COLLOQUIUM MATHEMATICUM

VOL. 94 2002 NO. 1

BLOW-UP OF SOLUTIONS FOR THE KIRCHHOFF EQUATION OF q-LAPLACIAN TYPE WITH NONLINEAR DISSIPATION

BY

ABBES BENAISSA (Sidi Bel Abbes) and SALIM A. MESSAOUDI (Dhahran)

Abstract. We establish the blow-up of solutions to the Kirchhoff equation of q-Laplacian type with a nonlinear dissipative term

$$(|u_t|^{l-2}u_t)_t - M(\|A^{1/2}u\|_2^2)Au + |u_t|^\beta u_t = |u|^p u, \quad x \in \Omega,\ t > 0.$$

1. Introduction. We consider the initial boundary value problem (IBVP) for the nonlinear Kirchhoff equation of q-Laplacian type

$$\text{(P)}\quad \begin{cases} (|u_t|^{l-2}u_t)_t - M(\|A^{1/2}u\|_2^2)Au + |u_t|^\beta u_t = |u|^p u, & x \in \Omega,\ t > 0,\\ u(x,t) = 0, \quad x \in \partial\Omega,\ t \ge 0,\\ u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x), & x \in \Omega, \end{cases}$$

where

$$Au = e^{-\Phi(x)}\operatorname{div}(e^{\Phi(x)}|\nabla_x u|^{q-2}\nabla_x u), \quad \|A^{1/2}u\|^2 = \int_\Omega e^{\Phi(x)}|\nabla_x u|^q\,dx,$$

$p > -1$, $\beta > -1$, $q > 1$ and $l > 1$ are constants, $M : \mathbb{R}_+ \to \mathbb{R}_+$ is a continuous function and $\Omega \subset \mathbb{R}^n$ is a bounded domain with boundary Γ so that the divergence theorem can be applied.

When $M \equiv 1$, $\Phi \equiv 0$ and $q = l = 2$, for the case of no dissipation (i.e. (P) without the term $|u_t|^\beta u_t$), it is well known that the source term $|u|^p u$ is responsible for finite blow-up (global nonexistence) of solutions with negative initial energy (see [3], [14], [18], [19]). The interaction between the damping term and the source has been first considered by Levine [18], [19]; for the case with linear dissipation of the form τu_t ($\tau > 0$), he showed that solutions with negative initial energy blow up in finite time. In [10] Georgiev and Todorova extended Levine's result to the case of nonlinear damping of the form $|u_t|^\beta u_t$. This result was generalized to an abstract setup by Levine and Serrin [20], Levine and Park [21] and Vitillaro [23]. In [17] Messaoudi extended the result of Levine to the situation where $\Phi \neq 0$.

When $\Phi \equiv 0$ and M is not a constant function, in the case $q = l = 2$, the equation without the damping and source terms is often called the *wave*

2000 *Mathematics Subject Classification*: 35L35, 35L70.

Key words and phrases: blow-up, Kirchhoff equation of q-Laplacian type, finite time.

equation of Kirchhoff type; it has been introduced by Kirchhoff [15] in order to study the nonlinear vibrations of an elastic string (in the case $l = 2$ and $q \neq 2$, the equation is called the *Kirchhoff equation of q-Laplacian type*) and the existence of local and global solutions in Sobolev and Gevrey classes was investigated by many authors (see [7], [8], [16], [6], [22], [13], [9], [11], [12] and [4]).

In [2] Bainov and Minchev studied the blowing up of solutions of the initial boundary value problem for the Kirchhoff equation with source and without damping term. Unfortunately this method does not seem to be applicable (also when $l \neq 2$, $q = 2$ and without damping term; $l = 2$, $q = 2$ and linear damping) to the case of more general nonlinear terms.

In the present paper, we investigate the blowing up of solutions of the initial-boundary value problem for the Kirchhoff equation of q-Laplacian type. We shall show that, for suitably chosen initial data, any classical solution blows up in finite time.

