponding to $\widetilde{g}$ and $g$, respectively. Let $\xi$ be a local unit normal vector field on $\widetilde{N}$ in $N$ and let $\varepsilon = \widetilde{g}(\xi,\xi) = \pm 1$. We can write the *Gauss formula* and the *Weingarten formula* of $\widetilde{N}$ in $N$ in the following form: $\nabla_X Y = \widetilde{\nabla}_X Y + \varepsilon \widetilde{H}(X,Y)\xi$ and $\nabla_X \xi = -\mathcal{A}X$, respectively, where $X, Y$ are vector fields tangent to $\widetilde{N}$, $\widetilde{H}$ is the *second fundamental tensor* of $\widetilde{N}$ in $N$, $\mathcal{A}$ is the *shape operator* of $\widetilde{N}$ in $N$ and $\widetilde{H}^k(X,Y) = g(\mathcal{A}^k X, Y)$, $\operatorname{tr}(\widetilde{H}^k) = \operatorname{tr}(\mathcal{A}^k)$, $k \geq 1$, $\widetilde{H}^1 = \widetilde{H}$ and $\mathcal{A}^1 = \mathcal{A}$. We denote by $\widetilde{R}$ and $R$ the Riemann–Christoffel curvature tensors of $\widetilde{N}$ and $N$, respectively. We let $U_{\widetilde{H}}$ be the set of all $x \in \widetilde{N}$ at which $\mathcal{A}^2$ is not a linear combination of $\mathcal{A}$ and Id. Note that $U_{\widetilde{H}} \subset U_{\widetilde{S}}$. The *Gauss equation* of $\widetilde{N}$ in $N$ has the form

$$\widetilde{R}(X_1,\ldots,X_4) = R(X_1,\ldots,X_4) + \frac{\varepsilon}{2}\,(\widetilde{H}\wedge\widetilde{H})(X_1,\ldots,X_4), \tag{40}$$

where $X_1,\ldots,X_4$ are vector fields tangent to $\widetilde{N}$. Let $x^r = x^r(y^\alpha)$ be the local parametric expression of $\widetilde{N}$ in $(N,g)$, where $y^\alpha$ and $x^r$ are local coordinates of $\widetilde{N}$ and $N$, respectively, and $\alpha,\beta,\gamma,\delta \in \{1,\ldots,n\}$ and $r \in \{1,\ldots,n+1\}$.

Let the ambient space $(N, g)$ be a semi-Riemannian space $N_s^{n+1}(c)$ of constant curvature. Now (40) reads

$$\widetilde{R}_{\alpha\beta\gamma\delta} = \varepsilon(\widetilde{H}_{\alpha\delta}\widetilde{H}_{\beta\gamma} - \widetilde{H}_{\alpha\gamma}\widetilde{H}_{\beta\delta}) + \frac{\tau}{n(n+1)}\,\widetilde{G}_{\alpha\beta\gamma\delta}, \tag{41}$$

where $\widetilde{R}_{\alpha\beta\gamma\delta}$, $\widetilde{G}_{\alpha\beta\gamma\delta}$ and $\widetilde{H}_{\alpha\delta}$ are the local components of $\widetilde{R}$, $\widetilde{G} = \frac{1}{2}\widetilde{g}\wedge\widetilde{g}$, and $\widetilde{H}$, respectively.

As a consequence of Theorem 1.1, Lemma 3.1 and Remark 3.2 we have the following

PROPOSITION 4.1. *Let $M$ be a hypersurface of $\mathbb{E}^{n+1}_s$, $n \geq 5$.*

(i) *If at $x \in M$ condition* (11) *is satisfied then* (38) *holds at $x$.*

(ii) *If at $x \in M$ the following conditions are satisfied*: (11), $\kappa \neq 0$ *and* $C \neq 0$, *then at $x$ we have* $R \cdot R \neq 0$, $R \cdot C \neq 0$ *and* $Q(S,C) \neq 0$.

(iii) *If at $x \in \overline{U} = U_H \cap U_L \subset M$ condition* (3) *is satisfied*, $\kappa \neq 0$, *and either* (4)(a), (4)(b), *or* (4)(c) *holds*, *then $M$ is nonsemisymmetric.*

We now present examples of nonsemisymmetric Ricci-semisymmetric hypersurfaces in $\mathbb{E}^{n+1}_s$, $n \geq 5$.

PROPOSITION 4.2. *Let* $(\overline{M}, \overline{g})$, $\dim \overline{M} = p \geq 2$, *be as defined in Example* 3.1 *and let* $(\widetilde{N}, \widetilde{g})$, $\dim \widetilde{N} = n - p \geq 1$, *be a semi-Riemannian manifold isometric to a hypersurface of* $N^{n-p+1}_s(c)$. *Let* $\overline{M} \times_F \widetilde{N}$ *be the warped product with $F$ and $c_0$ defined by* (29) *and* (30), *respectively*, *and*

$$c_0 = \frac{\tau}{(n-p)(n-p+1)}, \tag{42}$$

*where $\tau$ is the scalar curvature of* $N^{n-p+1}_s(c)$. *Then* $\overline{M} \times_F \widetilde{N}$ *can be realized locally as a hypersurface of* $\mathbb{E}^{n+1}_s$.

*Proof.* (i) Let $(M, \widetilde{g})$ be a hypersurface of $N^{n-p+1}_s(c)$. Thus (41) yields

$$\widetilde{R} = \frac{\varepsilon}{2}\, \widetilde{H} \wedge \widetilde{H} + \frac{\tau}{(n-p)(n-p+1)}\, \widetilde{G}, \tag{43}$$

where $\tau$ is the scalar curvature of the ambient space and $\widetilde{R}$ and $\widetilde{H}$ are the curvature tensor and the second fundamental tensor of $M$, respectively. By making use of (30), (42) and (43), we can write the formulas (5), (6) and (7) of [4] in the form

$$\begin{aligned}
R_{abcd} &= 0,\\
R_{\alpha ab\beta} &= -\frac{1}{2}\, T_{ab}\widetilde{g}_{\alpha\beta} = 0,\\
R_{\alpha\beta\gamma\delta} &= F\widetilde{R}_{\alpha\beta\gamma\delta} - \frac{\Delta_1 F}{4}\, \widetilde{G}_{\alpha\beta\gamma\delta}\\
&= \varepsilon(\sqrt{F}\, \widetilde{H}_{\alpha\delta}\sqrt{F}\, \widetilde{H}_{\beta\gamma} - \sqrt{F}\, \widetilde{H}_{\alpha\gamma}\sqrt{F}\, \widetilde{H}_{\beta\delta})\\
&\quad + F\left(\frac{\tau}{(n-p)(n-p+1)} - \frac{\Delta_1 F}{4F}\right)\widetilde{G}_{\alpha\beta\gamma\delta}\\
&= \varepsilon(H_{\alpha\delta}H_{\beta\gamma} - H_{\alpha\gamma}H_{\beta\delta}),
\end{aligned}$$

respectively, where $H_{\alpha\delta} = \sqrt{F}\, \widetilde{H}_{\alpha\delta}$. Let $H$ be the symmetric $(0,2)$-tensor $H$ on $\overline{M} \times_F \widetilde{N}$ with local components $H_{ab} = 0$, $H_{a\delta} = 0$ and $H_{\alpha\delta} = \sqrt{F}\, \widetilde{H}_{\alpha\delta}$. Using the fact that $\widetilde{\nabla}_\alpha \widetilde{H}_{\beta\delta} = \widetilde{\nabla}_\beta \widetilde{H}_{\alpha\delta}$, we can easily check that $H$ is a Codazzi tensor on $\overline{M} \times_F \widetilde{N}$. Thus the semi-Riemannian manifold $\overline{M} \times_F \widetilde{N}$ can be locally realized as a hypersurface of $\mathbb{E}^{n+1}_s$. Our proposition is thus proved.