We will then extend to the problem (P) the argument introduced in [10] to prove blow-up of solutions of a wave equation with nonlinear damping and source terms. We also extend the results of our previous paper [5], where $q = l = 2$ and the dissipative term is linear.

2. Main result. In order to state our main result, we make the following hypotheses:

(1) $M \in C(\mathbb{R}_+, \mathbb{R}_+)$ and $m\overline{M}(s) \geq sM(s)$ for all $s \geq 0$, where $\overline{M}(s) = \int_0^s M(k)\,dk$, $m \geq 1$,

$$E(0) = \frac{l-1}{l}\int_\Omega e^{\Phi(x)}|u_1|^l\,dx + \frac{1}{q}\,\overline{M}\Big(\int_\Omega e^{\Phi(x)}|\nabla_x u_0|^q\,dx\Big) - \frac{1}{p+2}\int_\Omega e^{\Phi(x)}|u_0|^{p+2}\,dx < 0, \tag{2}$$

$$p > \max\{l, \beta, qm-2\} \quad \text{and} \quad \beta > l-2 \ (l > 1). \tag{3}$$

THEOREM 2.1. *Assume that* (1)–(3) *hold. Then for any initial data satisfying* $(u_0, u_1) \in W_0^{1,q}(\Omega) \times L^l(\Omega)$, *the solution of* (P) *blows up in finite time.*

Proof. We set

$$E(t) = \frac{l-1}{l}\int_\Omega e^{\Phi(x)}|u_t|^l\,dx + \frac{1}{q}\,\overline{M}\Big(\int_\Omega e^{\Phi(x)}|\nabla_x u|^q\,dx\Big) - \frac{1}{p+2}\int_\Omega e^{\Phi(x)}|u|^{p+2}\,dx. \tag{4}$$

By multiplying the equation of (P) by $e^{\Phi(x)}u_t(x,t)$ and integrating over Ω, we get

$$E'(t) = -\int_\Omega e^{\Phi(x)}|u_t|^{\beta+2}\,dx. \tag{5}$$

Then it follows from (5) and (2) that

$$E(t) \le E(0) < 0, \tag{6}$$

and hence

$$u(t) \not\equiv 0 \quad \text{and} \quad \overline{M}\Big(\int_\Omega e^{\Phi(x)}|\nabla_x u|^q\,dx\Big) > 0. \tag{7}$$

We then define the function

$$G(t) = \int_\Omega e^{\Phi(x)}|u_t|^{l-2}u_t u\,dx$$

with

$$G'(t) = \int_\Omega e^{\Phi(x)}[|u_t|^l + u(|u_t|^{l-2}u_t)_t]\;dx.$$

By using the equation of (P), we arrive at

$$\begin{aligned} G'(t) = &\int_\Omega e^{\Phi(x)}|u_t|^l\,dx + \int_\Omega e^{\Phi(x)}|u|^{p+2}\,dx - \int_\Omega e^{\Phi(x)}u|u_t|^\beta u_t\,dx \\ &- M\Big(\int_\Omega e^{\Phi(x)}|\nabla_x u(x,t)|^q\,dx\Big)\int_\Omega e^{\Phi(x)}|\nabla_x u(x,t)|^q\,dx \\ \ge &\int_\Omega e^{\Phi(x)}|u_t|^l\,dx + \int_\Omega e^{\Phi(x)}|u|^{p+2}\,dx - \int_\Omega e^{\Phi(x)}u|u_t|^\beta u_t\,dx \\ &- m\overline{M}\Big(\int_\Omega e^{\Phi(x)}|\nabla_x u(x,t)|^q\,dx\Big) \end{aligned} \tag{8}$$

and by (4) we obtain

$$\begin{aligned} G'(t) \ge &\Big(1 + \frac{mq(l-1)}{l}\Big)\int_\Omega e^{\Phi(x)}|u_t|^l\,dx - qmE(t) \\ &+ \frac{p+2-qm}{p+2}\int_\Omega e^{\Phi(x)}|u|^{p+2}\,dx - \int_\Omega e^{\Phi(x)}u|u_t|^\beta u_t\,dx \\ = &\, H(t) + \frac{p+2-qm}{2(p+2)}\int_\Omega e^{\Phi(x)}|u|^{p+2}\,dx - \int_\Omega e^{\Phi(x)}u|u_t|^\beta u_t\,dx \end{aligned} \tag{9}$$