Proposition 4.1 and Example 3.2 imply

Corollary 4.1. *If* $(\widetilde{N},\widetilde{g})$, $n-p\geq 4$, *is a semisymmetric Einstein manifold not of constant curvature then the warped product* $\overline{M}\times_F\widetilde{N}$ *defined in Proposition* 4.1 *is a nonsemisymmetric Ricci-semisymmetric manifold which can be locally realized as a hypersurface of* $\mathbb{E}_s^{n+1}$.

Proposition 4.1 and Example 3.3 yield

Corollary 4.2. *If* $(\widetilde{N},\widetilde{g})$, $U_{\widetilde{S}}=\widetilde{N}$, $n-p\geq 4$, *is a non-Einstein Ricci-pseudosymmetric manifold such that on* $U_{\widetilde{S}}$,

$$\widetilde{R}\cdot\widetilde{R}=L_{\widetilde{S}}Q(\widetilde{g},\widetilde{S})\quad and\quad L_{\widetilde{S}}=\frac{\tau}{(n-p)(n-p+1)},$$

*then the warped product* $\overline{M}\times_F\widetilde{N}$ *defined in Proposition* 4.1 *is a nonsemisymmetric Ricci-semisymmetric manifold which can be locally realized as a hypersurface of* $\mathbb{E}_s^{n+1}$.

Remark 4.1. The above result is also true when $p=1$. However, in that case we must additionally assume that the space $N_s^{n-p+1}(c)$ is nonflat.

Proposition 4.3. *Let* $\widetilde{N}$ *be a hypersurface of* $N_s^{n+1}(c)$, $n\geq 4$. *If at a point* $x$ *the following conditions are satisfied*:

$$\widetilde{R}\cdot\widetilde{R}=0,\quad \widetilde{S}=\frac{\widetilde{\kappa}}{n}\widetilde{g}\quad and\quad \widetilde{R}-\frac{\widetilde{\kappa}}{(n-1)n}\widetilde{G}\neq 0,$$

*then at* $x$ *we have*

$$\widetilde{\kappa}=\frac{n-2}{n+1}\,\tau. \tag{44}$$

*Proof.* (9), by our assumptions, turns into

$$\left(\widetilde{\kappa}-\frac{n-2}{n+1}\tau\right)Q(\widetilde{g},\widetilde{R})=0.$$

Since $Q(\widetilde{g},\widetilde{R})\neq 0$ if and only if $\widetilde{R}-\frac{\widetilde{\kappa}}{(n-1)n}\widetilde{G}\neq 0$, the last equality implies (44), which completes the proof.

As an immediate consequence of the last proposition we have the following

Corollary 4.3. *Let* $\widetilde{N}$ *be a semisymmetric Einstein hypersurface of* $N_s^{2k+1}(c)$, $k\geq 2$. *Then on* $U_{\widetilde{R}}\subset\widetilde{N}$ *we have*

$$\widetilde{R}_{\alpha\beta\gamma\delta}=\varepsilon(\widetilde{H}_{\alpha\delta}\widetilde{H}_{\beta\gamma}-\widetilde{H}_{\alpha\gamma}\widetilde{H}_{\beta\delta})+\frac{\widetilde{\kappa}}{4k(k-1)}\widetilde{G}_{\alpha\beta\gamma\delta}. \tag{45}$$

Example 4.1 ([20]). We now present examples of semisymmetric Einstein hypersurfaces. Namely, in [20] Einstein hypersurfaces with $\mathcal{A}^2=$

$-b^2\,\mathrm{Id}$, $b \neq 0$, were classified. They are complex spheres, either $\mathbb{C}S^k(1/b)$ or $\mathbb{C}S^k(i/b)$, where, for any $\gamma \in \mathbb{C}$,

$$\mathbb{C}S^k(\gamma) = \{(z_1, \dots, z_{k+1}) \in \mathbb{C}^{k+1} : z_1^2 + \dots + z_{k+1}^2 = \gamma^2\}, \qquad k \geq 2.$$