where

$$\begin{aligned} H(t) \equiv &-qmE(t) + \Big(1 + \frac{mq(l-1)}{l}\Big)\int_\Omega e^{\Phi(x)}|u_t|^l\,dx \\ &+ \frac{p+2-qm}{2(p+2)}\int_\Omega e^{\Phi(x)}|u|^{p+2}\,dx. \end{aligned} \tag{10}$$

We observe that

$$(11)\qquad \Big|\int_\Omega e^{\Phi(x)} u|u_t|^\beta u_t\,dx\Big| \le \|e^{\Phi/(\beta+2)}u_t\|_{\beta+2}^{\beta+1}\|e^{\Phi/(\beta+2)}u\|_{\beta+2} \le B_1\|e^{\Phi/(\beta+2)}u_t\|_{\beta+2}^{\beta+1}\|e^{\Phi/(p+2)}u\|_{p+2}$$
$$\le B_1\|e^{\Phi/(\beta+2)}u_t\|_{\beta+2}^{\beta+1}\|e^{\Phi/(p+2)}u\|_{p+2}^{(p+2)/(\beta+2)}\|e^{\Phi/(p+2)}u\|_{p+2}^{-((p+2)/(\beta+2)-1)}$$

with

$$B_1 = \Big(\int_\Omega e^{\Phi(x)}\,dx\Big)^{\frac{p-\beta}{(p+2)(\beta+2)}}.$$

Since it follows from (4) that

$$\|e^{\Phi/(p+2)}u\|_{p+2}^{p+2} \ge (p+2)(-E(t)) \ge -E(t),$$

we have

$$(12)\qquad \|e^{\Phi/(p+2)}u\|_{p+2} \ge (-E(t))^{1/(p+2)} \ge (-E(0))^{1/(p+2)},$$

and from the Young inequality

$$B_1\|e^{\Phi/(\beta+2)}u_t\|_{\beta+2}^{\beta+1}\|e^{\Phi/(p+2)}u\|_{p+2}^{(p+2)/(\beta+2)}$$
$$\le \frac{\beta+1}{\beta+2}(\varepsilon^{-1}B_1)^{(\beta+1)/(\beta+2)}\|e^{\Phi/(\beta+2)}u_t\|_{\beta+2}^{\beta+2} + \frac{\varepsilon^{\beta+2}}{\beta+2}\|e^{\Phi/(p+2)}u\|_{p+2}^{p+2}$$

for any $\varepsilon > 0$. Therefore (11) takes the form

$$(13)\qquad \Big|\int_\Omega e^{\Phi(x)} u|u_t|^\beta u_t\,dx\Big| \le (\varepsilon^{-1}B_1)^{(\beta+1)/(\beta+2)}(-E(t))^{-\alpha}\|e^{\Phi/(\beta+2)}u_t\|_{\beta+2}^{\beta+2}$$
$$+ \varepsilon^{\beta+2}(-E(0))^{-\alpha}\|e^{\Phi/(p+2)}u\|_{p+2}^{p+2}$$

with $\alpha = 1/(\beta+2) - 1/(p+2) > 0$ since $p > \beta$. Thus, by choosing

$$\varepsilon^{\beta+2} = \frac{p+2-qm}{2(p+2)}(-E(0))^\alpha$$

in (13), we easily see, from (9), that

$$(14)\qquad G'(t) \ge \{H(t) - m_0(-E(t))^{-\alpha}\|e^{\Phi/(\beta+2)}u_t\|_{\beta+2}^{\beta+2}\}$$

with

$$m_0^{\beta+1} = \frac{2(p+2)}{p+2-qm}B_1^{\beta+2}(-E(0))^{-\alpha}.$$

Now, we define

$$(15)\qquad L(t) \equiv (-E(t))^{1-\alpha} + (1-\alpha)m_0^{-1}G(t),$$

where $\alpha = 1/(\beta+2) - 1/(p+2)$. Then differentiation of (15), using (5) and (14), yields

$$(16)\quad \begin{aligned} L'(t) &= (1-\alpha)(-E(t))^{-\alpha}(-E'(t)) + (1-\alpha)m_0^{-1}G'(t) \\ &= (1-\alpha)(-E(t))^{-\alpha}\|e^{\Phi/(\beta+2)}u_t\|_{\beta+2}^{\beta+2} + (1-\alpha)m_0^{-1}G'(t) \\ &\geq (1-\alpha)m_0^{-1}H(t). \end{aligned}$$

Moreover, since $H(t) \geq qm(-E(t)) > 0$ and $E(t) \leq E(0)$, we obtain

$$L'(t) \geq (1-\alpha)qmm_0^{-1}(-E(0)) > 0,$$

and consequently there exists a $t_0 \geq 0$ such that

$$(17)\qquad L(t) \geq L(t_0) > 0 \quad \text{for } t \geq t_0,$$

where we can take $t_0 = 0$ if

$$(18)\qquad L(0) = (-E(0))^{1-\alpha} + (1-\alpha)m_0^{-1}\int_\Omega e^{\Phi(x)}|u_1|^{l-2}u_1u_0\,dx > 0.$$

Next we estimate

$$\begin{aligned} \Big|\int_\Omega e^{\Phi(x)}|u_t|^{l-2}u_tu\,dx\Big| &\leq \|e^{\Phi(x)/l}u_t\|_l^{l-1}\|e^{\Phi(x)/l}u\|_l \\ &\leq B_2\|e^{\Phi(x)/l}u_t\|_l^{l-1}\|e^{\Phi(x)/(p+2)}u\|_{p+2} \end{aligned}$$

where

$$B_2 = \Big(\int_\Omega e^{\Phi(x)}\,dx\Big)^{\frac{p+2-l}{l(p+2)}}.$$

Therefore

$$\begin{aligned} L(t)^{1/(1-\alpha)} &\leq 2^{\alpha/(1-\alpha)}\{(-E(t)) + (m_0^{-1}|G(t)|)^{1/(1-\alpha)}\} \\ &\leq 2^{\alpha/(1-\alpha)}\{(-E(t)) \\ &\quad + (B_2m_0^{-1}\|e^{\Phi(x)/l}u_t\|_l^{l-1}\|e^{\Phi(x)/(p+2)}u\|_{p+2})^{1/(1-\alpha)}\} \\ &\leq 2^{\alpha/(1-\alpha)}\Big\{qm(-E(t)) + \frac{l-1}{l(1-\alpha)}\,\|e^{\Phi(x)/l}u_t\|_l^l \\ &\quad + \frac{1-l\alpha}{l(1-\alpha)}(B_2m_0^{-1}\|e^{\Phi(x)/(p+2)}u\|_{p+2})^{l/(1-l\alpha)}\Big\} \end{aligned}$$

where we have used the Young inequality in the last step. Moreover, since

$$\frac{l}{1-l\alpha} < p+2 \quad (l < \beta+2) \quad \text{and} \quad (-E(0))^{-1/(p+2)}\|e^{\Phi(x)/(p+2)}u\|_{p+2} \geq 1,$$

we have, from (9),

$$(19)\quad \begin{aligned} L(t)^{1/(1-\alpha)} &\leq 2^{\alpha/(1-\alpha)}\Big\{qm(-E(t)) + \frac{l-1}{l(1-\alpha)}\,\|e^{\Phi(x)/l}u_t\|_l^l \\ &\quad + \frac{1-l\alpha}{l(1-\alpha)}\,(B_2m_0^{-1})^{l/(1-l\alpha)}(-E(0))^{-(1-\frac{l}{(1-l\alpha)(p+2)})}\|e^{\Phi(x)/(p+2)}u\|_{p+2}^{p+2}\Big\} \\ &\leq m_1H(t) \end{aligned}$$

where

$$m_1 = 2^{\alpha/(1-\alpha)} \max\Big\{1, \frac{l-1}{(l+mq(l-1))(1-\alpha)},$$
$$2(p+2)(p+2-qm)^{-1}\,\frac{1-l\alpha}{l(1-\alpha)}\,(B_2 m_0^{-1})^{l/(1-l\alpha)}(-E(0))^{-(1-\frac{l}{(1-l\alpha)(p+2)})}\Big\}.$$