It is known that $\mathbb{C}S^k$ is an irreducible symmetric space (see [20] and references therein). $\mathbb{C}S^k(1/b)$ is a hypersurface in the indefinite sphere $S^{2k+1}_{k+1}(1/b)$ and $\mathbb{C}S^k(i/b)$ is a hypersurface in the indefinite hyperbolic space $H^{2k+1}_k(1/b)$, defined by

$$S^{2k+1}_{k+1}(1/b) = \Big\{(x_1, y_1, \dots, x_{k+1}, y_{k+1}) \in \mathbb{R}^{2k+2} : \sum_{j=1}^{k+1} x_j^2 - y_j^2 = 1/b^2\Big\},$$

$$H^{2k+1}_k(1/b) = \Big\{(x_1, y_1, \dots, x_{k+1}, y_{k+1}) \in \mathbb{R}^{2k+2} : \sum_{j=1}^{k+1} x_j^2 - y_j^2 = -1/b^2\Big\}.$$

$S^{2k+1}_{k+1}(1/b)$ has constant curvature $b^2$ and signature $(k+1, k)$, while $H^{2k+1}_k(1/b)$ has constant curvature $-b^2$ and signature $(k, k+1)$. By identifying $(x_1 + iy_1, \dots, x_{k+1} + iy_{k+1}) \in \mathbb{C}^{k+1}$ with $(x_1, y_1, \dots, x_{k+1}, y_{k+1}) \in \mathbb{R}^{2k+2}$, the complex spheres can be viewed as hypersurfaces defined by the single equation $\sum_j x_j y_j = 0$ in the sphere and hyperbolic space. It is clear that (45) holds on these hypersurfaces.

Example 4.2. Let $(\widetilde{N}, \widetilde{g})$, $\dim \widetilde{N} = n - p = 2k \geq 4$, $p \geq 1$, be a semi-Riemannian manifold isometric to an open nonempty part of the semisymmetric Einstein hypersurface of Example 4.1. We now consider the warped product $\overline{M} \times_F \widetilde{N}$ where $(\overline{M}, \overline{g})$ and $F$ are defined in Example 3.1, and the constant $c_0$ satisfies (42). In view of Corollary 4.1, $\overline{M} \times_F \widetilde{N}$ can be locally realized as a nonsemisymmetric Ricci-semisymmetric hypersurface of $\mathbb{E}^{n+1}_s$, $n \geq 5$. From Corollary 4.3 it follows that $c_0 = \tau/(2k(2k+1)) = \widetilde{\kappa}/(4k(k-1))$, where $\widetilde{\kappa}$ is the scalar curvature of $(\widetilde{N}, \widetilde{g})$. Thus on $\overline{M} \times_F \widetilde{N}$ we have $\kappa = -\widetilde{\kappa}/((n-p-2)F)$.

Example 4.3. Using Corollary 4.2, in the same way as in Example 4.2, we can show that the warped product $\overline{M} \times_F \widetilde{N}$, $\dim \overline{M} = p \geq 1$, $\dim \widetilde{N} = n - p = 3, 6, 12$ or $24$, defined in Example 3.3(ii) can be locally realized as a Ricci-semisymmetric hypersurface in $\mathbb{E}^{n+1}_s$, $n \geq 4$. If $n - p = 6, 12$ or $24$ the hypersurface is nonsemisymmetric.

Remark 4.2. (i) An example of a semisymmetric quasi-Einstein hypersurface of $\mathbb{E}^{n+1}_s$, $n \geq 4$, was given in [9, Example 5.1].

(ii) An example of a nonsemisymmetric Ricci-semisymmetric quasi-Einstein hypersurface of the Euclidean space $\mathbb{E}^{n+1}$, $n \geq 5$, was found recently in [1].

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