Thus, a combination of (16) and (19) leads to

$$L'(t) \geq (1-\alpha)(m_0 m_1)^{-1} L(t)^{1/(1-\alpha)}. \tag{20}$$

A simple integration of (20) over (t_0, t) then yields

$$L(t) \geq \{L(t_0)^{-\alpha/(1-\alpha)} - \alpha(m_0 m_1)^{-1}(t-t_0)\}^{-(1-\alpha)/\alpha} \tag{21}$$

for some $t \geq t_0$. As $L(t_0) > 0$, (21) shows that L becomes infinite in a time

$$T_{\max} \leq T_0 = m_0 m_1(\alpha)^{-1} L(t_0)^{-\alpha/(1-\alpha)} + t_0.$$

COROLLARY 2.1. *Under the assumptions of Theorem* 2.1, *suppose that*

$$L(0) \equiv (-E(0))^{1-\alpha} + (1-\alpha)m_0^{-1}\int_\Omega e^{\Phi(x)}|u_1|^{l-2}u_1 u_0\,dx > 0.$$

Then the lifespan T of the solution satisfies $T \leq m_0 m_1(\alpha)^{-1} L(0)^{-\alpha/(1-\alpha)}$, where α, m_0 and m_1 are positive constants as above.

REMARK 2.1. 1. We may replace $|u_t|^\beta u_t$ by $a(x)|u_t|^\beta u_t$, where $a \in L^\infty(\Omega)$ satisfies $a(x) \geq a_0 > 0$.

2. The same method works as well for a system of the form

$$\begin{cases} (|u_t|^{l-2}u_t)_t - M(\|A^{1/2}u\|_2^2 + \|B^{1/2}v\|_2^2)Au + a_1(x)|u_t|^\beta u_t = \mu_1 |u|^p u, \\ (|v_t|^{l'-2}v_t)_t - M(\|A^{1/2}u\|_2^2 + \|B^{1/2}v\|_2^2)Bv + a_2(x)|v_t|^\beta v_t = \mu_2 |v|^p v, \end{cases}$$

where

$$Au = e^{-\Phi(x)}\operatorname{div}(e^{\Phi(x)}|\nabla_x u|^{q-2}\nabla_x u), \qquad \|A^{1/2}u\|^2 = \int_\Omega e^{\Phi(x)}|\nabla_x u|^q\,dx,$$
$$Bv = e^{-\Psi(x)}\operatorname{div}(e^{\Psi(x)}|\nabla_x v|^{q'-2}\nabla_x v), \qquad \|B^{1/2}v\|^2 = \int_\Omega e^{\Psi(x)}|\nabla_x v|^{q'}\,dx,$$

with $\mu_1 > 0$, $a_1(x) > \lambda_1 > 0$, $\mu_2 > 0$ and $a_2(x) > \lambda_2 > 0$ where λ_1 and λ_2 are two positive constants (we can take $\mu_2 = a_2 \equiv 0$ or $\mu_2 > 0$ and $a_2 \equiv 0$).

REFERENCES

[1] A. B. Aliev and K. H. Khanmamedov, *Energy estimates for solutions of the mixed problem for linear second-order hyperbolic equations*, Math. Notes 59 (1996), 345–349.

[2] D. Bainov and E. Minchev, *An estimate of the interval of existence of solutions of the nonlinear Kirchhoff equation*, Comput. Math. Appl. 31 (1996), 63–65.

[3] J. M. Ball, *Remarks on blow up and nonexistence theorems for nonlinear evolution equations*, Quart. J. Math. Oxford 28 (1977), 473–483.

[4] A. Benaissa, *Existence globale et décroissance polynomiale de l'énergie des solutions des équations de Kirchhoff–Carrier moyennement dégénérées avec des termes nonlinéaires dissipatifs*, Bull. Belgian Math. Soc. 47 (2001), 607–622.

[5] A. Benaissa and S. A. Messaoudi, *Blowup of solutions of a nonlinear wave equation*, J. Appl. Math. 2 (2002), 105–108.

[6] P. D'Ancona and S. Spagnolo, *Global solvability for the degenerate Kirchhoff equation with real analytic data*, Invent. Math. 108 (1992), 247–262.

[7] R. W. Dickey, *Infinite systems of nonlinear oscillation equations*, J. Differential Equations 8 (1970), 16–26.

[8] —, *Infinite systems of nonlinear oscillation equations related to the string*, Proc. Amer. Math. Soc. 23 (1969), 459–468.

[9] F. Hirosawa, *Global solvability for the degenerate Kirchhoff equation with real-analytic data in* $\mathbb{R}^n$, Tsukuba J. Math. 21 (1997), 483–503.

[10] V. Georgiev and G. Todorova, *Existence of a solution of the wave equation with nonlinear damping and source terms*, J. Differential Equations 109 (1994), 295–308.

[11] D. Gourdin et M. Mechab, *Problème de Goursat non linéaire dans les classes de Gevrey, pour des équations de Kirchhoff généralisées*, J. Math. Pures Appl. 75 (1996), 569–593.

[12] K. Kajitani, *On the Cauchy problem for Kirchhoff equations of p-Laplacian type*, in: Proc. of Colloque of P.D.E. in Memory of Jean Leray, Sweden, 1999.

[13] K. Kajitani and K. Yamaguti, *On global real analytic solutions of the degenerate Kirchhoff equation*, Ann. Scuola Norm. Sup. Pisa 21 (1994), 279–297.

[14] V. K. Kalantarov and O. A. Ladyzhenskaya, *The occurrence of collapse for quasilinear equations of parabolic and hyperbolic type*, J. Soviet Math. 10 (1978), 53–70.

[15] G. Kirchhoff, *Vorlesungen über Mechanik*, Teubner, Leipzig, 1883.

[16] L. A. Medeiros and M. Miranda, *Solutions for the equation of nonlinear vibrations in Sobolev spaces of fractionary order*, Mat. Apl. Comput. 6 (1987), 257–276.

[17] S. A. Messaoudi, *Blow up in solutions of a semilinear wave equation*, Internat. J. Appl. Math. 1 (1999), 621–626.

[18] H. A. Levine, *Instability and nonexistence of global solutions of nonlinear wave equations of the form* $Pu_{tt} = -Au + \mathfrak{F}(u)$, Trans. Amer. Math. Soc. 192 (1974), 1–21.

[19] —, *Some additional remarks on the nonexistence of global solutions to nonlinear wave equations*, SIAM J. Math. Anal. 5 (1974), 138–146.

[20] H. A. Levine and J. Serrin, *Global nonexistence theorems for quasilinear evolution equations with dissipation*, Arch. Rational Mech. Anal. 137 (1997), 341–361.

[21] H. A. Levine, S. R. Park and J. Serrin, *Global existence and global nonexistence of solutions of the Cauchy problem for a nonlinearly damped wave equation*, J. Math. Anal. Appl. 228 (1998), 181–205.

[22] S. Spagnolo, *The Cauchy problem for Kirchhoff equations*, Rend. Sem. Mat. Fis. Milano 62 (1992), 17–51.

[23] E. Vittilaro, *Global nonexistence theorems for a class of evolution equations with dissipation*, Arch. Rational Mech. Anal. 137 (1997), 341–361.

Department of Mathematics
Djillali Liabes University
P.O. Box 89, Sidi Bel Abbes 22000, Algeria
E-mail: benaissa_abbes@yahoo.com

Mathematical Sciences Department
KFUPM
Dhahran 31261, Saudi Arabia
E-mail: messaoud@kfupm.edu.sa

Received 29 January 2002;
revised 28 March 2002 (4164)